Relative topological properties and relative topological spaces

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Abstract

This is one of the first surveys on relative topological properties. The emphasis is on relative separation axioms and on relative properties of compactness type. In particular, many relative versions of normality are discussed. Connections between relative compactness type properties and relative separation properties are scrutinized. Many new results and open problems are brought to light.

Keywords: Compactness of $Y$ in $X$; Normality of $Y$ in $X$; Relative topological property; Realnormality of $Y$ in $X$

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1. Introduction

In many topological arguments and constructions we have to deal with the following question: how is a given subspace $Y$ of a topological space $X$ located in $X$? In this article we describe a systematic approach to this problem; though it is very general, we do not pretend that it embraces all other possible approaches to the location problem.

The leading idea permeating the article can be briefly described as follows. With each topological property $P$ one can associate a relative version of it formulated in terms of location of $Y$ in $X$ in such a natural way that when $Y$ coincides with $X$ then this relative property coincides with $P$. Our basic conjecture is that the great majority of the results involving "absolute" topological properties can be interpreted as "location" results, that is as theorems on relative topological properties. This provides us with a guideline in our work on relative properties.

Situations involving relative topological properties have been encountered in topology on countless occasions. For example, some very important results on relative countable
compactness were obtained by Grothendieck [14]. Tkachuk [18] and Chigogidze [9] have considered relative topological dimensions. The first systematic exposition of relative topological properties along the lines of this article was given in [3]. Some new results on relative properties were recently obtained by Gordienko [12,13]. See also [6,7]. An important result on relative Lindelöfness, solving an old problem formulated by Ranchin [16], was obtained by Dow and Vermeer [10].

Quite often we encounter in General Topology standard pairs of spaces, consisting of a space and of a subspace of it. For example, a Tychonoff space $X$ can be treated as a subspace of the free topological group $F(X)$ of $X$, or as a subspace of the space of all closed subsets of $X$ in the Vietoris topology. The space $C_p(X)$ of continuous real-valued functions on a Tychonoff space $X$ is a subspace of $\mathbb{R}^X$, in the product topology. Any Hausdorff space can be considered as a subspace of its Katetov' extension. Each of these situations can be treated from the point of view of relative topological properties.

Below we provide a survey of results (some of them with proofs) on relative separation axioms (several versions of normality are of special interest here), and on relative compactness type properties (including, in particular, relative paracompactness and relative Lindelöfness). A rather comprehensive survey of recent results on relative cardinal invariants is given in [1]. Due to the lack of space, no attempt was made to make this survey comprehensive: many topics and corresponding references are omitted.

An important feature of relative topology is that our relativization effort may force a whole crowd of relative topological properties from a single "absolute" topological property. How many interesting versions an absolute property may have, and how one should define them, depends on the property itself, but there are at least two general approaches to defining relative properties, which can be applied to any topological invariant. It seems that the first of them provides for the weakest relative version of any topological property.

Let $Y$ be a subspace of a space $X$, and let $P$ be a topological property (an "absolute" one). Let us say that $Y$ has the property $P$ in $X$ from inside, if every subspace of the space $Y$ which is closed in $X$ has the property $P$ (in itself). If there is a subspace $Z$ of the space $X$ such that $Y$ is contained in $Z$, and the space $Z$ has the property $P$ in itself, we shall say that $Y$ has the property $P$ in $X$ from outside. Observe that if a property $P$ is closed hereditary, then having $P$ from outside implies having it from inside. Observe also that many theorems of General Topology are easily extended to the relative case if the "inside" version of relative properties is accepted.

Our notation and terminology are the same as in [11]. We assume everywhere that $Y$ is a subspace of a space $X$. A larger space than $Y$ is any space containing $Y$ as a subspace; in particular, $Y$ itself is such a space.

