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Full rank interpolatory subdivision schemes: Kronecker, filters and multiresolution

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ABSTRACT

In this extension of earlier work, we point out several ways how a multiresolution analysis can be derived from a finitely supported interpolatory matrix mask which has a positive definite symbol on the unit circle except at -1. A major tool in this investigation will be subdivision schemes that are obtained by using convolution or correlation operations based on replacing the usual matrix multiplications by Kronecker products.

1. Introduction

In the recent papers [1,2] we introduced and studied *full rank interpolatory* vector subdivision schemes. In particular, we investigated in [1] an extension of positivity of the symbol to the vector case and the implied convergence of associated subdivision schemes. Keeping in mind that the *symbol* of a finitely supported *mask* $\mathcal{A} = (\boldsymbol{A}_j : j \in \mathbb{Z})$ is the matrix valued Laurent polynomial $\boldsymbol{A}(z) = \sum_j \boldsymbol{A}_j z^j$, one of the main results in [1] can be formulated as follows.

Theorem 1. *If $\mathcal{A}$ is a finitely supported mask such that the associated symbol $\boldsymbol{A}(z)$ satisfies $\boldsymbol{A}(-1) = \boldsymbol{0}$, the interpolatory condition $\boldsymbol{A}(z) + \boldsymbol{A}(-z) = 2\boldsymbol{I}$, $z \in \mathbb{C}^*$ and is positive definite on $\{z \in \mathbb{C} : |z| = 1\} \setminus \{-1\}$, then there exists a canonical spectral factor $\mathcal{B}$ of $\mathcal{A}$ such that $\boldsymbol{A}(z) = \frac{1}{2}\boldsymbol{B}^H(z)\,\boldsymbol{B}(z)$ and an orthogonal $\mathcal{B}$-refinable function $\boldsymbol{G} \in L_2^{r\times r}(\mathbb{R})$.*

This result allowed us to introduce and investigate various different subdivision schemes which were in part classical stationary ones, but there also naturally appeared a nonstandard type of subdivision scheme which we called *correlated* since it consists of applying the subdivision scheme to the data sequence as well as to the "identity sequence" $\boldsymbol{I}\delta$ and then correlating the results:

$$\mathcal{S}_{\boldsymbol{B}}^n := \frac{1}{2^n}\left(S_{\mathcal{B}}^n \delta \boldsymbol{I}\right)^T \star S_{\mathcal{B}}^n, \tag{1}$$

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where $S_{\mathcal{B}}$ denotes the usual stationary subdivision scheme with respect to $\mathcal{B}$. The notational details will be explained in the next section.

Each of these subdivision schemes – if convergent – defines a *refinable function* where refinability has to be understood either in the classical or in a more general sense. These schemes are:

(1) the subdivision scheme $S_{\mathcal{A}}$ itself, based on the full rank interpolatory mask $\mathcal{A}$ whose symbol $\boldsymbol{A}(z)$ is assumed to be positive definite on the unit circle. The associated matrix refinable function (if it exists) is a cardinal function $\boldsymbol{F}$ and a partition of the identity, that is, $\boldsymbol{F}(k) = \delta_{k,0}\, \boldsymbol{I}$ and $\sum_k \boldsymbol{F}(\cdot - k) = \boldsymbol{I}$. However, the convergence of $S_{\boldsymbol{A}}$ or, equivalently, the existence of the cardinal refinable function $\boldsymbol{F}$ could not be concluded from the assumptions of Theorem 1. Whether or not this refinable function exists is still an open question.

(2) The subdivision scheme $S_{\mathcal{B}}$ based on the full rank mask $\mathcal{B}$ whose symbol is the canonical spectral factor, of $\boldsymbol{A}(z)$, i.e.

$$\boldsymbol{A}(z) = \frac{1}{2}\boldsymbol{B}^H(z)\boldsymbol{B}(z).$$

According to [1], the associated matrix refinable function $\boldsymbol{G}$ exists, is of full rank and is orthogonal so that the associated subdivision scheme converges in $L_2^r(\mathbb{R})$.

(3) The *correlated subdivision scheme* $\mathcal{S}_{\mathcal{B}}$ based on the full rank mask $\mathcal{B}$. The associated limit matrix function $\boldsymbol{F}_\star \in C_u^{r\times r}(\mathbb{R})$, whose existence was proved in [1], is refinable in the following sense:

$$\boldsymbol{F}_\star = \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \boldsymbol{B}_k^T \boldsymbol{F}_\star(2\cdot - j + k)\boldsymbol{B}_j. \tag{2}$$

Furthermore, $\boldsymbol{F}_\star$ is cardinal, $\boldsymbol{F}_\star(k) = \delta_{0k}\,\boldsymbol{I}$, and satisfies the partition of the identity property

$$\sum_{k\in\mathbb{Z}} \boldsymbol{F}_\star\,(\cdot - k) = \boldsymbol{I}.$$

Note that in the scalar case $r = 1$ the two functions $\boldsymbol{F}$ and $\boldsymbol{F}_*$ coincide as also do the respective subdivision schemes and refinement equations.

(4) The subdivision scheme $S_{\mathcal{C}}$ based on the mask $\mathcal{C}$ defined by means of the Kronecker product of symbols as

$$\boldsymbol{C}(z) = \frac{1}{2}\boldsymbol{B}(z) \otimes \boldsymbol{B}\left(z^{-1}\right), \tag{3}$$

and its associated *vector* refinable function Φ, that is, a solution of the refinement equation

$$\Phi = \sum_{k\in\mathbb{Z}} \boldsymbol{C}_k^T \Phi\,(2\cdot - k)$$

which could be derived directly from $\boldsymbol{F}_\star$.

In this paper, we will consider two more aspects. First, we will investigate more closely properties of the subdivision scheme $S_{\mathcal{C}}$ based on the full rank mask $\mathcal{C}$, proving that the subdivision scheme converges and that its associated full rank basic limit function $\boldsymbol{H}$ is stable and thus can be used to define a multiresolution analysis (usually abbreviated as "MRA"). Second, we will define multiresolution analyses and/or filter banks associated to all the above-mentioned "refinable" functions, pointing out some of the connections between them. All these MRAs will be suitable for *vector data processing*, and could be applied to vector valued time series, for example, in the analysis of EEG signals, cf. [3].

2. Notation and background

For $r \in \mathbb{N}$ we write an $r \times r$ matrix $\boldsymbol{A} \in \mathbb{R}^{r\times r}$ as $\boldsymbol{A} = \left[A_{jk} \,:\, j, k = 1, \ldots, r\right]$ and denote by $\ell_\infty^{r\times r}(\mathbb{Z})$ the Banach space of all $r \times r$ matrix valued bi-infinite sequences with bounded operator norm, considered as convolution operators on $\ell^{r\times 1}(\mathbb{Z})$. More precisely, $\mathcal{A} = \left(\boldsymbol{A}_j \,:\, j \in \mathbb{Z}\right) \in \ell_\infty^{r\times r}(\mathbb{Z})$, is defined by

$$\|\mathcal{A}\| := \|\mathcal{A}\|_\infty := \sum_{j\in\mathbb{Z}} \left|\boldsymbol{A}_j\right|_\infty < \infty, \qquad |\boldsymbol{A}|_\infty = \max_{1\le j\le r} \sum_{k=1}^{r} \left|A_{jk}\right|. \tag{4}$$

For notational simplicity we write $\ell_\infty^r(\mathbb{Z})$ for $\ell_\infty^{r\times 1}(\mathbb{Z})$ and denote vector sequences by lowercase letters. Moreover, $C_u^{r\times r}(\mathbb{R})$ will denote the Banach space of all uniformly continuous uniformly bounded $r \times r$ matrix valued functions on $\mathbb{R}$ with the norm

$$\|\boldsymbol{F}\|_\infty := \sup_{x\in\mathbb{R}} |\boldsymbol{F}(x)|_\infty < \infty.$$

