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Contractible bonds in graphs

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Abstract

This paper addresses a problem posed by Oxley (Matroid Theory, Cambridge University Press, Cambridge, 1992) for matroids. We shall show that if G is a 2-connected graph which is not a multiple edge, and which has no K_5-minor, then G has two edge-disjoint non-trivial bonds B for which G/B is 2-connected.

MSC: 05C38; 05C40; 05C70

Keywords: Bond; Minor; Contractible

1. Introduction

For a graph G we shall let $\varepsilon(G)$ and $v(G)$ denote the number of edges and vertices in G, respectively. For a set of edges or vertices A of $V(G)$, we let $\mathbf{G}(\mathbf{A})$ denote the subgraph induced by A. For sets of vertices $X \subseteq V(G)$ and $Y \subseteq V(G)$ we denote the set of edges having one endpoint in X and the other in Y by $[\mathbf{X}, \mathbf{Y}]$. A *cutset* is a set of edges $[X, \overline{X}]$ for some X. A cutset which is minimal is called a *bond* or *cocycle*; that is, $B = [X, \overline{X}]$ is a bond if and only if both $G(X)$ and $G(\overline{X})$ are connected subgraphs. A bond B is said to be *trivial* if $B = [\{v\}, V(G)\backslash\{v\}]$ for some vertex v. A collection of edge-disjoint bonds of a graph which partitions its edges is called a *bond decomposition*. If in addition all its bonds are non-trivial, then the decomposition is said to be *non-trivial*.

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For $A \subset E(G)$ we let $\mathbf{G}/\mathbf{A}$ denote the graph obtained by contracting the edges of A. For $v \in V(G/A)$ we denote by $> \mathbf{v} <_{\mathbf{A}}$ the vertices in the component of $G' = G(A) \cup V(G)$ corresponding to v. For an edge $e \in E(G/A)$ we let $> \mathbf{e} <_{\mathbf{A}}$ denote the corresponding edge in G. Similarly, for a subset of vertices (resp. edges) X of G/A we let $> \mathbf{X} <_{\mathbf{A}}$ denote the subset of vertices (resp. edges) $\bigcup_{x \in X} > x <_A$. For a subgraph H of G/H induced by $V(H)$ we let $> \mathbf{H} <_{\mathbf{A}}$ denote the subgraph of G induced by $> V(H) <_A$. For each vertex $v \in V(G)$ we associate the vertex $u \in V(G/A)$ where $v \in > u <_A$. We denote u by $\langle \mathbf{v} \rangle_{\mathbf{A}}$. Similarly, for an edge $e \in E(G) \backslash A$ we associate the edge $e' \in E(G/A)$ where $e = > e' <_A$. We denote e' by $\langle \mathbf{e} \rangle_{\mathbf{A}}$. For a subset of vertices $X \subseteq V(G)$ we let $\langle \mathbf{X} \rangle_{\mathbf{A}} = \{\langle v \rangle_A : v \in X\}$ and for a subset of edges $Y \subset E(G)$ we let $\langle \mathbf{Y} \rangle_{\mathbf{A}} = \{\langle e \rangle_A : e \in Y \backslash A\}$.

J. Oxley proposed the following problem in [7]:

1.1 Problem. *Let M be a simple connected binary matroid having cogirth at least* 4. *Does M have a circuit C such that $M \backslash C$ is connected*?

Here, by *cogirth* of a matroid M we mean the minimum cardinality of a cocircuit in M. For graphic matroids, this problem has been answered in the affirmative by a number of authors including Jackson [3], Mader [5], and Thomassen and Toft [8]. Recently, Goddyn and Jackson [1] proved that for any connected, binary matroid M having cogirth at least 5 which does not have either a F_7-minor or a F_7^*-minor, there is a cycle C for which $M \backslash C$ is connected. For cographic matroids, the above problem translates as follows. A circuit T in $M^*(G)$ corresponds to a bond in G. The matroid $M^*(G) \backslash T$ is connected if and only if either $|E(G/T)| = 1$ or G/T is loopless and 2-connected. Oxley's problem for cographic matroids can be restated as:

1.2 Problem. *Given G is a* 2*-connected*, 3*-edge connected graph with girth at least* 4, *does G contain a bond B such that G/B is* 2*-connected*?

We say that a collection of edges A in a 2-connected graph G is *contractible* if G/A is 2-connected. We say that a bond is *good* if it is both non-trivial and contractible. We call two edge-disjoint good bonds a *good pair* of bonds.

In [4], an example is given which shows that the answer to this problem is in general negative. The main result of this paper addresses Oxley's problem in the case of non-simple cographic matroids. Here there is a small example of a graph based on K_5 which has no contractible bonds: let B be a bond of cardinality 6 in K_5, and let G be the graph obtained from K_5 by duplicating each edge in $E(K_5) \backslash B$ and then subdividing both edges of each resulting digon exactly once (see Fig. 1). Then G is 2-connected with girth at least 4, but contracting any bond of G leaves a graph which is not 2-connected. We say that a digon is *isolated* if it is a multiple 2-edge consisting of two non-loop edges $\{e, f\}$ where no other edge has the same end vertices as e and f. In [2], the following theorem was proved which confirmed a conjecture of Jackson [3]:

1.3 Theorem. *Let G be a* 2*-connected graph having $k \in \{0, 1\}$ vertices of degree* 3 *and which has no Petersen graph minor and which is not a cycle. Then G has* $2 - k$ *edge-disjoint*

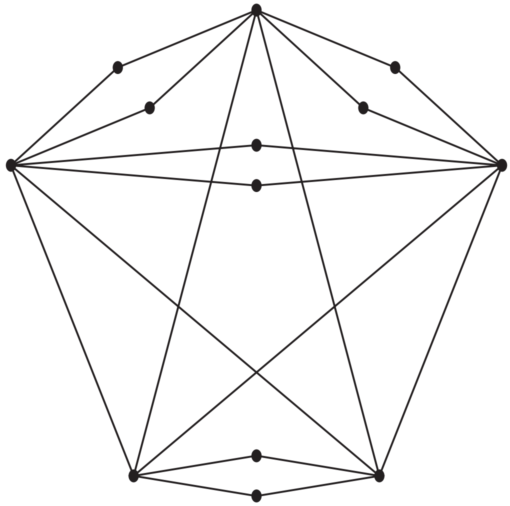

Fig. 1.

cycles C which are not isolated digons for which $G \backslash E(C)$ *is* 2*-connected, apart for possibly some isolated vertices*.

In this paper, the main result is the analog of the above result in the case of cographic matroids:

1.4 Theorem. *Let G be a* 2*-connected graph which is not a multiple edge and which has no triangles. If G has no* K_5*-minor, then it has a good pair of bonds*.

The proof strategy of the main theorem is to use the minimum counterexample approach, reducing as much as possible such a graph so that its structure is more apparent. The first step is to show that it is non-planar. Then we use a Wagner-type result for graphs without a K_5-minor to decompose the graph. In the initial stages of the proof, the problem of finding contractible bonds in planar graphs is examined. Certain lemmas are given here which play a central role in the main proof. Thereafter, we examine the case of non-planar graphs where we show that our graph G can be decomposed into a planar graph G_1 and another graph G_2 where G_1 and G_2 meet along a 3-vertex cut $\{v_1, v_2, v_3\}$. The bulk of the paper involves showing that certain contractible bonds for G_1 and G_2 can be 'spliced' together to form contractible bonds in G. The splicing is easier or harder depending on the mutual distances between $v_1, v_2,$ and v_3. We are able to succeed in our splicing operation for two main reasons; firstly, we have a great deal of flexibility in how we choose our contractible bonds in G_1, and secondly, by attaching "gadgets" to the vertices v_1, v_2, v_3, in G_1 and G_2, we are able to coerce the constructed contractible bonds in G_1 and G_2 to have certain favourable properties.