2. Relative separation axioms

We say that $Y$ is $T_1$ in $X$, if for each $y \in Y$ the set $\{y\}$ is closed in $X$. If for every two different points $y_1$ and $y_2$ of $Y$ there are disjoint open subsets $U_1$ and $U_2$ of $X$ such
that $y_1 \in U_1$ and $y_2 \in U_2$, we say that $Y$ is Hausdorff in $X$. If the above condition is satisfied whenever $y_1 \in Y$ and $y_2 \in X$, we should say that $Y$ is strongly Hausdorff in $X$. Of course, if $X$ is Hausdorff, then every subspace $Y$ of $X$ is strongly Hausdorff in $X$.

A subspace $Y$ is regular in $X$ (superregular in $X$), if for each $y \in Y$ and each closed in $X$ subset $P$ of $X$ such that $y \notin P$ there are disjoint open in $X$ sets $U$ and $V$ such that $y \in U$ and $P \cap Y \subseteq V$ (respectively such that $y \in U$ and $P \subseteq V$). Clearly, if $X$ is regular, then every subspace $Y$ of $X$ is superregular in $X$, and if $Y$ is regular in $X$, then $Y$ is regular (in itself). Neither one of the above implications can be reversed.

**Example 1.** Let us add to the natural topology of the real line $\mathbb{R}$ one new element—the complement to the set $P = \{1/n: n \in \omega \setminus \{0\}\}$. The resulting family is a subbase of a new topology $\mathcal{T}_t$ on the set $\mathbb{R}$. The set $\mathbb{R}$ endowed with the topology $\mathcal{T}_t$ is a space $X$. Let $Y = \{0\} \cup P$. Then $Y$ is a closed discrete subspace of $X$, hence $Y$ is regular. On the other hand, $Y$ is not regular in $X$, since the point $0 \in Y$ and the set $P$, which is a closed subset of $Y$, cannot be separated by open sets in $X$ (see [11]).

It is not clear which of the above two versions of relative regularity should be considered as the main one. Moreover, there is one more version of relative regularity. Let us say that $Y$ is strongly regular in $X$, if for each point $x \in X$ and each closed in $X$ subset $P$ which does not contain $x$, there are disjoint open sets $U$ and $V$ in $X$ such that $x \in U$ and $P \cap Y \subseteq V$. One can show that $Y$ can be superregular in $X$ without being strongly regular in $X$, and that vice versa. $Y$ can be strongly regular in $X$ without being superregular in $X$.

Let us introduce now some relative versions of normality. We will witness that this classical notion of General Topology is even more subject to splitting into many different natural versions under relativization than regularity.

Let us say that $Y$ is normal in $X$ (nearly normal in $X$) if for each pair $A, B$ of closed disjoint subsets of $X$ there are disjoint open subsets $U$ and $V$ in $X$ (respectively, open subsets $U$ and $V$ in $Y$) such that $A \cap Y \subseteq U$ and $B \cap Y \subseteq V$.

Clearly, if $Y$ is normal (in itself), then $Y$ is nearly normal in every larger space $X$. But a normal space need not be normal in a larger space.

**Example 2.** Let $X$ be the Niemytzki plane, and $Y$ the discrete line contained in it. Then $Y$ is closed in $X$ and normal in itself, and therefore $Y$ is nearly normal in $X$, but $Y$ is not normal in $X$—this follows from the classical argument showing the nonnormality of $X$ (see [11]).

It is also obvious that if $X$ is normal, then every subspace $Y$ of $X$ is normal in $X$. Therefore, $Y$ may be normal in $X$ without being normal in itself—this happens, for example, if $Y$ is a nonnormal subspace of a compact Hausdorff space $X$.

We will say that $Y$ is strongly normal in $X$, if for each pair $A, B$ of closed in $Y$ disjoint subsets of $Y$ there are open disjoint subsets $U$ and $V$ in $X$ such that $A \subseteq U$ and $B \subseteq V$.

It is good to keep in mind the following assertions, which are proved easily [3].
Proposition 3. If \( Y \subset Z \subset X \) and the space \( Z \) is normal, then \( Y \) is nearly normal in \( X \).

Proposition 4. If \( \overline{Y} = X \) and \( Y \) is nearly normal in \( X \), then \( Y \) is normal in \( X \).