For two matrix sequences we introduce the *convolution* "$*$" and the *correlation* "$\star$" defined, respectively, as

$$(\mathcal{A} * \mathcal{B})_j := \sum_{k\in\mathbb{Z}} \boldsymbol{A}_{j-k}\,\boldsymbol{B}_k, \qquad (\mathcal{A} \star \mathcal{B})_j := \sum_{k\in\mathbb{Z}} \boldsymbol{A}_{j+k}\,\boldsymbol{B}_k$$

and between a matrix function and a matrix sequence as

$$(\boldsymbol{F} * \mathcal{B}) := \sum_{k\in\mathbb{Z}} \boldsymbol{F}(\cdot - k)\, \boldsymbol{B}_k, \qquad (\boldsymbol{F} \star \mathcal{B}) := \sum_{k\in\mathbb{Z}} \boldsymbol{F}(\cdot + k)\, \boldsymbol{B}_k.$$

Similarly, we introduce the *Kronecker convolution* "$\circledast$" and the *Kronecker correlation* "$\bigcirc\!\!\!\!\star$" between matrix sequences and between matrix functions and matrix sequences, respectively, where the multiplication between matrices is now a *Kronecker product*: for two $r \times r$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ this is defined as

$$\boldsymbol{A} \otimes \boldsymbol{B} := [a_{ij}\boldsymbol{B}, i, j = 1, \ldots, r].$$

Closely related to Kronecker products is the operator vec($\boldsymbol{X}$) which, given a matrix $\boldsymbol{X} \in \mathbb{R}^{m\times n}$ with column vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^m$, is defined as

$$\mathrm{vec}(\boldsymbol{X}) = \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_n \end{bmatrix}.$$

The Kronecker convolution and correlation between matrix valued sequences are defined as

$$\left(\mathcal{A} \circledast \mathcal{B}\right)_j := \sum_{k\in\mathbb{Z}} \boldsymbol{A}_{j-k} \otimes \boldsymbol{B}_k, \qquad \left(\mathcal{A} \bigcirc\!\!\!\!\star\, \mathcal{B}\right)_j := \sum_{k\in\mathbb{Z}} \boldsymbol{A}_{j+k} \otimes \boldsymbol{B}_k,$$

and, between a function and a sequence, as

$$\boldsymbol{F} \circledast \mathcal{B} := \sum_{k\in\mathbb{Z}} \boldsymbol{F}(\cdot - k) \otimes \boldsymbol{B}_k, \qquad \boldsymbol{F} \bigcirc\!\!\!\!\star\, \mathcal{B} := \sum_{k\in\mathbb{Z}} \boldsymbol{F}(\cdot + k) \otimes \boldsymbol{B}_k.$$

Finally, the Kronecker convolution "$\circledast$" and correlation "$\bigcirc\!\!\!\!\star$" will be also considered between two matrix valued functions, say $\boldsymbol{F}$ and $\boldsymbol{G}$, producing the matrix valued functions whose elements are constructed via the Kronecker product and convolution or correlation of functions as

$$\boldsymbol{F} \circledast \boldsymbol{G} := \int_{\mathbb{R}} \boldsymbol{F}(t) \otimes \boldsymbol{G}(\cdot - t)\, \mathrm{d}t = \left[f_{jk} * \boldsymbol{G} : j, k = 1, \ldots, r\right],$$

and

$$\boldsymbol{F} \bigcirc\!\!\!\!\star\, \boldsymbol{G} := \int_{\mathbb{R}} \boldsymbol{F}(t) \otimes \boldsymbol{G}(\cdot + t)\, \mathrm{d}t = \left[f_{jk} \star \boldsymbol{G} : j, k = 1, \ldots, r\right],$$

respectively. Next we state an elementary but interesting observation about Kronecker type convolution and correlation.

Lemma 2. *Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be two finitely supported and continuous $r \times r$ matrix valued functions. Let $\boldsymbol{H} = \boldsymbol{F} \bigcirc\!\!\!\!\star\, \boldsymbol{G}$ and $c = a \bigcirc\!\!\!\!\star\, b$ for $a, b \in \ell_\infty^r(\mathbb{Z})$. Then,*

$$\boldsymbol{H} * c = (\boldsymbol{F} \star a) \bigcirc\!\!\!\!\star\, (\boldsymbol{G} \star b) \quad \text{and} \quad \boldsymbol{H} \star c = (\boldsymbol{F} * a) \bigcirc\!\!\!\!\star\, (\boldsymbol{G} * b). \tag{5}$$

If $\boldsymbol{H} = \boldsymbol{F} \circledast \boldsymbol{G}$ and $c = a \circledast b$, on the other hand, we have

$$\boldsymbol{H} * c = (\boldsymbol{F} * a) \circledast (\boldsymbol{G} * b) \quad \text{and} \quad \boldsymbol{H} \star c = (\boldsymbol{F} \star a) \circledast (\boldsymbol{G} \star b). \tag{6}$$

Proof. For $\boldsymbol{H} = \boldsymbol{F} \bigcirc\!\!\!\!\star\, \boldsymbol{G}$ and for $c = a \bigcirc\!\!\!\!\star\, b$ the matrix function $\boldsymbol{H} * c$ can be written as

$$\begin{aligned}
\boldsymbol{H} * c &= \sum_{k\in\mathbb{Z}} \boldsymbol{H}(\cdot - k) \sum_{j\in\mathbb{Z}} \boldsymbol{a}_{k+j} \otimes \boldsymbol{b}_j \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \boldsymbol{F}(t) \otimes \boldsymbol{G}(\cdot - k + t)\, \mathrm{d}t \sum_{j\in\mathbb{Z}} \boldsymbol{a}_{k+j} \otimes \boldsymbol{b}_j \\
&= \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} \boldsymbol{F}(t + k)\, \boldsymbol{a}_k \right) \otimes \left(\sum_{j\in\mathbb{Z}} \boldsymbol{G}(\cdot + j + t)\, \boldsymbol{b}_j \right) \mathrm{d}t \\
&= \left(\boldsymbol{F} \star a\right) \bigcirc\!\!\!\!\star\, \left(\boldsymbol{G} \star b\right)
\end{aligned}$$

where we used the well-known formula $(\boldsymbol{AB}) \otimes (\boldsymbol{CD}) = (\boldsymbol{A} \otimes \boldsymbol{C})\, (\boldsymbol{B} \otimes \boldsymbol{D})$, cf. [4], to complete the proof. Practically identical arguments can be used to verify the second identity in (5) and (6). □

Proposition 3. *Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be two finitely supported and continuous $r \times r$ matrix valued functions. Let $\boldsymbol{H} = \boldsymbol{F} \circledstar \boldsymbol{G}$ and $c \in \ell_\infty^{r^2}(\mathbb{Z})$. Then the matrix function $\boldsymbol{H} * c$ can be written as*

$$\boldsymbol{H} * c = \sum_{\ell=1}^{r} \left(\boldsymbol{F} \star u^\ell\right) \circledstar \left(\boldsymbol{G} \star v^\ell\right)$$

for suitable choices of $u^\ell, v^\ell \in \ell_\infty^r(\mathbb{Z})$. If, on the other hand, $\boldsymbol{H} = \boldsymbol{F} \circledast \boldsymbol{G}$ then

$$\boldsymbol{H} * c = \sum_{\ell=1}^{r} \left(\boldsymbol{F} * u^\ell\right) \circledast \left(\boldsymbol{G} * v^\ell\right).$$

Proof. As shown in [5], any sequence $c \in \ell_\infty^{r^2}(\mathbb{Z})$ can be written as the sum of r Kronecker convolutions of sequences in $\ell_\infty^r(\mathbb{Z})$ as $c = \sum_{\ell=1}^r u^\ell \circledstar v^\ell$ where the sequences $u^\ell,\ v^\ell$ are even *explicitly* given as

$$\boldsymbol{u}_k^\ell := \begin{pmatrix} (c_k)_\ell \\ (c_k)_{\ell+r} \\ (c_k)_{\ell+2r} \\ \vdots \\ (c_k)_{\ell+(r-1)r} \end{pmatrix}, \qquad \boldsymbol{v}_k^\ell = \delta_{k0} \boldsymbol{e}_\ell \quad k \in \mathbb{Z}, \ell = 1, \ldots, r. \tag{7}$$