2. Contractible bonds in planar graphs

A cycle C in a 2-connected graph G is said to be *removable* if it is not an isolated digon and $G\backslash E(C)$ is 2-connected apart from possibly some isolated vertices. A cycle which bounds a face of a plane graph is said to be *facial*. We say that a cycle in a 2-connected plane graph is *good* if it is both non-facial and removable. We call two edge-disjoint good cycles a *good pair* of cycles. The following theorems were shown in [6]:

2.1 Theorem. *Let G be a 2-connected plane graph which is not a cycle. Given G has $k \in \{0, 1\}$ vertices of degree 3, there exists $2 - k$ good cycles in G.*

2.2 Theorem. *Let G be a 2-connected plane graph having at most $k \in \{0, 1\}$ faces which are triangles. Assuming G is not a multiple edge, there exists $2-k$ edge-disjoint good bonds.*

The following lemmas play a central role in the proof of the main theorem.

2.3 Lemma. *Let G be a 2-connected plane graph with no vertices of degree 3. Let $v \in V(G)$ be a vertex of degree 4 where one or two isolated digons are incident with v. If G has no good cycle not containing v, then G is the union of a good pair of cycles, and each vertex has degree 2 or 4.*

Proof. Suppose G has no good cycle not containing v. By Theorem 2.1, G has a good pair of cycles, say C_1 and C_2 containing v and hence also edges of a digon incident to v, say D, having edges e and f and vertices u and v. We may assume that $e \in E(C_1)$. Suppose that C_1 contains no vertices of degree 5. Let $G' = G\backslash E(C_1)$. Then G' is 2-connected (apart from possibly some isolated vertices) and has no vertices of degree 3. It follows by Theorem 2.1 that if G' is not a cycle, then it has a good pair of cycles, one of which does not contain v. The cycle not containing v, say C_1', is seen to be good in G. This is because $G'\backslash E(C_1')$ is 2-connected except for possibly isolated vertices, and $G\backslash E(C_1')$ is obtained from $G'\backslash E(C_1')$ by replacing the edges of C_1. Since f and e are the edges of $G'\backslash E(C_1')$ and $E(C_1)$, respectively, and have the same endpoints, $G'\backslash E(C_1')$ is 2-connected except for possibly isolated vertices. Since by assumption no such cycle in G exists, G' must be a cycle, and in this case, G is the union of a good pair of cycles. We may therefore assume that C_1 contains at least one vertex of degree 5. Let w be the first vertex of degree 5 we encounter while travelling from v along C_1 where edge e of digon D is traversed first. Let P be the path representing the portion of C_1 traversed from v to w, and let $G' = G\backslash E(P)$. Then G' is 2-connected and has exactly one vertex of degree 3, namely v. By Theorem 2.1, there is a good cycle in G', and this cycle cannot contain v. Furthermore, this cycle is seen to be good in G, and this is contrary to our assumption. Thus no such vertex w can exist and this completes the proof of the lemma. $\square$

A path P in a 2-connected graph G is said to be *removable* if $G\backslash E(P)$ is 2-connected aside possibly for some isolated vertices.

2.4 Lemma. *Let G be a 2-connected plane graph having no vertices of degree 3. Let $v \in V(G)$ be a vertex of degree 5 which is incident with two isolated digons. If G has no* *good cycle not containing* v, *then G is the union of a good pair of cycles and a removable path from* v *to a vertex of degree* 5. *Moreover, all vertices of G have degree* 2 *or* 4, *except for* v *and another vertex of degree* 5, *and the removable path may chosen to contain any edge incident with* v.

Proof. We suppose that G has no good cycles not containing v. By Theorem 2.1, G has a good pair of cycles. Let C_1 and C_2 be two such cycles. Since there are two digons incident with v, the cycles C_1 and C_2 contain edges of one such digon. Suppose that C_1 contains no vertices of degree at least 5, apart from v. Then $G' = G\backslash E(C_1)$ is 2-connected (apart from possibly some isolated vertices) and has exactly one vertex of degree 3, namely v. By Theorem 2.1, there exists a good cycle C' in G'. Such a cycle does not contain v, and is also seen to be good in G. To see this, one can use the same argument as was used in the proof of Lemma 2.3. Since this is contrary to our assumption, C_1 must contain a vertex of degree at least 5, apart from v. Let w be the first vertex of degree at least 5 that we encounter while travelling along C_1 from v. Let P be the path representing the portion of C_1 traversed from v to w, and let $G' = G\backslash E(P)$. Then $d_{G'}(v) = 4$ and there are 1 or 2 digons incident with v. If G' has a good cycle not containing v, then such a cycle is clearly good in G. Thus no such cycle exists in G' and hence Lemma 2.3 implies that G' is the union of a good pair of cycles. These cycles are also a good pair in G. Observing that each (non-isolated) vertex in G' has degree 2 or 4, and each internal vertex of P has degree 2 or 4 in G, we conclude that each vertex of G has degree 2 or 4, except for v and w which have degree 5. The above arguments also demonstrate that for any edge incident with v, there is a good cycle containing it, and such a cycle must contain w. Thus for any edge incident with v we can choose the removable path P so that it contains this edge. □

2.5 Lemma. *Let G be a* 2*-connected plane graph having no vertices of degree* 3. *Let* $v \in V(G)$ *be a vertex of degree* 6 *where* v *is incident with three isolated digons. If G has no good cycle not containing* v, *then we have two possibilities for G*:

(i) *G is the edge-disjoint union of three good cycles, and all vertices of G have degree* 2 *or* 4, *except for* v *and at most one other vertex of degree* 6.
(ii) *G is the edge-disjoint union of three good cycles and a removable path between two vertices of degree* 5. *Moreover, all vertices of G have degree* 2 *or* 4, *apart from* v *and two vertices of degree* 5.

Proof. We suppose that G has no good cycle which does not contain v. By Theorem 2.1, G has a good pair cycles, say C_1 and C_2 which contain v and hence also edges of a digon incident to v. Suppose C_1 contains no vertices of degree at least 5, apart from v. Let $G' = G\backslash E(C_1)$. Then G' is 2-connected (apart from possibly some isolated vertices), and has no vertices of degree 3. Moreover, $d_{G'}(v) = 4$, and v is incident with exactly one digon in G'. If G' contains a good cycle which avoids v, then such a cycle is also good in G. To see this, one can use the similar arguments as were used in the proof of Lemma 2.3. Thus no such cycles exist in G', and hence by Lemma 2.3 the edges of G' are partitioned by a good pair cycles. These cycles together with C_1 decompose the edges of G into good cycles. Consequently, each vertex of G has degree 2, 4, or 6. Suppose G has two vertices of degree 6 apart from v, say w and z. Let P be the path from w to z in C_1 which contains v. Let $G'' = G\backslash E(P)$. Then G'' is 2-connected (apart from possibly some isolated vertices), has no vertices of degree 3, and $d_{G'}(w) = d_{G'}(z) = 5$, and $d_{G'}(v) = 4$. The vertex v is incident with one isolated digon in G'', and G'' contains no good cycles which avoid v. In this case, Lemma 2.3 implies that G'' is the union of a good pair of cycles. This is impossible since both w and z have odd degree (equal to 5) in G''. We conclude that two such vertices w and z cannot exist in G, and consequently, G has at most one other vertex of degree 6, apart from v. Then (i) holds.