Let us say that \( Y \) is perfectly located in \( X \), if for each open subset \( U \) of \( X \) containing \( Y \) there is a countable family \( \mathcal{P} \) of closed sets in \( X \) such that \( Y \subset \bigcup \mathcal{P} \subset U \) (see [11]). The next result was obtained in [3].

Theorem 5. If \( Y \) is nearly normal in \( X \), and \( Y \) is perfectly located in \( X \), then the space \( Y \) is normal.

The next three assertions from [3] are obvious.

Proposition 6. If \( Y \) is strongly normal in \( X \), then \( Y \) is normal.

Proposition 7. If \( \overline{Y} = X \), and \( Y \) is normal (in itself), then \( Y \) is strongly normal in \( X \).

Proposition 8. If \( Y \) is closed in \( X \) and \( Y \) is normal in \( X \), then \( Y \) is strongly normal in \( X \).

Our first impression might be that the strong normality of \( Y \) in \( X \) is too strong a property. Proposition 7 shows that it might be not too strong after all. The next theorem has first appeared in [3], in Russian. It generalizes a well known classical result. Note that it cannot be reduced to that result, it contains more information about the spaces involved. We reproduce a simple proof of this theorem here to provide a better reference source.

Theorem 9. If \( Y \) is regular in \( X \) and the space \( Y \) is Lindelöf, then \( Y \) is strongly normal in \( X \).

Proof. Let \( A \) and \( B \) be two disjoint closed in \( Y \) subsets of \( Y \). For each \( a \in A \) and each \( b \in B \) fix neighbourhoods \( O(a) \) and \( O(b) \) of \( a \) and \( b \) in \( X \) such that \( \overline{O(a)} \cap B = \emptyset \) and \( \overline{O(b)} \cap A = \emptyset \). Since \( Y \) is Lindelöf, there are countable subsets \( A_1 \subset A \) and \( B_1 \subset B \) such that

\[
A \subset \bigcup \{ O(a): a \in A_1 \} \quad \text{and} \quad B \subset \bigcup \{ O(b): b \in B_1 \}.
\]

The rest follows from the proof of Lemma 1.5.14 in [14]. \( \Box \)

Corollary 10. If \( X \) is a regular space, then every Lindelöf subspace of \( X \) is strongly normal in \( X \).

The next result is formulated for the first time.
Theorem 11. A subspace $Y$ is strongly normal in $X$ if and only if $Y$ is normal in itself and for each continuous real-valued function $f$ on $Y$ there is a real-valued function $f^*$ on $X$ continuous at all points of $Y$ which is an extension of $f$, that is, such that $f^*(x) = f(x)$ for each $x \in Y$.

Proof. Necessity. Assume that $Y$ is strongly normal in $X$. Then $Y$ is normal (Proposition 6). Let $\mathcal{T}$ be the topology of the space $X$. The family $\mathcal{P} = \{A: A \subset X \setminus Y\} \cup \mathcal{T}$ is a subbase of a new topology $\mathcal{T}^*$ on the set $X$. Endowing $X$ with the topology $\mathcal{T}^*$, we get a space $X_Y$. From the strong normality of $Y$ in $X$ it follows easily that the space $X_Y$ is normal. Clearly, $Y$ is closed in $X_Y$, and $X$ and $X_Y$ generate the same topology on $Y$. Therefore, every continuous function $f: Y \to \mathbb{R}$ can be extended to a continuous function $g: X_Y \to \mathbb{R}$ [11]. Take any $y \in Y$. Since the family $\{U \in \mathcal{T}: y \in U\}$ is a base of $X_Y$ at $y$, the function $g$ is continuous at $y$ with regards to the original topology $\mathcal{T}$ of the space $X$.

Sufficiency. Take any two closed in $Y$ disjoint nonempty subsets $A$ and $B$ of $Y$. Since $Y$ is normal, there is a continuous function $f: Y \to \mathbb{R}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Extend this function to a function $g: X \to \mathbb{R}$ continuous at each point of $Y$. Then the interiors $U$ and $V$ of the sets $\{x \in X: g(x) < 1/2\}$ and $\{x \in X: g(x) > 1/2\}$ are disjoint open subsets of $X$ such that $A \subset U$ and $B \subset V$. Thus, $Y$ is strongly normal in $X$. □

Corollary 12. If $Y$ is normal (in itself) and dense in $X$, then every continuous real-valued function on $Y$ can be extended to a real-valued function on $X$ which is continuous at each point of $X$.