Thus, for $\boldsymbol{H} = \boldsymbol{F} \circledstar \boldsymbol{G}$ and for $c \in \ell_\infty^{r^2}(\mathbb{Z})$ Lemma 2 completes the proof. For the second identity we just note that u^ℓ, v^ℓ can be chosen in a way similar to (7) such that $c = \sum_\ell u^\ell \circledast v^\ell$. $\square$

The *subdivision operator* $S_{\mathcal{A}}$ based on the *mask* $\mathcal{A} \in \ell_\infty^{r \times r}(\mathbb{Z})$ is defined as the operator that maps any $c = (\boldsymbol{c}_k : k \in \mathbb{Z}) \in \ell_\infty^r(\mathbb{Z})$ to

$$(S_{\mathcal{A}} c)_j = \sum_{k \in \mathbb{Z}} \boldsymbol{A}_{j-2k} \boldsymbol{c}_k, \quad j \in \mathbb{Z}.$$

Thus, for any finitely supported $\mathcal{A} \in \ell_0^{r \times r}(\mathbb{Z})$, the subdivision operator $S_{\mathcal{A}}$ is a continuous linear map from $\ell^r(\mathbb{Z})$ to $\ell^r(\mathbb{Z})$ which can be easily extended to an operator from $\ell^{r \times s}(\mathbb{Z})$ to $\ell^{r \times s}(\mathbb{Z})$, $s \geq 1$, by acting on the column vectors of matrix sequences separately. Throughout this paper we will tacitly assume that the mask is *real* and finitely supported, hence $\boldsymbol{A}_j = \boldsymbol{0}$ for $j \notin [-N, N]$ for some suitable $N \in \mathbb{N}$.

The *subdivision scheme* then consists of iterative applications of the subdivision operator to an initial sequence $c^0 := c$ yielding

$$c^{n+1} := S_{\mathcal{A}} c^n = S_{\mathcal{A}}^{n+1} c^0, \quad n \geq 0.$$

A subdivision scheme is said to be uniformly *convergent* if for any $c \in \ell_\infty^r(\mathbb{Z})$ there exists a uniformly continuous vector valued function $\boldsymbol{f}_c$ such that

$$\lim_{n \to \infty} \sup_{j \in \mathbb{Z}} \left\| \left(S_{\mathcal{A}}^n c\right)_j - \boldsymbol{f}_c\left(2^{-n} j\right) \right\|_\infty = 0. \tag{8}$$

An equivalent description of convergence is to demand the existence of the *basic limit function* as the limit of the matrix sequence $S_{\mathcal{A}}^n \delta \boldsymbol{I}$, where $(\delta \boldsymbol{I})_k = \delta_{k,0} \boldsymbol{I}$, that is, the existence of a uniformly continuous matrix valued function $\boldsymbol{F}$ such that

$$\lim_{n \to \infty} \sup_{j \in \mathbb{Z}} \left\| \left(S_{\mathcal{A}}^n \delta \boldsymbol{I}\right)_j - \boldsymbol{F}\left(2^{-n} j\right) \right\|_\infty = 0. \tag{9}$$

In fact, in the case of convergence of the subdivision scheme we have that

$$\boldsymbol{f}_c = \boldsymbol{F} * c = \sum_{j \in \mathbb{Z}} \boldsymbol{F}(\cdot - j)\, \boldsymbol{c}_j.$$

The basic limit function is *refinable* with respect to $\mathcal{A}$, which means that it satisfies the functional equation

$$\boldsymbol{F} = \boldsymbol{F} * \boldsymbol{A}(2\cdot) = \sum_{j \in \mathbb{Z}} \boldsymbol{F}\left(2 \cdot - j\right)\, \boldsymbol{A}_j. \tag{10}$$

In the L_2^r-setting (8) is replaced by

$$\lim_{n \to \infty} 2^{-n/2} \left\| \mu^n\left(\boldsymbol{f}_c\right) - S_{\mathcal{A}}^n c \right\|_2 = 0, \quad \boldsymbol{f}_c \in L_2^r(\mathbb{R})$$

where the *mean value operator* at level $n \in \mathbb{N}$ is defined for $\boldsymbol{f} \in L_2^r(\mathbb{R})$ as

$$\mu^n(\boldsymbol{f})(k) := 2^n \int_{2^{-n}k}^{2^{-n}(k+1)} \boldsymbol{f}(t)\,\mathrm{d}t, \quad k \in \mathbb{Z},$$

cf. [6–8].

As usual, we associate both to the mask $\mathcal{A}$ and to the subdivision operator $S_{\mathcal{A}}$ the *symbol* $\boldsymbol{A}(z)$, which is a matrix valued Laurent polynomial of the form

$$\boldsymbol{A}(z) = \sum_{k\in\mathbb{Z}} \boldsymbol{A}_k z^k, \quad z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$$

which in turn naturally defines the matrix valued trigonometric polynomial

$$\widehat{\boldsymbol{A}}(\theta) = \frac{1}{2}\boldsymbol{A}\left(\mathrm{e}^{-\mathrm{i}\theta}\right), \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

The *rank* of the mask $\mathcal{A}$ or of the associated subdivision scheme $S_{\mathcal{A}}$ is the number

$$R(\mathcal{A}) := \dim \left\{ \boldsymbol{y} \in \mathbb{R}^r \;:\; \left(\sum_{j\in\mathbb{Z}} \boldsymbol{A}_{2j}\right)\boldsymbol{y} = \left(\sum_{j\in\mathbb{Z}} \boldsymbol{A}_{2j+1}\right)\boldsymbol{y} = \boldsymbol{y} \right\}, \tag{11}$$

satisfying, for convergent schemes, $1 \le R(\mathcal{A}) \le r$, cf. [9]. A subdivision scheme $S_{\mathcal{A}}$ is said to be *of full rank* if $R(\mathcal{A}) = r$. As pointed out in [2], full rank schemes appear most naturally in the context of *interpolatory* vector subdivision schemes which are characterized by the property that for any $c \in \ell_\infty^r(\mathbb{Z})$

$$(S_{\mathcal{A}}c)_{2j} = \boldsymbol{c}_j, \quad j \in \mathbb{Z}, \tag{12}$$

or, equivalently, $\boldsymbol{A}_{2j} = \delta_{j0}\boldsymbol{I}, j \in \mathbb{Z}$. We can also describe full rank and interpolatory properties of a subdivision mask in terms of the symbol $\boldsymbol{A}(z)$, recalling that:

(1) $\mathcal{A}$ is of full rank iff $\boldsymbol{A}(1) = 2\boldsymbol{I}$ and $\boldsymbol{A}(-1) = \boldsymbol{0}$,
(2) $\mathcal{A}$ is interpolatory iff $\boldsymbol{A}(z) + \boldsymbol{A}(-z) = 2\boldsymbol{I}$,
(3) an interpolatory subdivision scheme is of full rank iff $\boldsymbol{A}(1) = 2\boldsymbol{I}$,

see again [2]. If a convergent subdivision scheme is interpolatory, then its associated basic limit function is *cardinal*, i.e., it satisfies $\boldsymbol{F}(k) = \delta_{0k}\boldsymbol{I}$, vanishing at all integers except zero where its value is the identity matrix $\boldsymbol{I}$.

Throughout the paper we will always assume that we are given a mask $\mathcal{A}$ satisfying the assumptions of Theorem 1, that is, $\boldsymbol{A}(-1) = \boldsymbol{0}$, $\boldsymbol{A}(z) + \boldsymbol{A}(-z) = 2\boldsymbol{I}$, $z \in \mathbb{C}^*$, and that $\boldsymbol{A}$ is (strictly) positive definite on the unit circle except at -1. Under these circumstances the existence of the mask $\mathcal{B}$ and the existence of the associated orthogonal refinable function $\boldsymbol{G}$ are ensured.