Suppose now that C_1 contains at least one vertex of degree at least 5, apart from v. Let P be a path traversed by moving along C_1 from v until one first reaches a vertex of degree at least 5, say u. Let $G' = G\backslash E(P)$. Then G' is 2-connected, $d_{G'}(v) = 5$, and v is incident with two isolated digons. We have that G' contains no good cycles which avoid v, as such cycles are seen to be good in G. By Lemma 2.4, G' is the union of a good pair of cycles C_1' and C_2', and a removable path P' from v to a vertex of degree 5 in G', say w. Furthermore, each (non-isolated) vertex of G' has degree 2 or 4, apart from v and w which have degree 5. If $u = w$, then $d_G(u) = 6$, and G has no vertices of odd degree. Then we can show, as in the previous paragraph, that (i) holds. We suppose therefore that $u \neq w$. This means that G has exactly 2 odd degree vertices which are u and w and every other vertex has degree 2 or 4 apart from v which has degree 6. Then $d_{G'}(u) = 4$, and $d_{G'}(w) = 5$, and one of the cycles C_1' or C_2' contains both u and w. We may assume that C_1' contains u and w. Let P'' be the path from u to w in $C_1''\backslash\{v\}$, and let $G'' = G\backslash E(P'')$. We have that G'' is 2-connected (apart from possibly some isolated vertices), v is incident with three isolated digons in G'', and G'' has no odd degree vertices. Repeating previous arguments, we deduce that G'' is the edge-disjoint union of three good cycles, say C_i'', $i = 1, 2, 3$. Moreover, all (non-isolated) vertices have degree 2 or 4, apart from v and at most one other vertex of degree 6. If v is the only vertex of degree 6 in G'', then all the vertices of G have degree 2 or 4, apart from u, w, and v which have degrees 5, 5, and 6, respectively. Then (ii) is seen to hold. If G'' has another vertex of degree 6, apart from v, then this vertex must be w. Thus $d_G(w) = 7$, $d_G(u) = 5$, $d_G(v) = 6$, and all other vertices of G have degree 2 or 4. Since $d_G(u) = 5$, one of the cycles C_i'', $i = 1, 2, 3$ (which are good in G), say C_1'', does not contain u (but contains v). Now C_1'' contains no vertices of degree 5, and thus by the first part of the proof, G is the edge-disjoint union of three good cycles. This yields a contradiction. We conclude that in this case, G has exactly one vertex of degree 6, namely v, and hence all the vertices of G have degree 2 or 4, with the exception of u, w, and v which have degrees 5, 5, and 6, respectively. In this case, (ii) holds with C_i'', $i = 1, 2, 3$ and P''. $\square$

2.6 Lemma. *Let G be a 2-connected graph and suppose S is a proper subset of edges such that $G\backslash S$ is connected and $G^* = G/S$ is 2-connected. Suppose that B^* is a contractible subset of edges in G^*. Let $B = > B^* <_S$. If B is not contractible in G, then G/B contains loops.*

Proof. Let S, B, and B^* be as in the statement of the lemma. We suppose that B is not contractible in G, and $G' = G/B$ contains no loops. Let $S' = \langle S\rangle_B$. If G' contains 2 or

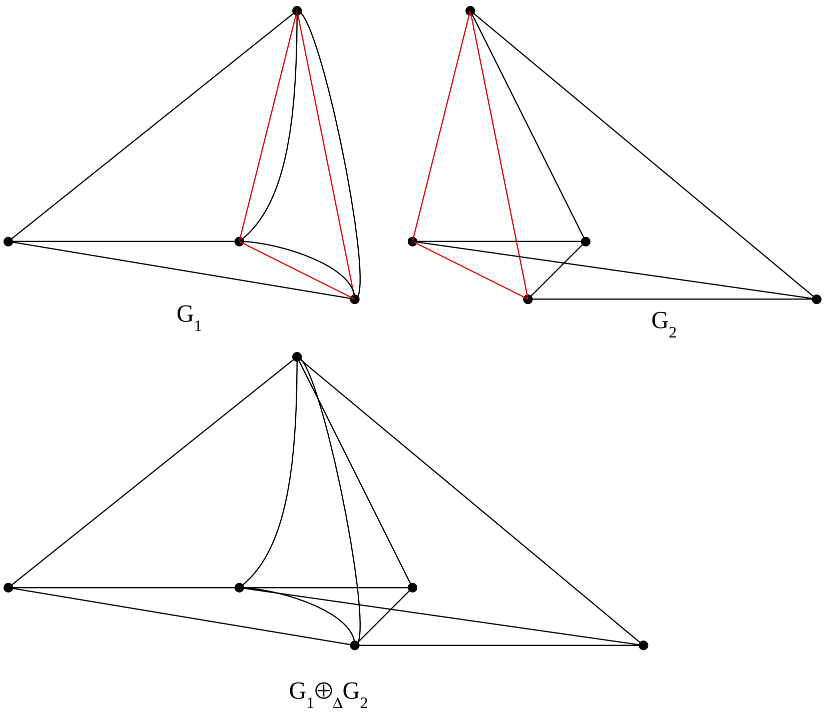

Fig. 2. $\triangle$-sum of G_1 and G_2.

more blocks K' where $E(K') \not\subseteq S'$, then G'/S' has 2 or more blocks. However,

$$G'/S' = G/B/S' = (G/S)/B^* = G^*/B^*$$

which is 2-connected. So at most one such block exists. Thus if G' has more than one block, then we can find a block K' of G' where $E(K') \subseteq S'$. If K' is not a loop, then the edges of $> K' <_B$ form a cutset in G, which means that the edges of S must also be a cutset in G. However, this is impossible since $G \backslash S$ is connected. Thus K' is a loop. So if B is not contractible in G then G/B must contains loops, and moreover, G/B minus its loops is a 2-connected graph. $\square$

2.1. *The $\triangle$-sum of two graphs*

Following the definition given in [9], we define a $\triangle$–*sum* of two graphs G_1 and G_2 with $\varepsilon(G_i) \geqslant 7$, $i = 1, 2$ to be the graph obtained by 'glueing' together G_1 and G_2 along the edges of a given triangle in both G_1 and G_2 and then deleting the edges of this triangle (see Fig. 2). We denote such a graph by $\mathbf{G_1} \oplus_{\triangle} \mathbf{G_2}$.

2.7 Lemma. *Let G be a $\triangle$-sum of planar graphs $G = G_1 \oplus_\triangle G_2$ where G_1 is a plane graph. Let $B = [X, \overline{X}]$ be a bond of G and let C be a cycle which bounds a face of G_1. Then $|B \cap E(C)| \leqslant 2$.*

Proof. Let $G = G_1 \oplus_\triangle G_2$ where the $\triangle$-sum occurs along a triangle $T = uvw$. Let C be a cycle which bounds a face of G_1 and let $B = [X, \overline{X}]$ be a bond of G. Suppose $|B \cap E(C)| \geqslant 3$, and $e_1 = x_1y_1,\ e_2 = x_2y_2$, and $e_3 = x_3y_3$ are three edges in $B \cap E(C)$. We may assume that $x_i \in X,\ i = 1, 2, 3$, and we meet the edges e_1, e_2, e_3 in this order as we move along C. So while traversing C we meet the vertices $x_1, y_1, y_2, x_2, x_3, y_3$ in this order (noting that it is possible that $y_1 = y_2$ or $x_2 = x_3$). Since B is a bond, both $G(X)$ and $G(\overline{X})$ are connected. So there exists a path P from x_1 to x_2 in $G(X)$ and a path Q from y_1 to y_3 in $G(\overline{X})$. Either $P \subset G_1$ or $E(P) \cap E(G_1)$ is a vertex disjoint union of two paths P_1 and P_2 where $P_j = u_{j1}u_{j2}\cdots u_{jn_j},\ j = 1, 2$, and $u_{11} = x_1,\ u_{2n_2} = x_2$. If the latter occurs, then $u_{1n_1},\ u_{21} \in \{u, v, w\}$. Since $T = uvw$ is a triangle of G_1, it follows that $u_{1n_1}u_{21} \in E(G_1)$, and $P' = P_1 \cup P_2 \cup \{u_{1n_1}u_{21}\}$ is a path in G_1 from x_1 to x_2. Since Q does not intersect P it does not intersect P' either. However, since G_1 is plane, any path from y_1 to y_3 in G_1 must cross P' and this yields a contradiction. If $P \subset G_1$, the same conclusion holds. We conclude that no such cycle C can exist. $\square$

3. Reductions on a minimum counterexample

We suppose that Theorem 1.4 is false and suppose that G is a minimal counterexample where $\varepsilon(G)$ is minimum subject to $v(G)$ being minimum. By Theorem 2.2 we may assume that G is non-planar.

We call a path P between two vertices of degree at least 3 a *thread* if it is an edge, or if all its internal vertices have degree 2. We define the *length* of P to be the number of its edges and we denote it by $|P|$.

Claim 1. *G has no thread of length 3 or greater.*

Proof. Suppose $T = u_0e_0u_1\cdots e_{k-1}u_k$ is a thread where $k \geqslant 3$. Let $G' = (G \backslash \{u_1, \ldots, u_{k-1}\}) \cup \{u_0u_k\}$. Suppose G' contains no triangles. Then by the minimality of G, the graph G' has a good pair of bonds, say B_1 and B_2. We may assume that $u_0u_k \notin B_1$. Then B_1 and $C = [\{u_1, \ldots, u_{k-1}\}, \overline{\{u_1, \ldots, u_{k-1}\}}]$ are a good pair of bonds in G.