In the class of Tychonoff spaces, a better result than Corollary 12 can be proved.

Theorem 13. If $X$ is a Tychonoff space and $Y$ is a dense subspace of $X$, then every continuous real-valued function on $Y$ can be extended to a real-valued function on $X$, which is continuous at each point of $Y$.

It may be convenient to introduce a special name for the situation described in Theorems 11 and 13 and Corollary 12. Let us say that $Y$ is weakly $C$-embedded into $X$, if every continuous real-valued function on $Y$ can be extended to a real-valued function on $X$ which is continuous at each point of $Y$.

Problem 14. Let $Y$ be a dense subspace of a regular $T_1$-space $X$. Is then $Y$ weakly $C$-embedded into $X$? What if in addition the space $Y$ is Tychonoff?

Problem 15. When is a Tychonoff space $Y$ weakly $C$-embedded into every larger Tychonoff space $X$?

The results obtained so far provide us with the following partial answer to this question.
Corollary 16. If $Y$ is a Lindelöf subspace of a regular space $X$, then $Y$ is weakly $C$-embedded into $X$.

Example 17 [12]. Let $X$ be the Tychonoff plank, that is, 
$$X = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\},$$
with the subspace topology. Let $Y = \{(\omega_1, n): n \in \omega\}$. Then $Y$ is a countable, closed discrete subspace of $X$, but there is a bounded real-valued continuous function $f$ on $Y$ which cannot be extended to a continuous real-valued function on $X$. Indeed, if $f$ takes value 1 when $n$ is odd, and value 0 when $n$ is even, then $f$ is continuous on $Y$, but it cannot be extended to a continuous function on $X$.

Example 18. Let $I$ be the closed unit interval and $\tau$ an uncountable cardinal. Fix a point $a$ in the Tychonoff cube $I^\tau$, and put $Y = I^\tau \setminus \{a\}$. Then $Y$ is not normal, but $Y$ is weakly $C$-embedded into every larger Tychonoff space.

The next assertion is easy to prove.

Proposition 19. If every closed subspace of a space $X$ is weakly $C$-embedded into $X$, then $X$ is normal.

The second part of the next result follows from Theorem 11.

Theorem 20. If $X$ is hereditarily normal, then every subspace $Y$ of $X$ is strongly normal in $X$ and weakly $C$-embedded into $X$.

There is yet another version of relative normality with interesting connections to some classical notions of General Topology.

A subset $A$ of $X$ is said to be concentrated on $Y$, if $A$ is contained in the closure in $X$ of the trace $A \cap Y$ of the set $A$ on $Y$. Closed subsets of $X$ concentrated on $Y$ are precisely the closures in $X$ of subsets of $Y$. Let us say that a space $X$ (the larger space!) is normal on a subspace $Y$, if for every two disjoint closed subsets $A$ and $B$ of $X$ concentrated on $Y$, there are disjoint open sets $U$ and $V$ in $X$ such that $A \subset U$ and $B \subset V$. The next two propositions are easy to prove.

Proposition 21. If $X$ is normal, then $X$ is normal on every subspace $Y$ of $X$.

Proposition 22. If $X$ is normal on $Y$, then $Y$ is normal in $X$.

Since the strong normality of $Y$ in $X$ implies that the space $Y$ is normal, and a subspace of a normal space need not be normal, it follows from Proposition 21 that $X$ is normal on $Y$ does not imply that $Y$ is strongly normal in $X$. On the other hand, there is a dense countable subspace $Y$ of a Tychonoff space $X$ such that $X$ is not normal on $Y$, while $Y$ is strongly normal in $X$ by Theorem 9. Hence, the next assertion is of some interest.
Proposition 23. Let $Y$ be closed in $X$. Then the next three conditions are equivalent:

(a) $X$ is normal on $Y$;
(b) $Y$ is normal in $X$;
(c) $Y$ is strongly normal in $X$.