3. Properties of the subdivision scheme $S_{\mathcal{C}}$

Using the spectral factor $\mathcal{B}$ of $\mathcal{A}$, introduced in Theorem 1, we define the mask $\mathcal{C} \in \ell^{r^2\times r^2}(\mathbb{Z})$ by means of the *Kronecker correlation* as

$$\mathcal{C} = \frac{1}{2}(\mathcal{B} \circledast \mathcal{B}), \qquad \boldsymbol{C}_j = \frac{1}{2}\left(\sum_k \boldsymbol{B}_{j+k} \otimes \boldsymbol{B}_k\right), \quad j \in \mathbb{Z}. \tag{13}$$

As already shown in [1], this mask is of full rank, i.e., $\boldsymbol{C}(1) = 2\boldsymbol{I}$, $\boldsymbol{C}(-1) = \boldsymbol{0}$. To show that also $\mathcal{C}$ leads to a convergent subdivision scheme, we define the Kronecker autocorrelation function

$$\boldsymbol{H} := \boldsymbol{G} \circledast \boldsymbol{G}(-\cdot) = \int_{\mathbb{R}} \boldsymbol{G}(t) \otimes \boldsymbol{G}(t-\cdot)\,\mathrm{d}t$$

and first note that

$$\widehat{\boldsymbol{H}}(\xi) = \widehat{\boldsymbol{G}}(\xi) \otimes \overline{\widehat{\boldsymbol{G}}(\xi)} = \left[\widehat{g}_{jk}(\xi)\,\overline{\widehat{\boldsymbol{G}}(\xi)} \;:\; j,k = 1,\ldots,r\right],$$

hence, by Hölder's inequality, $\widehat{\boldsymbol{H}} \in L_1^{r^2\times r^2}(\mathbb{R})$ and therefore $\boldsymbol{H} \in C_u^{r^2\times r^2}(\mathbb{R})$, the space of uniformly continuous $r^2 \times r^2$ matrix valued functions. Moreover, since $\mathcal{C}$ is finitely supported we also have that $\boldsymbol{H}$ is of finite support.

Proposition 4. *The subdivision scheme $S_{\mathcal{C}}$ converges with basic limit function $\boldsymbol{H} \in C_u^{r\times r}(\mathbb{R})$.*

Proof. Continuity of $\boldsymbol{H}$ has already been observed above, and so we proceed showing that $\boldsymbol{H}$ is a refinable matrix function. To that end, we consider $\widehat{\boldsymbol{H}} = \widehat{\boldsymbol{G}} \otimes \overline{\widehat{\boldsymbol{G}}}$ and use the refinability of $\boldsymbol{G}$ to realize that for $\xi \in \mathbb{R}$

$$\begin{aligned}\frac{1}{2}\widehat{\boldsymbol{H}}(\xi/2)\,\widehat{\boldsymbol{C}}(\xi/2) &= \frac{1}{2}\left(\widehat{\boldsymbol{G}}(\xi/2)\otimes\overline{\widehat{\boldsymbol{G}}(\xi/2)}\right)\left(\frac{1}{2}\widehat{\boldsymbol{B}}(\xi/2)\otimes\overline{\widehat{\boldsymbol{B}}(\xi/2)}\right)\\ &= \frac{1}{2}\left(\widehat{\boldsymbol{G}}(\xi/2)\,\widehat{\boldsymbol{B}}(\xi/2)\right)\otimes\frac{1}{2}\overline{\left(\widehat{\boldsymbol{G}}(\xi/2)\,\widehat{\boldsymbol{B}}(\xi/2)\right)}\\ &= \widehat{\boldsymbol{G}}\,(\xi)\ \overline{\widehat{\boldsymbol{G}}\,(\xi)}\\ &= \widehat{\boldsymbol{H}}\,(\xi)\,.\end{aligned}$$

To show convergence, we first observe that a simple inductive proof gives that for $n \geq 0$

$$\prod_{j=0}^{n}\frac{1}{2}\widehat{\boldsymbol{C}}\left(2^{-j}\xi\right) = \left(\prod_{j=0}^{n}\frac{1}{2}\widehat{\boldsymbol{B}}\left(2^{-j}\xi\right)\right)\otimes\left(\prod_{j=0}^{n}\frac{1}{2}\overline{\widehat{\boldsymbol{B}}\left(2^{-j}\xi\right)}\right). \tag{14}$$

Indeed, the case $n = 0$ is just the product formula above for Kronecker products, while in the general case we note that

$$\begin{aligned}\prod_{j=0}^{n+1}\frac{1}{2}\widehat{\boldsymbol{C}}\left(2^{-j}\xi\right) &= \left(\prod_{j=0}^{n}\frac{1}{2}\widehat{\boldsymbol{C}}\left(2^{-j}\xi\right)\right)\left(\frac{1}{2}\widehat{\boldsymbol{B}}\left(2^{-n-1}\xi\right)\otimes\frac{1}{2}\overline{\widehat{\boldsymbol{B}}\left(2^{-n-1}\xi\right)}\right)\\ &= \left(\prod_{j=0}^{n}\frac{1}{2}\widehat{\boldsymbol{B}}\left(2^{-j}\xi\right)\right)\otimes\left(\prod_{j=0}^{n}\frac{1}{2}\overline{\widehat{\boldsymbol{B}}\left(2^{-j}\xi\right)}\right)\left(\frac{1}{2}\widehat{\boldsymbol{B}}\left(2^{-n-1}\xi\right)\otimes\frac{1}{2}\overline{\widehat{\boldsymbol{B}}\left(2^{-n-1}\xi\right)}\right)\\ &= \left(\prod_{j=0}^{n+1}\frac{1}{2}\widehat{\boldsymbol{B}}\left(2^{-j}\xi\right)\right)\otimes\left(\prod_{j=0}^{n+1}\frac{1}{2}\overline{\widehat{\boldsymbol{B}}\left(2^{-j}\xi\right)}\right)\end{aligned}$$

by another application of the product formula. Hence, since the infinite product for $\widehat{\boldsymbol{B}}$ converges to $\widehat{\boldsymbol{G}}$, the limit of (14) exists for $n \to \infty$ and is $\widehat{\boldsymbol{G}} \otimes \overline{\widehat{\boldsymbol{G}}} = \widehat{\boldsymbol{H}}$. But convergence of the cascade algorithm is equivalent to convergence of the subdivision scheme and this completes our proof. $\square$

Since $S_{\mathcal{C}}$ is a *full rank scheme* and converges by Proposition 4, we can apply [10, Proposition 1] to conclude that

$$\sum_{k\in\mathbb{Z}}\boldsymbol{H}\,(\cdot - k) = \boldsymbol{I} \tag{15}$$

and so the rows and columns of $\boldsymbol{H}$ are linearly independent, respectively. Moreover, we note that all the *rows* $\boldsymbol{h}_j^T$, $j = 1, \ldots, r^2$ of $\boldsymbol{H}$, i.e., $\boldsymbol{H}^T = \left[\boldsymbol{h}_1, \ldots, \boldsymbol{h}_{r^2}\right]$, are refinable vector functions with respect to $\mathcal{C}$, so that for $j = 1, \ldots, r^2$ we have

$$\boldsymbol{h}_j^T(x) = \sum_{k\in\mathbb{Z}}\boldsymbol{h}_j^T\,(2x-k)\ \boldsymbol{C}_k, \quad \text{or} \quad \boldsymbol{h}_j(x) = \sum_{k\in\mathbb{Z}}\boldsymbol{C}_k^T\,\boldsymbol{h}_j\,(2x-k)\,.$$

As already observed in [11], we again face the situation that in matrix refinable functions the *columns* are linearly independent while the *rows* are refinable.