We suppose instead that G' contains a triangle (which must contain u_0u_k). Let G'' be the graph obtained from G' by deleting u_0u_k and adding a vertex u together with the edges uu_0 and uu_k. The graph G'' has no triangles since G has no edge between u_0 and u_k; for otherwise it would have a triangle (since G' has a triangle). Thus by assumption, G'' has a good pair of bonds, say B_1 and B_2. If $B_i,\ i \in \{1, 2\}$ do not contain the edges uu_0 or uu_k, then they are a good pair in G. If for some $i \in \{1, 2\}$ B_i contains one of the edges incident to u, for example u_0u, then $B_i' = (B_i \backslash \{uu_0\}) \cup \{e_0\}$ is a contractible bond in G. So the bonds B_1, B_2 give rise to a good pair of bonds in G. $\square$

Claim 2. *Between any two vertices of G there is at most one thread.*

Proof. Suppose P_1 and P_2 are threads between two vertices u and v. By Claim 1, a thread of G has at most one internal vertex. Thus, given that G is triangle-free, both P_1 and P_2 have the same length. Let G' be the graph obtained from G by deleting all the internal vertices of P_2. Then G' is 2-connected, triangle-free, and therefore has a good pair of bonds. Such bonds are easily seen to be extendable to a good pair of bonds in G. □

For positive integers m and n we let $\mathbf{K}_{m,n}$ denote the complete bipartite graph with parts of size m and n. We let G_8 denote the *Wagner graph* which is the graph obtained from an 8-cycle $v_1v_2\cdots v_8v_1$ by adding the chords $v_iv_{i+4},\ i=1,2,3,4$.

Claim 3. *G is not a subdivision of $K_{3,3}$ or G_8.*

Proof. Using Claim 1, this is a straightforward exercise which is left to the reader. □

3.1. The graph $hom(G)$

For a graph G none of whose components are cycles, we define a graph $hom(\mathbf{G})$ to be the graph obtained from G by suppressing all its vertices of degree 2. For a subgraph H of G we define $hom(\mathbf{G}|\mathbf{H})$ to be the subgraph of $hom(G)$ induced by $V(hom(G)) \cap V(H)$.

Claim 4. *$hom(G)$ is 3-connected.*

Proof. It suffices to show that G has no 2-separating set apart from the neighbours of a vertex of degree 2. Suppose the assertion is false, and there exists a 2-separating set of G, $\{v_1, v_2\}$ which separates 2 subgraphs G_1 and G_2; that is, $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v_1, v_2\}$, where $G_i,\ i = 1, 2$ is not a single vertex joined to v_1 and v_2. We have $E(G) = E(G_1) \cup E(G_2)$. We shall consider two cases.

Case 1: Suppose $e = v_1v_2 \in E(G)$ (and thus $e \in E(G_1) \cap E(G_2)$). Then both G_1 and G_2 are 2-connected and triangle-free, and moreover, $\varepsilon(G_i) < \varepsilon(G),\ i = 1, 2$. For $i = 1, 2$ the graph G_i has a good pair of bonds B_{i1} and B_{i2}. We may assume that $e \notin B_{11} \cup B_{21}$. One sees that B_{11} and B_{21} is a good pair of bonds in G.

Case 2: Suppose $v_1v_2 \notin E(G)$. If $G_i \cup \{v_1v_2\}$ does not contain a triangle, for $i = 1, 2$, then we can repeat more or less the same arguments as in Case 1. So we suppose it has a triangle. Then v_1v_2 is an edge of this triangle. Let $G_i' = G_i \cup \{u_i, u_iv_1, u_iv_2\},\ i = 1, 2$, where $u_i,\ i = 1, 2$ are new vertices added to G_i having neighbours v_1 and v_2. The graph G_i' is triangle-free for $i = 1, 2$ and has a good pair of bonds, say B_{i1}' and B_{i2}'. If $B_{ij}',\ j \in \{1, 2\}$ contain no edges incident to u_i, then they are seen to be a good pair of bonds in G. We may assume that B_{11}' and B_{12}' contain edges incident to u_1. We suppose without loss of generality that $u_1v_1 \in B_{11}$ and $u_1v_2 \in B_{12}'$. Let $B_{ij}' = [P_{1j}', Q_{1j}'],\ i, j = 1, 2$. We can assume that at least one of B_{21}' or B_{22}' contains an edge incident to u_2. Suppose without loss of generality that B_{21}' contains u_2v_1. We may assume that $v_1 \in P_{11}'$ (and $u_1, v_2 \in Q_{11}'$), $v_2 \in P_{12}'$ (and $u_1, v_1 \in Q_{12}'$), and $v_1 \in P_{21}'$ (and $u_2, v_2 \in Q_{21}'$). The set $A_1 = [(Q_{12}' \cup P_{21}')\backslash\{u_1, u_2\},\ (P_{12}' \cup Q_{21}')\backslash\{u_1, u_2\}]$ is seen to be a good bond in G. Similarly, if B_{22}' contains u_2v_2, then, assuming $v_2 \in P_{22}$, the set $A_2 = [(P'_{11} \cup Q'_{22})\backslash\{u_1, u_2\},\ (Q'_{11} \cup P'_{22})\backslash\{u_1, u_2\}]$ is a good bond of G. We conclude that regardless of whether B'_{22} contains $u_2 v_2$ or not, G will have a good pair of bonds. This concludes Case 2.

The proof of the claim follows from Cases 1 and 2. $\square$

4. Good separations

A *separation* (or *separating set*) of a graph G is a set of vertices $S \subset V(G)$ such that $G\backslash S$ has more components than G. A separation with k vertices is called a k-separation. We say that two subgraphs G_1 and G_2 are separated by a separation S if $E(G_1) \cap E(G_2) = \emptyset$, $V(G_1) \cap V(G_2) \subseteq S$, $V(G_i)\backslash S \neq \emptyset,\ i = 1, 2$, and any path from a vertex of G_1 to a vertex of G_2 must contain a vertex of S. Extending this, we say that k subgraphs $G_1, \ldots, G_k$ are separated by a separating set S if any pair of subgraphs $G_i, G_j,\ i \neq j$ is separated by S.

We call a separating set $\{v_1, v_2, v_3\}$ which separates two subgraphs G_1 and G_2 a *good separation* if $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{v_1, v_2, v_3\}$, and it satisfies an additional three properties:

(i) $G_1 \cup \{v_1v_2, v_2v_3, v_1v_3\}$ is planar and has a plane representation where the triangle $v_1v_2v_3$ bounds a 3-face.
(ii) $|V(hom(G|G_1))\backslash\{v_1, v_2, v_3\}| \geqslant 2$.
(iii) There is no good bond of G contained in G_1.

Our principle aim in this section is to show that G has good separations. We shall use a variation of Wagners theorem which can be found in [9].

4.1 Theorem. *Let G be a 3-connected non-planar graph without a K_5-minor and which is not isomorphic to $K_{3,3}$ or G_8. Assume G to have a designated triangle T or edge e. Then G is a $\triangle$-sum $G_1 \oplus_\triangle G_2$ where G_2 contains T or e, whichever applies, and G_1 is planar.*

Our aim is to show that G has a good separation. To this end, we shall need the following lemma:

4.2 Lemma. *Let G be a 3-connected non-planar graph without a K_5-minor, and which is not isomorphic to G_8. Then there exists a 3-separating set $\{v_1, v_2, v_3\}$ which separates three subgraphs G_1, G_2, G_3 where $G = G_1 \cup G_2 \cup G_3$, $V(G_1) \cap V(G_2) \cap V(G_3) = \{v_1, v_2, v_3\}$, and $G_i \cup \{v_1v_2, v_2v_3, v_1v_3\}$ is planar for $i = 1, 2$.*

Proof. By induction on $|E(G)|$. Suppose that G is a 3-connected, non-planar graph which is not isomorphic to G_8 and which has no K_5-minor. If G is isomorphic to $K_{3,3}$, then the lemma is is seen to be true. We shall therefore assume that G is not isomorphic to $K_{3,3}$. In addition, we assume that the lemma holds for any graph having fewer edges than G which satisfies the requirements of the lemma. By Theorem 4.1, G can be expressed as a $\triangle$-sum $G_1 \oplus_\triangle G_2$ where G_1 is planar. If G_2 is planar, then G would be planar since a $\triangle$-sum of two planar graphs is also planar. Thus G_2 is non-planar, and moreover it is 3-connected and contains no K_5-minor. Also, G_2 is not isomorphic to $K_{3,3}$ or G_8 since it contains the triangle