Let us call a space $X$ densely normal, if there is a dense subspace $Y$ in $X$ such that $X$ is normal on $Y$. There is an interesting relationship between densely normal spaces and $\kappa$-normal spaces introduced by Stchepin in [17]. We recall that canonical closed sets are the closures of open sets. A space $X$ is $\kappa$-normal, if every two disjoint canonical closed sets in it can be separated by disjoint neighbourhoods. Obviously, if $Y$ is dense in $X$, then each canonical closed subset of $X$ is concentrated on $Y$. Therefore, the following implication holds.

Theorem 24. Every densely normal space is $\kappa$-normal.

The next natural question is open.

Problem 25. Is every $\kappa$-normal regular space densely normal?

Note that the product of any family of separable metrizable spaces is densely normal, while it is normal only in a trivial situation.

Stchepin has observed [17] that in every $\kappa$-normal space $X$ every two disjoint canonical closed subsets of $X$ can be separated by a continuous real-valued function (taking value 0 on one of them and value 1 on another). From this we easily get the next result.

Theorem 26. If a regular $T_1$-space $X$ is normal on $Y$, then the space $Y$ is Tychonoff.

Obviously, we should also discuss, to what extent Urysohn's Lemma is true for different versions of relative normality, which form does it take.

Let $f$ be a real-valued function on $X$. We shall say that $f$ is $Y$-continuous, if it is continuous at each point $y$ of $Y$. If for every two nonempty disjoint closed sets $A$ and $B$ in $X$ there is a $Y$-continuous function $f : X \to \mathbb{R}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that $Y$ is realnormal in $X$. Note, that we get an equivalent definition, if we replace the last two conditions by the following ones: $f(A \cap Y) \subseteq \{0\}$ and $f(B \cap Y) \subseteq \{1\}$.

If for every two disjoint closed subset $A$ and $B$ in $X$ there is a continuous real-valued function $f$ on $Y$ such that $f(A \cap Y) \subseteq \{0\}$ and $f(B \cap Y) \subseteq \{1\}$, then $Y$ is called weakly realnormal in $X$.

Obviously, if $Y$ is normal, then $Y$ is weakly realnormal in $X$. It is also clear that if $Y$ is realnormal in $X$, then $Y$ is normal in $X$. It follows (see Example 2) that weak realnormality of $Y$ in $X$ and realnormality of $Y$ in $X$ are not equivalent. In this connection, the next corollary of Theorem 13 is of interest.

Theorem 27. If $X$ is a Tychonoff space and $Y$ is dense in $X$ and weakly realnormal in $X$, then $Y$ is realnormal in $X$. 
Theorem 28. If $X$ is normal on $Y$, $X$ is a regular $T_1$-space and $Y$ is dense in $X$, then $Y$ is realnormal in $X$.

Proof. The space $X$ is densely normal and therefore $\kappa$-normal (see Theorem 24). Let $A$ and $B$ be any two nonempty subsets of $Y$ such that the closures of $A$ and $B$ in $X$ are disjoint. Then, since $Y$ is dense in $X$ and $X$ is normal on $Y$, there are disjoint closed canonical subsets $P$ and $H$ in $X$ such that $A \subset P$ and $B \subset H$. Since $X$ is $\kappa$-normal, there is a continuous real-valued function $f$ on $X$ such that $f(P) = \{0\}$ and $f(H) = \{1\}$ [17]. This completes the proof. □

After the definitions above, it would be natural to call $Y$ strongly realnormal in $X$, if for every two disjoint nonempty subsets $A$ and $B$ of $Y$ closed in $Y$ there is a $Y$-continuous function $f : X \to \mathbb{R}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Note, however, that the following result is an easy corollary of Theorem 11.

Theorem 29. $Y$ is strongly realnormal in $X$ if and only if $Y$ is strongly normal in $X$.

In the next section we will see that, in contrast to the above result, normality of $Y$ in $X$ is not equivalent to realnormality of $Y$ in $X$ in the class of Hausdorff spaces, even when $Y$ is dense in $X$. But the following question remains open.