Another important property of $\boldsymbol{H}$ is stated in the following result.

Theorem 5. *The function* $\boldsymbol{H}$ *is stable, i.e.,* $\boldsymbol{H} * c = \boldsymbol{0}$ *implies that* $c = 0$.

Proof. We recall, once more from [1], that the Fourier transform $\widehat{\boldsymbol{G}}$ of $\boldsymbol{G}$ has the property that there exists a constant $\rho > 0$ such that $\left|\det\widehat{\boldsymbol{G}}(\xi)\right| \geq \rho$ for $\xi \in [-\pi, \pi]$ — this fact was used in the proof of [1, Theorem 2.2], where the function was named Φ and the underlying argument to give a lower estimate for the infinite product is very similar to the one used in [12]. It is based on the fact that under the assumptions on $\mathcal{A}$ the trigonometric polynomial $\widehat{\boldsymbol{B}}$ is *strictly* positive definite on $[-\pi/2, \pi/2]$ so that the determinant of the infinite product can be estimated from below by a positive constant.

Since $\widehat{\boldsymbol{H}} = \widehat{\boldsymbol{G}} \otimes \overline{\widehat{\boldsymbol{G}}}$, the determinant formula for Kronecker products, $\det(\boldsymbol{X}\otimes\boldsymbol{Y}) = (\det\boldsymbol{X})^r\,(\det\boldsymbol{Y})^r$, $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{r\times r}$, yields that for $\xi \in [-\pi, \pi]$ we have

$$\left|\det\widehat{\boldsymbol{H}}(\xi)\right| = \left|\det\widehat{\boldsymbol{G}}(\xi)\right|^r\,\left|\det\overline{\widehat{\boldsymbol{G}}(\xi)}\right|^r = \left|\det\widehat{\boldsymbol{G}}(\xi)\right|^{2r} \geq \rho^{2r}$$

and therefore $\widehat{\boldsymbol{H}}^{-1}$ is a well-defined continuous function on $[-\pi, \pi]$. Now suppose that for some sequence c we have $\boldsymbol{H} * c = \boldsymbol{0}$, then, taking the Fourier transform,

$$\widehat{\boldsymbol{H}}(\xi)\,\widehat{c}(\xi) = \boldsymbol{0}, \quad \widehat{c}(\xi) := \sum_{k\in\mathbb{Z}}\boldsymbol{c}_k\,\mathrm{e}^{-\mathrm{i}k\xi}, \quad \xi \in \mathbb{R}.$$

But then also

$$\mathbf{0} = \widehat{\boldsymbol{H}}^{-1}(\xi)\,\widehat{\boldsymbol{H}}(\xi)\,\widehat{c}(\xi) = \widehat{c}(\xi)$$

for $\xi \in [-\pi, \pi]$ and since the trigonometric series is 2π-periodic it follows that $\widehat{c} = 0$, hence $c = 0$. □

We next recall that the matrix valued function $\boldsymbol{H}$ contains $\boldsymbol{F}_\star$, more precisely, the vector $\Phi = \mathrm{vec}\,(\boldsymbol{F}_\star)$. Using (2), the relationship is

$$\begin{aligned}\Phi &= \mathrm{vec}\,(\boldsymbol{F}_\star) = \mathrm{vec}\left(\frac{1}{2}\sum_{j,k\in\mathbb{Z}} \boldsymbol{B}_k^T\,\boldsymbol{F}_\star\,(2\cdot -j)\,\boldsymbol{B}_{j+k}\right)\\ &= \frac{1}{2}\sum_{j,k\in\mathbb{Z}} \mathrm{vec}\left(\boldsymbol{B}_k^T\,\boldsymbol{F}_\star\,(2\cdot -j)\,\boldsymbol{B}_{j+k}\right) = \sum_{j,k\in\mathbb{Z}} \frac{1}{2}\left(\boldsymbol{B}_{j+k}^T \otimes \boldsymbol{B}_k^T\right)\mathrm{vec}\,(\boldsymbol{F}_\star(2\cdot -j))\\ &= \sum_{j\in\mathbb{Z}}\left(\frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{B}_{j+k}\otimes \boldsymbol{B}_k\right)^T \mathrm{vec}(\boldsymbol{F}_\star)(2\cdot -j) = \sum_{j\in\mathbb{Z}} \boldsymbol{C}_j^T\,\Phi\,(2\cdot -j)\,.\end{aligned}$$

Thus, Φ is a linear combination of the refinable row vectors of $\boldsymbol{H}$ or, in other words, there exists a vector $\boldsymbol{y}$ such that $\Phi = \boldsymbol{H}^T\boldsymbol{y}$. Since $\boldsymbol{F}_\star$ is cardinal, we have that for $k \in \mathbb{Z}$

$$\delta_k \mathrm{vec}(\boldsymbol{I}) = \mathrm{vec}\,(\boldsymbol{F}_\star(k)) = \boldsymbol{H}^T(k)\,\boldsymbol{y}$$

and summation over $k \in \mathbb{Z}$ yields together with (15) that

$$\mathrm{vec}(\boldsymbol{I}) = \left(\sum_{k\in\mathbb{Z}} \boldsymbol{H}^T(k)\right)\boldsymbol{y} = \boldsymbol{y},$$

hence $\Phi = \boldsymbol{H}^T\,\mathrm{vec}(\boldsymbol{I})$. Thus, we can consider the two nested chains of spaces

$$T_n = \mathrm{span}_{\,\mathbb{R}^{r^2}}\left\{\boldsymbol{H}\left(2^n\cdot -j\right), j \in \mathbb{Z}\right\} \tag{16}$$

and

$$\mathcal{T}_n = \mathrm{span}_{\,\mathbb{R}^{r^2}}\left\{\Phi^T\left(2^n\cdot -j\right), j \in \mathbb{Z}\right\} \tag{17}$$

of vector and scalar valued functions, respectively. They are connected by

$$\begin{aligned}\Phi^T * \boldsymbol{c}\left(2^n\cdot\right) &= \sum_{j\in\mathbb{Z}} \Phi^T\left(2^n\cdot -j\right)\boldsymbol{c}_j = \sum_{j\in\mathbb{Z}} \mathrm{vec}(\boldsymbol{I})^T\boldsymbol{H}\left(2^n\cdot -j\right)\boldsymbol{c}_j\\ &= \mathrm{vec}(\boldsymbol{I})^T\,\boldsymbol{H} * \boldsymbol{c}\left(2^n\cdot\right).\end{aligned}$$

Remark 6. The vector field Φ could appear to be a good basis for a scalar multiresolution analysis of the multiwavelet type. Unfortunately, even the minimal requirement, namely stability of Φ cannot be guaranteed. More precisely: even if $\boldsymbol{F}$ is a stable function, the vector field vec($\boldsymbol{F}$) will usually not be stable. The simplest counterexample is to choose $\boldsymbol{F} = f\,\boldsymbol{I}$ where f is a stable scalar refinable function, for example a cardinal B-spline with simple knots. Note that a degree one cardinal B-spline is also interpolatory.