$v_1v_2v_3$. The graph G_2 has less edges than G since by the definition of $\triangle$-sum, $|E(G_1)| \geqslant 7$, and hence

$$|E(G_2)| = |E(G)| - |E(G_1)| + 6 < |E(G)|.$$

Consequently, by the inductive assumption, the lemma holds for G_2, and it contains a 3-separating set $\{u_1, u_2, u_3\}$ which separates three subgraphs G_{21}, G_{22}, and G_{23} where $G_{21} \cup G_{22} \cup G_{23} = G_2$, $V(G_{21}) \cap V(G_{22}) \cap V(G_{23}) = \{u_1, u_2, u_3\}$, and $G_{2j} \cup \{u_1u_2, u_2u_3, u_1u_3\}$ is planar for $j = 1, 2$. We have that $\{v_1, v_2, v_3\} \subset V(G_{2j})$, for some j. If this holds for $j = 1$ or $j = 2$, then $G_1 \oplus_\triangle G_{2j}$ is planar. The set $\{u_1, u_2, u_3\}$ is seen to be the desired 3-separation of G. The proof of the lemma now follows by induction. $\square$

Claim 5. *G has a good separation* $\{v_1, v_2, v_3\}$.

Proof. By Lemma 4.2, there exists a 3-separating set $\{v_1, v_2, v_3\}$ which separates three subgraphs G_1, G_2, G_3 where $V(G_1) \cap V(G_2) \cap V(G_3) = \{v_1, v_2, v_3\}$, and $G_i \cup \{v_1v_2, v_2v_3, v_1v_3\}$ is planar for $i = 1, 2$. We suppose that $|V(hom(G|G_i)) \backslash \{v_1, v_2, v_3\}| = 1$ for $i = 1, 2$ and let $V(hom(G|G_i)) \backslash \{v_1, v_2, v_3\} = \{u_i\}$, $i = 1, 2$. Since $hom(G)$ is 3-connected, there exists three threads T_{i1}, T_{i2}, T_{i3} from u_i to v_1, v_2, v_3, respectively, which meet only at u_i. Suppose $|T_{11}| + |T_{12}| + |T_{13}| \geqslant |T_{21}| + |T_{22}| + |T_{23}|$. Let $G' = G \backslash (V(G_2) \backslash \{v_1, v_2, v_3\})$. The graph G' is 2-connected and contains a good pair of bonds which can easily be extended to a good pair of bonds of G. We conclude that for some $i \in \{1, 2\}$ we have $|V(hom(G|G_i)) \backslash \{v_1, v_2, v_3\}| \geqslant 2$. We may assume that this holds for $i = 1$. Suppose there is a good bond B of G contained in G_1. Then neither G_2 nor G_3 contains a good bond of G. If $|V(hom(G|G_2) \backslash \{v_1, v_2, v_3\}| \geqslant 2$, then G_2 can play the role of G_1 as in the definition of a good separation and we are done. We suppose therefore that $|V(hom(G|G_2) \backslash \{v_1, v_2, v_3\}| = 1$. Then, using the same arguments as before, we have $|V(hom(G|G_3)) \backslash \{v_1, v_2, v_3\}| \geqslant 2$. If G_3 is planar, then G_3 can play the role of G_1 as in the definition of a good separation and we are done. We suppose therefore that G_3 is non-planar. Then it has a 3-separating set $\{w_1, w_2, w_3\}$ similar to $\{v_1, v_2, v_3\}$ which separates 3 subgraphs H_1, H_2, H_3 where H_1 and H_2 are planar, and $|V(hom(G|H_1)) \backslash \{w_1, w_2, w_3\}| \geqslant 2$. If there is a good bond C of G where C is contained in H_1, then B and C would be a good pair of bonds. Thus H_1 contains no good bonds, and $\{w_1, w_2, w_3\}$ would be the desired separating set. $\square$

4.1. *The type of a good separation*

Suppose $\{v_1, v_2, v_3\}$ is a good separation of G. Suppose that in G_1 for each $i \neq j$ we have $dist_{G_1}(v_i, v_j) = 1$ or $dist_{G_1}(v_i, v_j) \geqslant 3$. Let $G_1' = G_1 \cup \{v_1v_2, v_2v_3, v_1v_3\}$. Then G_1' is a 2-connected planar graph with one triangle namely $v_1v_2v_3$. By Theorem 2.2, G_1' has a good bond B' which contains no edges of this triangle. Thus B' is also good in G, and this contradicts the choice of G_1. Hence in a good separation $\{v_1, v_2, v_3\}$ it holds for at least one pair of vertices v_i, v_j that $dist_{G_1}(v_i, v_j) = 2$.

We say that a good separation $\{v_1, v_2, v_3\}$ is of *type* k, $k \in \{1, 2, 3\}$ if there are exactly k pairs of vertices v_i, v_j, $i \neq j$ where $dist_G(v_i, v_j) = 2$. Since G contains no triangles, if $dist_{G_1}(v_i, v_j) = 2$, then $dist_{G_2}(v_i, v_j) \geqslant 2$ (similarly, if $dist_{G_2}(v_i, v_j) = 2$, then $dist_{G_1}(v_i, v_j) \geqslant 2$).

4.2. The graphs G'_1 and G'_2

We shall define a graph $\mathbf{G}'_1$ obtained from G_1 in the following way: For every pair of vertices v_i, v_j $i \neq j$ if $dist_{G_1}(v_i, v_j) = 2$, then provided there is no vertex of degree 2 in G_1 with neighbours v_i and v_j, we shall add such a vertex to G_1 and label it $\mathbf{w}^1_{ij}$. If such a vertex already exists in G_1, then we give it the same label w^1_{ij}. If $dist_{G_1}(v_i, v_j) \neq 2$, then provided there is no edge between v_i and v_j in G_1, we shall add such an edge to G_1.

We define a graph $\mathbf{G}'_2$ from G_2 in a corresponding way(with analogous vertices $\mathbf{w}^2_{ij}$) with one additional requirement. If $\{v_1, v_2, v_3\}$ is a separation of type 3, then provided G_2 does not have a vertex of degree 3 with v_1, v_2, v_3 as its neighbours, we shall add such a vertex and label it $\mathbf{w}_2$. If such a vertex already exists in G_2, then we shall give it the same label w_2.

By Claim 2, G_1 and G_2 cannot both have vertices of degree 2 with common neighbours v_i, v_j. If such a vertex exists in G_1 or G_2, then we label it by w_{ij} in G. The three different possibilities for G'_1 and G'_2 are depicted in Fig. 3.

Given $\{v_1, v_2, v_3\}$ is a good separation, we may assume throughout that G'_1 has a plane representation where v_1, v_2, v_3 belong to a face which we denote by $\mathbf{F}$. We have that $|F| = 4, 5$, or 6 depending on whether the separation has type 1, 2, or 3. We let $\mathbf{K}$ denote the cycle which bounds F. For all $i \neq j$, let $\mathbf{F}_{ij}$ denote the face of G'_i containing v_i and v_j (where $F_{ij} \neq F$), and let $\mathbf{K}_{ij}$ denote the cycle which bounds F_{ij}. We denote the dual of G'_1 by $\mathbf{H}'_1$ and we let $\mathbf{u}$ be the vertex of H'_1 corresponding to the face F in G'_1. The vertex u has exactly three neighbours which we denote by $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$. For each vertex $v \in V(G'_1)$ we let $\mathbf{\Phi}(\mathbf{v})$ denote the face in H'_1 corresponding to v. For $i = 1, 2, 3$ we let $\mathbf{\Phi}_i = \mathbf{\Phi}(v_i)$.

4.3. Wishbones and minimal good separations

A *wishbone* is a graph consisting of a vertex joined to three other vertices by disjoint threads, where at least one of the threads has length 2.