Problem 30. Let $X$ be regular, and let $Y$ be normal in $X$ and dense in $X$. Is then $Y$ realnormal in $X$? What if we drop “dense”? What if we assume $X$ to be Tychonoff?

In connection with Problem 30, note the next result which easily follows from Theorem 29 and Proposition 7.

Corollary 31. If $Y$ is normal in itself and dense in $X$, then $Y$ is strongly realnormal in $X$.

At this point, it is worthwhile to mention the following obvious fact.

Proposition 32. If $X$ is a $T_1$-space and $Y$ is weakly realnormal in $X$, then $Y$ is a Tychonoff space.

The last result suggests the next question, closely related to Problem 30.

Problem 33. Let $X$ be a regular $T_1$-space, and let $Y$ be normal in $X$ and dense in $X$. Is then $Y$ Tychonoff?

3. On relative compactness type properties

In this section, we discuss how several relative compactness type properties influence relative separation properties. A few other important results involving relative compactness properties are also mentioned.
We say that \( Y \) is compact in \( X \), if in every open covering of \( X \) there is a finite subfamily \( \gamma \) such that \( Y \subset \bigcup \gamma \). If every open covering of \( X \) contains a countable subfamily \( \gamma \) such that \( Y \subset \bigcup \gamma \), \( Y \) is called Lindelöf in \( X \).

It is not difficult to show [16] that if \( X \) is a regular \( T_1 \)-space, then \( Y \) is compact in \( X \) if and only if the closure of \( Y \) in \( X \) is compact, which makes the notion of relative compactness almost trivial in the case of regular \( T_1 \)-spaces.

This is not so, if we only assume that \( X \) is Hausdorff [16]. But it is easily proved that if \( X \) is Hausdorff, and \( Y \) is compact in \( X \) and dense in \( X \), then \( X \) is an \( H \)-closed space (that is, \( X \) is closed in every larger Hausdorff space) [16].

The next two results were established in [16] (see also [4]).

**Theorem 34.** If \( X \) is Hausdorff and \( Y \) is compact in \( X \), then \( Y \) is normal in \( X \).

**Theorem 35.** If \( X \) is regular and \( Y \) is Lindelöf in \( X \), then \( Y \) is normal in \( X \).

Theorem 35 should be compared to Theorem 9. Note, that Theorem 9 does not generalize to the case, when the regularity restriction on the location of \( Y \) in \( X \) is replaced by the assumption that \( X \) is Hausdorff [3]. Theorem 35 also does not generalize to the case, when \( X \) is assumed to be Hausdorff and \( Y \) is assumed to be regular in \( X \) [3]. Note also that we cannot claim in the conclusion of Theorem 35 that \( Y \) is strongly normal in \( X \) [3]. But the next question (to be compared with Problem 30) remains open.

**Problem 36.** Let \( Y \) be Lindelöf in a regular space \( X \). Is then \( Y \) realnormal in \( X \)? What if, in addition, \( Y \) is dense in \( X \)?

It is easy to show that \( Y \) may be Lindelöf in a Hausdorff space \( X \), while there is no Lindelöf space \( Z \) such that \( Y \subset Z \subset X \), that is, \( Y \) is not Lindelöf in \( X \) from outside. To construct a similar example when \( X \) is regular is much more difficult. This was done by Dow and Vermeer [10]. Therefore, the next result, based on Theorem 9, is not strong enough to provide a positive answer to Problem 36.

**Theorem 37.** If \( X \) is regular and \( Y \) is Lindelöf in \( X \) from outside, then \( Y \) is realnormal in \( X \).

**Problem 38.** Let \( X \) be regular and let \( Y \) be Lindelöf in \( X \). Is then \( Y \) normal in \( X \) from outside? That is, is there a normal space \( Z \) such that \( Y \subset Z \subset X \)? What if we also assume that \( Y \) is dense in \( X \)?

The next obvious proposition allows us to derive an important corollary from Theorem 34.