4. Multiresolution analyses, filters and filter banks

4.1. Interpolatory multiresolution analysis based on $\boldsymbol{F}$

We start by discussing how to build an MRA and associated filter banks based on the refinable cardinal function $\boldsymbol{F} := \lim_{n\to\infty} S_{\mathcal{A}}^n \delta\boldsymbol{I}$ under the assumption that $S_{\mathcal{A}}$ is uniformly convergent. Of course, this "prediction–correction" approach is not new and essentially dates back at least to Faber [13] where the decay of wavelet coefficients (in the sense explained below) of piecewise linear scalar interpolants was used to construct continuous, nowhere differentiable functions. For the finitely supported and continuous $r \times r$ matrix function $\boldsymbol{F}$ cardinality means that $\boldsymbol{F}(k) = \delta_{k0}\,\boldsymbol{I}$, $k \in \mathbb{Z}$, while refinability with respect to $\mathcal{A}$ means that

$$\boldsymbol{F} = \boldsymbol{F} * \boldsymbol{A}(2\cdot) = \sum_{k\in\mathbb{Z}} \boldsymbol{F}\,(2\cdot -k)\,\boldsymbol{A}_k.$$

The MRA generated by $\boldsymbol{F}$ consists of the spaces

$$\mathcal{V}_k := \mathrm{span}_{\,\mathbb{R}^r}\left\{\boldsymbol{F}\left(2^k\cdot -j\right) : j \in \mathbb{Z}\right\} = \left(\boldsymbol{F} * \ell^r(\mathbb{Z})\right)\left(2^k\cdot\right), \quad k \in \mathbb{N}_0,$$

where the nestedness $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots$ is equivalent to the refinability of $\boldsymbol{F}$. Moreover, the limit

$$\mathcal{V}_\infty := \lim_{k\to\infty} \mathcal{V}_k = \bigcup_{k\in\mathbb{Z}} \mathcal{V}_k$$

is dense in $C_u^r(\mathbb{R})$, the space of uniformly continuous function vectors of dimension r since, in addition,

$$\sum_{k\in\mathbb{Z}} \boldsymbol{F}(\cdot - k) = \boldsymbol{I}.$$

This can be proved by a "test function" argument as in [8,9]. Hence, for the refinable cardinal $\boldsymbol{F}$, the spaces $\mathcal{V}_k$ satisfy all requirements of a multiresolution analysis, cf. [14, p. 221], since:

(1) $\mathcal{V}_k \subset \mathcal{V}_{k+1}$, $k \in \mathbb{N}_0$, and $\mathcal{V}_\infty$ is dense in $C_u^r(\mathbb{R})$,
(2) any $\mathcal{V}_0$ is translation invariant,
(3) a function belongs to $\mathcal{V}_k$ if its dilated version belongs to $\mathcal{V}_{k+1}$,
(4) the generating cardinal function $\boldsymbol{F}$ is *stable*, i.e., for any $c \in \ell_\infty^r(\mathbb{Z})$

$$\|c\|_{\ell_\infty^r(\mathbb{Z})} \le \|\boldsymbol{F} * c\|_{L_\infty^r(\mathbb{R})} \le C\, \|c\|_{\ell_\infty^r(\mathbb{Z})}, \quad C = \|\boldsymbol{F}\|_{L_\infty^r(\mathbb{R})} < \infty. \tag{18}$$

The next step consists of defining the cardinal projections $\mathbb{P}_k : C_u^r(\mathbb{R}) \to \mathcal{V}_k$, defined as the interpolant

$$\mathbb{P}_k\boldsymbol{g} = \sum_{j\in\mathbb{Z}} \boldsymbol{F}\left(2^k \cdot - j\right) \boldsymbol{g}\left(2^{-k}j\right) =: \boldsymbol{F} * \sigma_{2^{-k}\mathbb{Z}}\boldsymbol{g}\left(2^k\cdot\right), \quad \boldsymbol{g} \in C_u^r(\mathbb{R}),$$

with the *sampling operator* $\sigma_h\boldsymbol{f} := (\boldsymbol{f}(jh) : j \in \mathbb{Z})$. Note that these projections are even more suitable than the inner products in usual wavelet analysis since most functions are available in sampled form. In particular, the canonical projection $\mathbb{P}_0$ is

$$\mathbb{P}_0\boldsymbol{g} = \sum_{k\in\mathbb{Z}} \boldsymbol{F}(\cdot - k)\ \boldsymbol{g}(k).$$

The wavelet spaces are now defined by means of the canonical projections.

Definition 7. The *wavelet spaces* $\mathcal{W}_k$, $k \ge 0$, are defined as

$$\mathcal{W}_k = \{\boldsymbol{f} \in \mathcal{V}_{k+1} \ :\ \mathbb{P}_k\boldsymbol{f} = 0\} = \left\{\boldsymbol{f} \in \mathcal{V}_{k+1} \ :\ \boldsymbol{f}\left(2^{-k}j\right) = 0, j \in \mathbb{Z}\right\}.$$

Writing $\boldsymbol{f} \in \mathcal{V}_{k+1}$ as

$$\boldsymbol{f} = \boldsymbol{F} * \boldsymbol{c}\left(2^{k+1}\cdot\right) = \sum_{j\in\mathbb{Z}} \boldsymbol{F}\left(2^{k+1}\cdot - j\right) \boldsymbol{c}_j,$$

it is easily seen that $\boldsymbol{f}\left(2^{-k}\mathbb{Z}\right) = 0$ if and only if $\boldsymbol{c}_{2j} = 0$, hence:

$$\mathcal{W}_k = \operatorname{span}_{\mathbb{R}^r} \left\{\boldsymbol{F}\left(2^k \cdot - j\right) \ :\ j \in 2\mathbb{Z} + 1\right\}, \tag{19}$$

so that wavelets and scaling functions coincide except on the translational factor. The wavelet decomposition by itself is quite simple as well: starting with $\boldsymbol{f} \in \mathcal{V}_n$, we first decompose it into

$$\boldsymbol{f} = \mathbb{P}_{n-1}\boldsymbol{f} + (I - \mathbb{P}_{n-1})\boldsymbol{f}$$

where the first one belongs to $\mathcal{V}_{n-1}$ while the second one clearly vanishes at $2^{n-1}\mathbb{Z}$ so that it belongs to $\mathcal{W}_{n-1}$. Continuing with decomposing $\mathbb{P}_{n-1}\boldsymbol{f}$ recursively $L > 0$ times, we get the complete wavelet decomposition

$$\boldsymbol{f} = \mathbb{P}_{n-L}\boldsymbol{f} + \mathbb{Q}_{n-L}\boldsymbol{f} + \mathbb{Q}_{n-L+1}\boldsymbol{f} + \cdots + \mathbb{Q}_{n-1}\boldsymbol{f}$$

where $\mathbb{Q}_{n-k} := (I - \mathbb{P}_{n-k})\,\mathbb{P}_{n-k+1} = \mathbb{P}_{n-k+1} - \mathbb{P}_{n-k}$ is the projection operator on the wavelet space $\mathcal{W}_{n-k}$.

In order to express this process in terms of filter banks, then by means of discrete wavelet decomposition and reconstruction schemes, we start with decomposition and assume that

$$\boldsymbol{f} = \mathbb{P}_1\boldsymbol{f} = \sum_{k\in\mathbb{Z}} \boldsymbol{F}(2\cdot - k)\ \boldsymbol{c}_k \in \mathcal{V}_1$$

is a given function to be decomposed where the decomposition has to be performed in terms of the coefficient sequence $\boldsymbol{c}$. One part is almost trivial, namely

$$\mathbb{P}_0\boldsymbol{f} = \sum_{k\in\mathbb{Z}} \boldsymbol{F}(\cdot - k)\ \boldsymbol{c}_{2k},$$

hence the decomposition low pass filter is just subsampling:

$$\boldsymbol{c}_k^1 = (\downarrow_2 c)_k = \boldsymbol{c}_{2k}, \quad k \in \mathbb{Z}. \tag{20}$$