Claim 6. *Let $\{v_1, v_2, v_3\}$ be a good separation. Then G_1 does not contain an induced subgraph which is a wishbone.*

Proof. Suppose that G_1 contains a wishbone T as an induced subgraph. We shall assume that T consists of a vertex a joined to vertices a_1, a_2, a_3 by threads T_1, T_2, and T_3, respectively. If for some $i \neq j$ we have $|T_i| \geqslant 2$ and $|T_j| \geqslant 2$, then letting $S = V(T) \backslash \{a_1, a_2, a_3\}$ one sees that $B = [S, \overline{S}]$ is a good bond of G. This gives a contradiction, as $\{v_1, v_2, v_3\}$ is a good separation and hence G_1 contains no good bonds of G. Thus $|T_i| \geqslant 2$ for at most one value of i, and we can assume without loss of generality that $|T_1| \geqslant 2$ and $|T_2| = |T_3| = 1$. By Claim 1, we have that G has no threads of length 3 or longer, and as such $|T_1| = 2$. Let $T_1 = aba_1$. If a_2 and a_3 are not joined by a thread of length 2, then $B = [\{a, b\}, \overline{\{a, b\}}]$ is a good bond of G

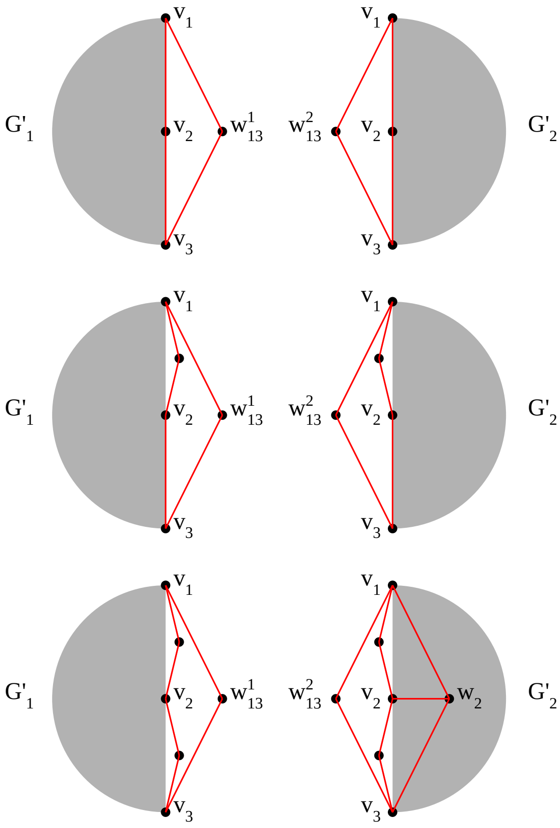

Fig. 3. The graphs G_1' and G_2' as defined for G of type 1, 2, or 3.

which is contained in G_1. Again, this yields a contradiction. Thus there is a thread of length 2 between a_2 and a_3. Let $G' = G \backslash \{a, b\}$. We have that G' is 2-connected and therefore has a good pair of bonds, say B_1' and B_2'. Let $B_i' = [X_i', V(G') \backslash X_i'],\ i = 1, 2$. For $i = 1, 2$ we can assume that $|X_i' \cap \{a_1, a_2, a_3\}| \leqslant 1$. We have that $\langle a_2 \rangle_{B_i'} \neq \langle a_3 \rangle_{B_i'},\ i = 1, 2$ as a_2 and a_3 are joined by a thread. Thus if $a_1, a_2, a_3 \notin X_i'$, then B_i' is a good bond of G. Suppose for $i = 1, 2, 3$ it holds that $a_i \notin X_1' \cap X_2'$. Then the bonds $B_i',\ i = 1, 2$ can easily be modified to yield a good pair of bonds of G. We therefore suppose that for some $i \in \{1, 2, 3\}$ that $a_i \in X_1' \cap X_2'$. If $a_1 \in X_1' \cap X_2'$, then $[X_1', V(G) \backslash X_1']$ and $[X_2' \cup \{b\}, V(G) \backslash (X_1' \cup \{b\})]$ are a good pair of bonds. Suppose that $a_2 \in X_1' \cap X_2'$ or $a_3 \in X_1' \cap X_2'$. Then $[X_1', V(G) \backslash X_1']$ and $[X_2' \cup \{b\}, V(G) \backslash (X_2' \cup \{b\})]$ are a good pair of bonds of G. We conclude that G_1 contains no induced subgraph which is a wishbone. $\square$

We say that a good separation $\{v_1, v_2, v_3\}$ is *minimal* if there is no other good separation contained in $V(G_1)$.

Claim 7. *Let* $\{v_1, v_2, v_3\}$ *be a minimal good separation of* G. *Then for* $i = 1, 2, 3$ *the vertex* v_i *has at least* 2 *neighbours in* $V(hom(G|G_1))\backslash\{v_1, v_2, v_3\}$.

Proof. Suppose the claim is false and assume without loss of generality that v_1 only has one neighbour in $V(hom(G|G_1))\backslash\{v_1, v_2, v_3\}$. We may assume that v_1 is joined by a thread T to a vertex a where $d_{G_1}(a) \geqslant 3$. Since $\{v_1, v_2, v_3\}$ is a good separation, we have $|V(hom(G|G_1))\backslash\{v_1, v_2, v_3\}| \geqslant 2$. If $|V(hom(G|G_1))\backslash\{v_1, v_2, v_3\}| > 2$, then $\{a, v_2, v_3\}$ would be a good separation of G, contradicting the fact that $\{v_1, v_2, v_3\}$ is minimal. Thus $hom(G|G_1)$ has exactly five vertices v_1, v_2, v_3, a, and an additional vertex b. Since $hom(G)$ is 3-connected, b is joined by three disjoint threads T_1, T_2, T_3 to a, v_2, and v_3 respectively. By Claim 6, G_1 has no induced subgraph which is a wishbone. Thus $|T_i| = 1,\ i = 1, 2, 3$ and $ba, bv_2, bv_3 \in E(G)$. Since $d_{G_1}(a) \geqslant 3$, we have that a is joined to at least one of v_2 or v_3 by a thread T. If $|T| = 1$, then G_1 contains a triangle. Consequently, $|T| = 2$. If a is not joined to both v_2 and v_3 by threads, then G_1 would have an induced subgraph containing T which is a wishbone. Thus a is joined to both v_2 and v_3 by threads of length 2. Let $S = V(G_1)\backslash\{v_1, v_2, v_3, b\}$. Then $[S, \overline{S}]$ is seen to be a good bond contained in G_1. This contradicts the fact that $\{v_1, v_2, v_3, \}$ is a good separation. We conclude that v_1 has at least 2 neighbours in $V(G_1'\backslash K)$, and the same applies to v_2 and v_3. $\square$

5. G_1-good bonds and H_1-good cycles

Suppose $\{v_1, v_2, v_3\}$ is a good separation. Then G_1 contains no good bonds of G. This means that G_1' has no good bond $B = [X, V(G_1')\backslash X]$ such that $X \subset V(G_1')\backslash V(K)$. In the dual H_1', this means that H_1' has no good cycle which does not contain u. We say that a good bond $B' = [X, Y]$ in G_1' is $\boldsymbol{G}_1$*-good* if $X\backslash V(K) \neq \emptyset$, and $Y\backslash V(K) \neq \emptyset$. A cycle in H_1' corresponding to a G_1-good bond is called a $\boldsymbol{H}_1$*-good* cycle. That is, a good cycle C' in H_1' is H_1-good if both its interior and exterior contain faces $\Phi(v)$ where $v \in V(G_1')\backslash V(K)$.

According to Lemmas 2.3–2.5, we can find a decomposition of H_1' into two or more good cycles and at most one removable path (between vertices of degree 5). We have exactly four possibilities:

(a) A decomposition into two good cycles ($d_{H_1'}(u) = 4$).
(b) A decomposition into two good cycles and a removable path ($d_{H_1'}(u) = 5$).
(c) A decomposition into three good cycles ($d_{H_1'}(u) = 6$).
(d) A decomposition into three good cycles and a removable path ($d_{H_1'}(u) = 6$).

If all the cycles in the decomposition are H_1-good, then we say that the decomposition is H_1-good.