**Proposition 39.** If \( Y \) is normal in a \( T_1 \)-space \( X \), then \( Y \) is a regular space.

**Corollary 40.** If \( Y \) is compact in a Hausdorff space \( X \), then \( Y \) is a regular space.
Corollary 40 gives rise to a very natural question: is it true that if \( Y \) is compact in a Hausdorff space \( X \), then \( Y \) is a Tychonoff space? It was shown in [7], that this is not the case. It follows from Theorem 34 and Proposition 32 that normality of \( Y \) in a Hausdorff space \( X \) does not imply, in general, that \( Y \) is weakly realnormal in \( X \).

Another natural conjecture is that every regular \( T_1 \)-space \( Y \) is compact in some larger Hausdorff space. This conjecture was also dealt with in [7], where it was shown to be false. The argument runs as follows. Let \( Y \) be compact in a Hausdorff space \( X \). Then, for every \( x \in X \), the subspace \( Y_x = Y \cup \{x\} \) is also compact in \( X \). Therefore, \( Y_x \) is regular, by Corollary 40. Let us now assume that \( Y \) is closed in every larger regular space. It follows from the above argument that \( Y \) is closed in \( X \). Then, since \( Y \) is compact in \( X \), \( Y \) is compact in itself. Thus, we have established the next proposition:

**Proposition 41.** Let \( Y \) be a regular \( T_1 \)-space closed in every larger regular space. Then \( Y \) is compact in a Hausdorff space \( X \) if and only if \( Y \) is compact in itself.

In view of the above results, the next problem seems to be very interesting.

**Problem 42.** Find an “inner” characterization of regular \( T_1 \)-spaces \( Y \) for which there exists a larger Hausdorff space \( X \) such that \( Y \) is compact in \( X \).

Such an \( X \) as in Problem 42 should be called a relative Hausdorff compactification of \( Y \) if, in addition, \( Y \) is dense in \( X \). Note that if \( Y_\alpha \) is compact in \( X_\alpha \) for each \( \alpha \in A \) then the product of the spaces \( Y_\alpha \) is compact in the product of the spaces \( X_\alpha \)—this was shown by Ranchin in [16].

In connection with Problem 42, yet another two results from [7] should be mentioned. Recall that a space \( X \) is called Urysohn, if for every two different points \( x \) and \( y \) of \( X \) there are neighbourhoods \( U \) and \( V \) of \( x \) and \( y \) respectively, such that \( \overline{U} \cap \overline{V} = \emptyset \). Clearly, Urysohn spaces are Hausdorff.

**Theorem 43** [7]. If \( Y \) is compact in a Urysohn space \( X \) and dense in \( X \), then \( X \) is normal on \( Y \), and, therefore, \( X \) is \( \kappa \)-normal.

**Theorem 44** [7]. If \( Y \) is compact in a Urysohn space \( X \), then \( Y \) is a Tychonoff space.

One of the new ideas, stemming from the concept of a relative topological property, is conveyed by the following general question.

**General Problem 45.** Let \( P \) be a hereditary class of spaces. For a given relative property \( Q \), characterize topological spaces \( Y \), which have the property \( Q \) in every \( X \) in \( P \) such that \( Y \subset X \).

Some most natural questions we get, if we take \( P \) to be the class of all Hausdorff spaces, or the class of all regular \( T_1 \)-spaces, or the class of all Tychonoff spaces. For example, when is a \( T_1 \)-space \( Y \) normal in every larger regular \( T_1 \)-space \( X \)? Or, when is
A Hausdorff space $Y$ regular in every larger Hausdorff space $X$? Such spaces $Y$ might be called absolutely relatively normal spaces and absolutely relatively regular spaces, respectively. A particularly interesting version of Problem 45 we obtain, when $P$ is a hereditary class of spaces (then $Y$ has also to be in that class).

Some unexpected and very interesting results were obtained in this direction by Gordienko [13].

**Theorem 46** [13]. If a regular $T_1$-space $Y$ is normal in every larger regular $T_1$-space $X$, then every closed discrete subspace of $Y$ is countable, that is, the extent of $Y$ is countable.

This result provides a key step in the proof of the following theorem of Gordienko, which can be considered as a solution of a natural version of the famous normal Moore space problem.