The refinement equation of $\boldsymbol{F}$ allows us to rewrite $\mathbb{P}_0\boldsymbol{f}$ as

$$\mathbb{P}_0\boldsymbol{f} = \boldsymbol{F} * S_{\mathcal{A}}c^1(2\cdot),$$

hence, the high pass filter becomes

$$\mathbb{Q}_0\boldsymbol{f} = (I - \mathbb{P}_0)\boldsymbol{f} = \boldsymbol{F} * \left(\boldsymbol{c} - S_{\mathcal{A}}c^1\right)(2\cdot),$$

and since $\left(c - S_{\mathcal{A}}c^1\right)_{2k} = 0$, we indeed have that

$$\begin{aligned}(I - \mathbb{P}_0)\boldsymbol{f} &= \sum_{j\in\mathbb{Z}} \boldsymbol{F}\left(2\cdot -(2j+1)\right)\left(\boldsymbol{c}_{2j+1} - \sum_{k\in\mathbb{Z}} \boldsymbol{A}_{2j+1-2k}\boldsymbol{c}_{2k}\right)\\ &= \sum_{j\in\mathbb{Z}} \boldsymbol{F}\left(2\cdot -(2j+1)\right)\left(\boldsymbol{c}_{2j+1} - \left(\sum_{k\in\mathbb{Z}} \boldsymbol{A}_{2j+1-k}\boldsymbol{c}_{k} - \boldsymbol{c}_{2j+1}\right)\right)\\ &= \sum_{j\in\mathbb{Z}} \boldsymbol{F}\left(2\cdot -(2j+1)\right)\left(\widetilde{\boldsymbol{A}} * \boldsymbol{c}\right)_{2j+1}\end{aligned}$$

where $\widetilde{\boldsymbol{A}} = 2\boldsymbol{I}\delta_0 - \boldsymbol{A}$. Hence, the wavelet coefficients are given by means of convolution and subsampling as

$$\boldsymbol{d}_j^1 = \left(\widetilde{\boldsymbol{A}} * \boldsymbol{c}\right)_{2j+1}, \quad j \in \mathbb{Z}.$$

Conversely, reconstruction is simple as well by setting

$$\begin{aligned}&\boldsymbol{c}_{2j} = \boldsymbol{c}_j^1\\ &\boldsymbol{c}_{2j+1} = (S_{\mathcal{A}}c^1)_{2j+1} + \boldsymbol{d}_j^1 = \left(S_{\mathcal{A}}c^1 + \uparrow_2 d^1\right)_{2j+1}\end{aligned}$$

where, as usual, given a sequence a, the upsampling operator constructs a new sequence b such that $b_{2j} = a_j$, $b_{2j+1} = 0$.

With the translation filter τ, $\tau\boldsymbol{c} := \boldsymbol{c}(\cdot + 1)$, we thus can describe the filter bank in the usual way as

$$c \to \odot \begin{array}{l} \nearrow \boxed{\tau\widetilde{\boldsymbol{A}}} \to \boxed{\downarrow_2} \to d^1 \to \boxed{\uparrow_2} \to \boxed{\tau^{-1}} \searrow \\ \searrow \boxed{I} \to \boxed{\downarrow_2} \to c^1 \to \boxed{\uparrow_2} \to \boxed{\boldsymbol{A}} \nearrow \end{array} \oplus \to c \tag{21}$$

The primal/dual high and low pass filters are

$$\boldsymbol{Q} = \tau\widetilde{\boldsymbol{A}}, \qquad \boldsymbol{P} = I, \qquad \widetilde{\boldsymbol{Q}} = \tau^{-1}, \qquad \widetilde{\boldsymbol{P}} = \boldsymbol{A},$$

with symbols

$$\begin{aligned}&\boldsymbol{Q}(z) = z\left(2\boldsymbol{I} - \boldsymbol{A}(z)\right), \qquad \boldsymbol{P}(z) = \boldsymbol{I},\\ &\widetilde{\boldsymbol{Q}}(z) = z^{-1}\boldsymbol{I}, \qquad \widetilde{\boldsymbol{P}}(z) = \boldsymbol{A}(z),\end{aligned}$$

so that

$$\widetilde{\boldsymbol{Q}}(z)\,\boldsymbol{Q}(z) + \widetilde{\boldsymbol{P}}(z)\,\boldsymbol{P}(z) = 2\boldsymbol{I} - \boldsymbol{A}(z) + \boldsymbol{A}(z) = 2\boldsymbol{I}$$

which is nothing but the *perfect reconstruction* property of the filter bank.

4.2. Multiresolution analysis based on $\boldsymbol{H}$

Due to the refinability of the matrix function $\boldsymbol{H}$,

$$\boldsymbol{H} = \sum_{j\in\mathbb{Z}} \boldsymbol{H}(2\cdot -j)\,\boldsymbol{C}_j,$$

the spaces

$$T_n := \operatorname{span}_{\mathbb{R}^{r^2}}\left\{\boldsymbol{H}(2^n\cdot -k) \;:\; k \in \mathbb{Z}\right\}$$

are obviously nested. Also, since $\boldsymbol{H}$ is stable and a partition of the identity by Theorem 5 and (15), we can construct a multiresolution analysis based on it. To construct an MRA in $L_2^{r^2}(\mathbb{R})$ based on $\boldsymbol{H}$, we make use of an interesting connection between the scaling spaces $\{V_n\}$ and $\{T_n\}$ generated by $\boldsymbol{G}$ and $\boldsymbol{H}$, respectively. Recall that the construction of an MRA based on the orthogonal matrix function $\boldsymbol{G}$, in particular, the construction of the wavelet, has been described in [11].

Definition 8. For $n \in \mathbb{Z}$ the *Cassata product space* $V_n \circledast V_n$ of V_n with itself is defined via

$$V_n \circledast V_n := \left\{ \boldsymbol{f} \in L_2^{r^2}(\mathbb{R}) \ : \ \boldsymbol{f} = \sum_{\ell=1}^{r} m^\ell \circledstar n^\ell, \ m^\ell, n^\ell \in V_n, \ \ell = 1, \ldots, r \right\}.$$

If $V_n \subset L_2^{r \times r}(\mathbb{R})$ then the Hölder inequality automatically gives the Cassata product space $V_n \circledast V_n$ as a subspace of $L_1^{r \times r}(\mathbb{R})$. On the other hand, the same Fourier transform argument given before Proposition 4 to show that $\boldsymbol{H}$ was uniformly continuous, could be applied here to show that the elements of $V_n \circledast V_n$ even have to be uniformly continuous. In fact, we give a more direct and explicit proof of that fact now in Proposition 9 where we show that the Cassata product space is generated by $\boldsymbol{H}$ via convolutions.

Proposition 9. *For each $n \in \mathbb{Z}$ the multiresolution analysis spaces T_n generated by $\boldsymbol{H}$ coincide with the Cassata product space $V_n \circledast V_n$ of V_n, generated by $\boldsymbol{G}$.*

Proof. Let $\boldsymbol{f} \in V_n \circledast V_n$ and let $m^\ell = \boldsymbol{G} * c^\ell(2^n \cdot)$ and $n^\ell = \boldsymbol{G} * d^\ell(2^n \cdot)$, $\ell = 1, \ldots, r$ be the respective representations in V_n. By the second identity of (5) we have that

$$\sum_{\ell=1}^{r} m^\ell \circledstar n^\ell = \sum_{r=1}^{\ell} (\boldsymbol{G} \circledstar \boldsymbol{G}) \star (c^\ell \circledstar d^\ell)(2^n \cdot) = \boldsymbol{H} * \sum_{r=1}^{\ell} \left((c^\ell \circledstar d^\ell)(-\cdot)\right)(2^n \cdot)$$

belongs to T_n. Conversely, if $\boldsymbol{f} \in T_n$, it can be written as

$$\boldsymbol{f} = \sum_{j \in \mathbb{Z}} \boldsymbol{H}(2^n \cdot -j)\boldsymbol{s}_j = (\boldsymbol{H} * s)(2^n \cdot)$$

where $s = \{\boldsymbol{s}_j\} \in \ell(\mathbb{Z})^{r^2}$. By making use of Proposition 3 we can write

$$\boldsymbol{f} = \sum_{\ell=1}^{r} \left(\boldsymbol{G} \star u^\ell\right) \circledstar \left(\boldsymbol{G} \star v^\ell\right)(2^n \cdot)$$

which proves that $\boldsymbol{f}$ can be written as the sum of Kronecker correlations of functions belonging to V_n, that is $\boldsymbol{f} \in V_n \circledast V_n$. $\square$