5.1. *Swapping cycles*

Suppose C_1' and C_2' are two edge-disjoint cycles in H_1' which contain u. Suppose $w, w' \in V(C_1') \cap V(C_2')$ where $w, w' \neq u$. For $i = 1, 2$ we let $C_i'[ww']$ denote the path in $C_i'\backslash\{u\}$ between w and w', and let $C_i'[wuw']$ denote the path in C_i' between w and w' which contains

u. If $C_i'[ww'],\ i = 1, 2$ contain no vertices of $V(C_1') \cap V(C_2')$ other than w and w', we can define two new cycles

$$C_1'' = C_1'[wuw'] \cup C_2'[ww'], \quad C_2'' = C_2'[wuw'] \cup C_1'[ww'].$$

We call $C_i'',\ i = 1, 2$ the cycles obtained by *swapping* C_1' and C_2' between w and w'.

We can also define a swap between a cycle and a path. Let C be a cycle of H_1' containing u and let P be a path in H_1' with terminal vertices w_0 and w_t which is edge-disjoint from C. Suppose $w, w' \in V(C) \cap V(P)$ and $C[ww']$ and $P[ww']$ contain no vertices of P apart from w and w'. We can define a new cycle C' and path P'. Assuming w occurs first while travelling from w_0 to w_t along P, we let

$$C' = C[wuw'] \cup P[ww'], \quad P' = P[w_0w] \cup C[ww'] \cup P[w'w_t].$$

5.1 Lemma. *If $\{v_1, v_2, v_3\}$ is a minimal good separation, then there exists a H_1-good decomposition of H_1'.*

Proof. We suppose that $\{v_1, v_2, v_3\}$ is a minimal good separation. Then there is a decomposition $\mathcal{D}$ of H_1' as specified by one of (a)–(d). We may assume that $\mathcal{D}$ is maximal in the sense that one cannot replace any members of $\mathcal{D}$ so as to obtain a decomposition with a greater number of H_1-good cycles. We suppose that $\mathcal{D}$ is not H_1-good. Let $C_1' \in \mathcal{D}$ be a cycle which is not H_1-good. We can assume that the interior of C_1' contains no faces $\Phi(v)$, where $v \in V(G_1')\backslash V(K)$. We may also assume that the interior also contains exactly one of the faces $\Phi_i,\ i \in \{1, 2, 3\}$ say Φ_1. By Claim 7, the vertex v_1 has at least two neighbours in $V(hom(G|G_1))\backslash\{v_1, v_2, v_3\}$. Thus C_1' contains a vertex $w \neq u, u_1, u_2, u_3$ and two edges $e', e'' \in E(C_1')$ incident with w where $e' \in \Phi(v_1')$ and $e'' \in \Phi(v_1'')$, the vertices v_1', v_1'' being neighbours of v_1 in $V(G_1')\backslash V(K)$. We have that $d_{H_1'}(w) \geqslant 4$, and thus there is a path or cycle of $\mathcal{D}\backslash\{C_1'\}$ which contains w.

We suppose there is a cycle $C_2' \in \mathcal{D}\backslash\{C_1'\}$ which contains w. We observe that faces $\Phi(v_1')$, and $\Phi(v_1'')$ both belong to the interior of C_2' or both belong to the exterior. Since $u \in V(C_1') \cap V(C_2')$, at least one of u's neighbours u_1, u_2, or u_3 belongs to both C_1' and C_2'. This means that we can find a vertex $w' \in V(C_1') \cap V(C_2')\backslash\{w, u\}$ where $C_2'[ww']$ contains no vertices of C_1' other than w and w'. We perform a swap on C_1' and C_2' between w and w' yielding two cycles C_1'' and C_2'' where

$$C_1'' = C_1'[wuw'] \cup C_2'[ww'], \quad C_2'' = C_2'[wuw'] \cup C_1'[ww']$$

(see Fig. 4). The cycle $C_{12}' = C_1'[ww'] \cup C_2'[ww']$ contains exactly one of the faces $\Phi(v_1'),\ \Phi(v_1'')$ in its interior (and hence exactly one in its exterior). Thus C_1'' contains exactly one of these faces in its interior, and one in its exterior. The same also applies to C_2''. We shall show that C_1'' and C_2'' are H_1-good. To show this, it suffices to show that they are removable. Let $H_1'' = H_1'\backslash E(C_1'')$, and let $v \in V(H_1'')$ be an arbitrary vertex where $d_{H_1''}(v) \geqslant 3$. Let $\mathcal{D}' = (\mathcal{D}\backslash\{C_1', C_2'\}) \cup \{C_1'', C_2''\}$. We note that $\mathcal{D}'$ contains at most one path since $\mathcal{D}$ contains at most one path. Thus there is a cycle $C' \in \mathcal{D}'\backslash\{C_1''\}$ containing v, since $d_{H_1''}(v) \geqslant 3$. We have that $u, v \in V(C')$ and consequently u and v belong to the same block of H_1''. If H_1'' has no vertices v where $d_{H_1''}(v) \geqslant 3$, then H_1'' consists of a cycle plus possibly

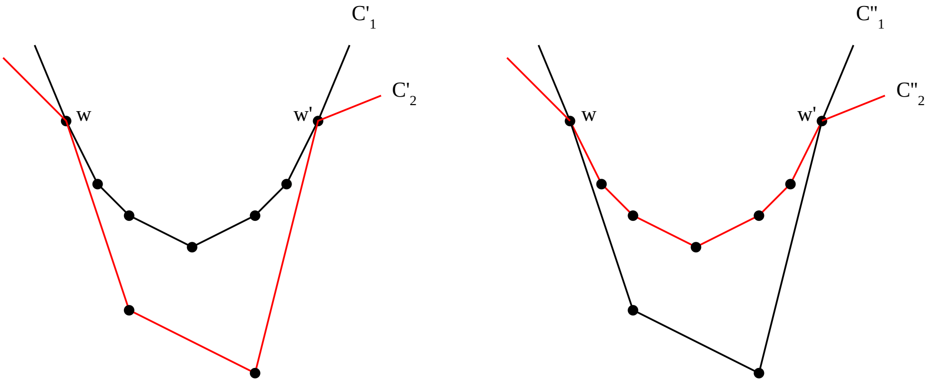

Fig. 4. Swapping C_1' and C_2'.

some isolated vertices. In either case, H_1'' consists of one non-trivial block plus possibly some isolated vertices. This shows that C_1'' is removable in H_1', and the same applies to C_2''. We conclude that both C_1'' and C_2'' are H_1-good. However, this means that $\mathcal{D}'$ has more H_1-good cycles than $\mathcal{D}$, contradicting the maximality of $\mathcal{D}$.

From the above, we deduce that $\mathcal{D}\backslash\{C_1'\}$ contains no cycles which contain w. Thus $\mathcal{D}$ contains a path P' which contains w. If C_1' contains a vertex of P' other than w or u, then we could swap C_1' and P' between two vertices so as to obtain an H_1-good cycle C_1'' and a removable path P''. Then $(\mathcal{D}\backslash\{C_1', P'\}) \cup \{C_1'', P''\}$ would have more H_1-good cycles than $\mathcal{D}$, contradicting the maximality of $\mathcal{D}$. Thus C_1' contains no such vertex, and in particular this means that C_1' cannot contain both of the terminal vertices w_0, w_t of P'. In particular, this means that $w_0, w_t \neq w$. However, since both terminal vertices have degree 5, there is a cycle of $\mathcal{D}\backslash\{P', C_1'\}$, say C_2', containing both of these vertices. Let $P'' = C_2'[w_0 w_t]$. Then $H_1'' = H_1'\backslash E(C_1') \cup E(P'')$ is 2-connected, has no vertices of degree 3, and has no removable cycle which does not contain u. Thus by Lemma 2.3, H_1'' is the union of two good cycles, say C_2'', C_3''. Both C_2'' and C_3'' contain w_0, w_t, and at least one of them, say C_2'', contains w. We can swap C_1' and C_2'' in H_1' to obtain two H_1-good cycles C_1'' and C_2'''. If C_3'' is not H_1-good, then we can swap C_2''' and C_3'' to obtain two H_1-good cycles. In either case, we obtain a H_1—good decomposition. □

For a path in H_1', we call the corresponding subgraph in G_1' a *semi-bond*. A decomposition of G_1' consisting of two or more good bonds and at most one contractible semi-bond is said to be G_1-good if each of the bonds in the decomposition are G_1-good. That is, a decomposition of G_1' is G_1-good if and only if the corresponding decomposition of H_1' is H_1-good. The previous lemma immediately implies that we can find G_1-good decompositions in G_1'.