**Theorem 47** [13]. A Moore space $Y$ is normal in every larger regular space $X$ if and only if $Y$ is separable and metrizable.

An unexpected result of similar kind, providing a new point of view onto compactness, was obtained by Arhangel’skii and Tartir in [8].

**Theorem 48** [8]. A Hausdorff space $Y$ is regular in every larger Hausdorff space if and only if $Y$ is compact.

Theorem 46 and General Problem 45 suggest the following question.

**Problem 49.** Characterize regular (Tychonoff) spaces $Y$ that are normal in every larger regular (Tychonoff) space $X$.

Probably, it will be natural to restrict ourselves in Problem 49 to $T_1$-spaces. Note that Theorem 46 is not reversible: the space $\omega_1 \times \{0, 1\}$ is countably compact (therefore, its extent is countable), while it is easy to find a larger Tychonoff space, in which it is not normal.

**Problem 50.** Characterize regular (Tychonoff) spaces, that are strongly normal in every larger regular (Tychonoff) space.

To formulate another interesting result in the spirit of Problem 45, we now introduce relative versions of paracompactness.

Let us say that $Y$ is paracompact in $X$ ($1$-paracompact in $X$), if for every open covering $\gamma$ of $X$ there exists a family $\mu$ of open sets in $X$ such that $\mu$ is locally finite at each point of $Y$, $\mu$ refines $\gamma$, that is, for each $V \in \mu$ there is $U \in \gamma$ containing $V$, and $Y \subseteq \bigcup \mu$ (such that $\bigcup \mu = X$, respectively).
Clearly, if $X$ is paracompact, then every subspace $Y$ of $X$ is 1-paracompact in $X$. Of course, 1-paracompactness of $Y$ in $X$ implies paracompactness of $Y$ in $X$, but the converse is not true [3].

Let us call $Y$ nearly paracompact in $X$, if for each open covering $\gamma$ of $X$ one can find a covering $\mu$ of $Y$ by open subsets of $Y$, which is locally finite at each $y \in Y$ and refines $\gamma$. The next assertion is easy to prove.

**Proposition 51.** If $Y$ is dense in $X$ and nearly paracompact in $X$, then $Y$ is paracompact in $X$.

The assumption that $Y$ is dense in $X$ cannot be dropped in the last statement. Indeed, if $Y$ is paracompact (in itself), then $Y$ is nearly paracompact in every larger space $X$, while such $Y$ need not be paracompact in $X$, even if $X$ is Tychonoff [3].

**Theorem 52** [3]. If $Y$ is Lindelöf in a regular space $X$, then $Y$ is paracompact in $X$.

Theorem 52 cannot be strengthened to the conclusion that $Y$ is 1-paracompact in $X$. Indeed, if $Y$ is a countable space and $Y$ is dense in a regular space $X$, then $Y$ is 1-paracompact in $X$ if and only if $X$ is Lindelöf [3]. Thus, any separable nonnormal Tychonoff space provides us with a counterexample to such a strengthening of Theorem 52.

Gordienko has established a result, which might be viewed as a converse to Theorem 52.

**Theorem 53** [13]. If a regular space $Y$ is paracompact in every larger regular space $X$, then $Y$ is Lindelöf.

It is obvious from the discussion above, that “paracompact in” in Theorem 53 cannot be replaced by “nearly paracompact in”. The proof of Theorem 53 is based on Theorem 44 and on the next fact.

**Theorem 54** [13]. If $Y$ is paracompact in $X$ and $X$ is regular, then $Y$ is normal in $X$.

One can find more results on relative paracompactness in [3]. An interesting open problem, concerning the above versions of relative paracompactness, is whether one can generalize to this case the well known Michael’s criteria of paracompactness (see [15,11]). In conclusion, we formulate another natural general question—a general version of Problem 42.

**General Problem 55.** Let $\mathcal{P}$ be a class of spaces, and $\mathcal{Q}$ a relative topological property. Characterize topological spaces $Y$ such that $Y$ has $\mathcal{Q}$ in $X$ for some $X \in \mathcal{P}$, which is larger than $Y$. 
References