Based on Proposition 9, we can easily achieve that each of complementary wavelet spaces $R_n = T_{n+1} \ominus T_n$ of the MRA $\{T_n\}$ is indeed the sum of three spaces. In fact, denoting with $\{W_n\}$ the sequence of wavelet spaces associated to $\{V_n\}$, we have that:

$$T_{n+1} = T_n \oplus R_n$$

but, on the other hand,

$$\begin{aligned} T_{n+1} &= V_{n+1} \circledast V_{n+1} = (V_n \oplus W_n) \circledast (V_n \oplus W_n) \\ &= \underbrace{(V_n \circledast V_n)}_{T_n} \oplus \underbrace{(V_n \circledast W_n) \oplus (W_n \circledast V_n) \oplus (W_n \circledast W_n)}_{R_n}. \end{aligned}$$

This formula, which is very similar to the types of decompositions that appear in tensor product wavelets, allows obtaining a fast decomposition/reconstruction algorithm for function vectors belonging to $L_2^{r^2}(\mathbb{R})$: One first decomposes a signal $s \in \ell^{r^2}(\mathbb{Z})$ by means of (7) and then applies the *orthogonal* multiresolution based on $\boldsymbol{G}$ on the components separately. Note that one part of this decomposition is simple and even data independent since $\boldsymbol{v}^\ell$ essentially consists of unit vectors and δ-sequences.

4.3. Filters and filter banks based on $\boldsymbol{F}_\star$

For the function $\boldsymbol{F}_\star$, which is only refinable in the more general sense of (2), the situation is somewhat different. Here, the concept of filters and filter banks associated to an MRA will only yield a set of *discrete* operations working on *discrete vector data* $c \in \ell_\infty^r(\mathbb{Z})$. On level n this sequence will be related to sampling values of a function, say $\boldsymbol{g} \in C^r(\mathbb{R})$, at the grid $2^{-n}\mathbb{Z}$, i.e., $\boldsymbol{c}_j^n = \boldsymbol{g}\left(2^{-n}j\right)$ where the upper index denotes the sampling level. Since $\boldsymbol{F}_\star$ is a cardinal function, we can define the level n interpolant

$$\boldsymbol{f}_n := \boldsymbol{F}_\star * c^n\left(2^n \cdot\right) = \sum_{j \in \mathbb{Z}} \boldsymbol{F}_\star\left(2^n \cdot -j\right) \boldsymbol{c}_j^n, \tag{22}$$

which has the property that

$$\boldsymbol{f}_n\left(2^{-n}k\right) = \boldsymbol{c}_k^n = \boldsymbol{g}\left(2^{-n}k\right). \tag{23}$$

To build a matrix filter bank, we will again make use of a prediction–correction strategy as in [13] that always identifies a vector c^n with the associated function $\boldsymbol{f}_n$. Note that whether the spaces V_n spanned by $\boldsymbol{F}_\star(2^n \cdot -k)$ are nested or not (and there is no strong reason to expect them to be nested) is completely irrelevant at this point. To build the prediction–correction scheme we first downsample c^n to get the subsequence c^{n-1} as

$$c^{n-1} = \downarrow_2 c^n, \qquad \text{i.e. } \boldsymbol{c}_j^{n-1} = \boldsymbol{c}_{2j}^n, \quad j \in \mathbb{Z},$$

and once more relate it to the interpolant on level $n-1$,

$$\boldsymbol{f}_{n-1} := \boldsymbol{F}_\star * c^{n-1}\left(2^{n-1}\cdot\right), \qquad \boldsymbol{f}_{n-1}\left(2^{1-n}k\right) = \boldsymbol{g}\left(2^{1-n}k\right), \quad k \in \mathbb{Z}. \tag{24}$$

In fact, (22) and (24) can easily be extended to a general decomposition process based on the downsampled sequences $c^k = \downarrow_2 c^{k+1}$, $k = 0, \ldots, n-1$, and the associated interpolants

$$\boldsymbol{f}_k := \boldsymbol{F}_\star * c^k\left(2^{-k}\cdot\right),$$

satisfy the nested interpolation property

$$\boldsymbol{f}_k\left(2^{-k}j\right) = \boldsymbol{c}_j^k = \boldsymbol{g}\left(2^{-k}j\right).$$

Here "nested" means that if $\boldsymbol{f}_k$ interpolates $\boldsymbol{g}$ at some grid point, then so does $\boldsymbol{f}_\ell$ for $k \le \ell \le n$.

We will use the subsampled sequences as *predictors* for the values of $\boldsymbol{g}$; of course, c^{n-1} will usually predict $\boldsymbol{g}$ at $2^{1-n}\mathbb{Z}$ but will normally fail to give the correct value at $2^{-n}\mathbb{Z} \setminus 2^{1-n}\mathbb{Z}$. This is the reason why a *correction* has to be applied which is computed by substituting the refinement equation (2) for $\boldsymbol{F}_\star$ into $\boldsymbol{f}_{n-1}$

$$\boldsymbol{f}_{n-1} = \sum_{j\in\mathbb{Z}} \boldsymbol{F}_\star\left(2^{n-1}\cdot -j\right) \boldsymbol{c}_j^{n-1} = \frac{1}{2}\left(S_{\mathcal{B}}\delta\boldsymbol{I}\right)^T \star \boldsymbol{F}_\star * S_{\mathcal{B}} c^{n-1}\left(2^n\cdot\right)$$

and then considering the difference from $\boldsymbol{f}_n$ — but only at *integer points* where we can easily compare the functions. And indeed, for $k \in \mathbb{Z}$ we get that

$$\begin{aligned}(\boldsymbol{f}_n - \boldsymbol{f}_{n-1})\left(2^{-n}(2k+1)\right) &= \sum_{j\in\mathbb{Z}} \boldsymbol{F}_\star(2k+1-j)\,\boldsymbol{c}_j^n - \frac{1}{2}\left(S_{\mathcal{B}}\delta\boldsymbol{I}\right)^T \star \boldsymbol{F}_\star * S_{\mathcal{B}} c^{n-1}(2k+1) \\ &= \boldsymbol{c}_{2k+1}^n - \left(\mathcal{S}_{\boldsymbol{B}} c^{n-1}\right)_{2k+1} = \left(\left(I - \mathcal{S}_{\boldsymbol{B}}\downarrow_2\right) c^n\right)_{2k+1},\end{aligned}$$

where, as in [1], we consider the *correlated* subdivision operator $\mathcal{S}_{\boldsymbol{B}}$ of (1) and recall from there that $\mathcal{S}_{\boldsymbol{B}} = \mathcal{S}_{\boldsymbol{B}}^1 = S_{\mathcal{A}}$, so that the "wavelet" coefficients of this interpolatory multiresolution analysis on the integers are computed as

$$\boldsymbol{d}_k^n := \left(\left(I - S_{\mathcal{A}}\downarrow_2\right)\boldsymbol{c}^n\right)_{2k+1}, \quad k \in \mathbb{Z}. \tag{25}$$

The reconstruction of c^n from c^{n-1} and d^n is straightforward, namely

$$c^n = S_{\mathcal{A}} c^{n-1} + \tau\uparrow_2 d^n, \qquad \tau c := \left(\boldsymbol{c}_{j+1} \,:\, j \in \mathbb{Z}\right).$$

This is the standard interpolatory multiresolution analysis based on the interpolatory mask $\mathcal{A}$.

Remark 10. We encountered a rather curious behavior of multiresolution analyses that clearly can be observed in the vector subdivision case only: though we cannot prove the existence of a refinable function relative to $\mathcal{A}$, the filter bank based on $S_{\mathcal{A}}$ can be perfectly explained and motivated in terms of a uniformly continuous cardinal function $\boldsymbol{F}_\star$. In particular, all results on approximation order of $\boldsymbol{F}_\star$ immediately would lead to decay estimates for the wavelet coefficients from the filter bank. In the scalar case all the effects pointed out here are only apparent in a very trivial sense since $\boldsymbol{F}$, $\boldsymbol{F}_\star$ and even $\boldsymbol{H}$ coincide there.

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