5.2 Lemma. *If* $\{v_1, v_2, v_3\}$ *is a minimal good separation, then there exists a* G_1*-good decomposition of* G_1'.

We shall need a slight refinement of the previous lemma.

5.3 Lemma. *Suppose* $|K| = 6$ *and* $|K_{23}| = 5$ *where* $K_{23} = v_2 x y v_3 w_{23}^1 v_2$. *Then one can choose a* G_1*-good decomposition consisting of bonds* $B'_{1i},\ i = 1, 2, 3$ *and semi-bond S so that* $yv_3 \notin S$.

Proof. Suppose $|K| = 6$ and $|K_{23}| = 5$. Let $e \in E(H'_1)$ be the edge in H'_1 corresponding to yv_3. We can find a decomposition $\mathcal{D}$ of H'_1 consisting of three good cycles $C'_i,\ i = 1, 2, 3$ and a removable path P' where $e \notin E(P')$. We choose $\mathcal{D}$ to have as many H_1-good cycles as possible subject to $e \notin E(P')$. We can now swap cycles and paths in the same way as was done in the proof of Lemma 5.1 to obtain the desired H_1-good decomposition. □

6. Cross-bonds

For a good separation $\{v_1, v_2, v_3\}$, we call a bond B of G a *cross-bond* if either B is a good bond of G'_i for $i = 1$ or 2, or $B \subseteq B'_1 \cup B'_2$ where B'_i is a good bond of G'_i for $i = 1, 2$. A *block* of a graph is maximal connected subgraph which has no cut-vertex (separating vertex). Every graph has a unique *block decomposition*, where any two blocks share at most one vertex.

Claim 8. *Let* $\{v_1, v_2, v_3\}$ *be a minimal good separation of G and let B be a cross-bond of G.*

(i) *If* $\langle v_1\rangle_B,\ \langle v_2\rangle_B$, *and* $\langle v_3\rangle_B$ *all belong to one block of* G/B, *then* G/B *is itself a block, and B is a good bond of G.*
(ii) *If no block of* G/B *contains all of* $\langle v_1\rangle_B, \langle v_2\rangle_B$, *and* $\langle v_3\rangle_B$, *then* G/B *consists of exactly two blocks which meet at a cut-vertex of* G/B *which is one of* $\langle v_1\rangle_B, \langle v_2\rangle_B$, *or* $\langle v_3\rangle_B$.
(iii) *If* $\langle v_i\rangle_B = \langle v_j\rangle_B$ *for some* $i \neq j$, *then* G/B *is itself a block, and B is a good bond of G.*

Proof. Let B be a cross-bond. If B is a good bond of G'_i for some i, then B is seen to be good in G and (i)–(iii) hold in this case. We suppose therefore that $B \subseteq B'_1 \cup B'_2$ where B'_i is a good bond of G'_i for $i = 1, 2$. We let $B_i = B'_i \cap E(G_i),\ i = 1, 2$.

We showed in Section 4 that $dist_{G_1}(v_i, v_j) = 2$, for some $i \neq j$. We can assume without loss of generality that $dist_{G_1}(v_1, v_3) = 2$ and $w_{13}^i \in V(G'_i),\ i = 1, 2$. Now since B'_i is contractible in G'_i, it holds that $\langle v_3\rangle_{B'_i} \neq \langle v_1\rangle_{B'_i}$ (since $w_{13}^i \in V(G'_i)$). Thus $\langle v_3\rangle_{B_i} \neq \langle v_1\rangle_{B_i}$ and not all the vertices $v_i,\ i = 1, 2, 3$ contract into a single vertex in G/B_i. This also implies that $\langle v_1\rangle_{B\cap B_1} \neq \langle v_3\rangle_{B\cap B_1}$.

We shall first show that G/B contains no loops. Suppose that $e = xy \in E(G_1)\backslash B$ contracts into a loop $\langle e\rangle_B$ in G/B. Then $\langle X\rangle_B = \langle y\rangle_B$ and there is a path $P \subseteq G(B)$ between x and y. If $P \subseteq G_1$, then $\langle X\rangle_{B'_1} = \langle y\rangle_{B'_2}$, and consequently $\langle e\rangle_{B'_1}$ would be a loop of G/B'_1, a contradiction since B'_1 is good. Thus $P \not\subseteq G_1$ and a portion of P, say path Q, is contained in G_2. The path Q has terminal vertices v_i and v_j for some $i \neq j$. P is the union of three paths: $P = P_1 \cup P_2 \cup Q$ where we may assume that P_1 has terminal vertices x and v_i and P_2 has terminal vertices y and v_j. Since $Q \subseteq G_2$, it holds that $\langle v_i\rangle_{B'_2} = \langle v_j\rangle_{B'_2}$

and hence $w_{ij}^2 \notin V(G_2')$. By the construction of G_2', it follows that $v_i v_j \in E(G_2')$, and hence $v_i v_j \in B_2'$ since B_2' is good (otherwise, edge $v_i v_j$ becomes a loop in G_2'/B_2'). Consequently, $v_i v_j \in B_1'$, and $P_1 \cup P_2 \cup \{v_i v_j\}$ is a path in $G_1'(B_1')$ between x and y. This would mean that $\langle e \rangle_{B_1'}$ is a loop in G_1'/B_1' yielding a contradiction (since B_1' is good). If instead $e \in E(G_2)\backslash B$, then we obtain a contradiction with similar arguments. This shows that G/B contains no loops.

To show (i), suppose that $\langle v_i \rangle_B,\ i = 1, 2, 3$ belong to the same block of G/B say X, and suppose that G/B has at least two blocks. Then G/B has another block Y which is not a loop and contains at most one of the vertices $\langle v_i \rangle_B,\ i = 1, 2, 3$. Using the above, one can show that K is not a loop. Then Y contains a vertex $\langle a \rangle_B$ where $\langle a \rangle_B \notin V(X)$. Suppose that $a \in V(G_1)$. Since $G_1'/(B_1 \cap B)$ is 2-connected, $\langle a \rangle_{B_1 \cap B},\ \langle v_1 \rangle_{B_1 \cap B}$, and $\langle v_3 \rangle_{B_1 \cap B}$ belong to the same block of $G_1/(B_1 \cap B)$. However, since Y contains only at most one of the vertices $\langle v_i \rangle_B,\ i = 1, 2, 3$, it must hold that $\langle v_1 \rangle_B = \langle v_3 \rangle_B$, yielding a contradiction. We conclude that $a \notin V(G_1)\backslash\{v_1, v_2, v_3\}$, and in a similar fashion, one can show that $a \notin V(G_2)\backslash\{v_1, v_2, v_3\}$. Thus no such vertex a exists, and hence no such block Y exists. We conclude that G/B is itself a block (hence 2-connected), and thus B is good.

The above argument also shows that each block of G/B must contain at least two of the vertices $\langle v_i \rangle_B,\ i = 1, 2, 3$. Thus if $\langle v_i \rangle_B = \langle v_j \rangle_B$ for some $i \neq j$, then G/B has only one block, itself, and hence B is good. This proves (iii).

If G/B has more than one block, then by the above argument it has exactly two blocks, separated by a vertex which is one of the vertices $\langle v_i \rangle_B,\ i = 1, 2, 3$. This proves (ii). □

Claim 9. *Let* $\{v_1, v_2, v_3\}$ *be a good separation and let B be a cross-bond of* G. *If for all* $i \neq j,\ \langle v_i \rangle_B \neq \langle v_j \rangle_B$ *and there exists a path from* $\langle v_i \rangle_B$ *to* $\langle v_j \rangle_B$ *in* $(G/B)\backslash\langle v_k \rangle_B$ *where* $k \neq i, j$, *then B is good.*

Proof. Let B be a cross-bond, and suppose that $\forall i \neq j, \langle v_i \rangle_B \neq \langle v_j \rangle_B$ and there exists a path from $\langle v_i \rangle_B$ to $\langle v_j \rangle_B$ in $(G/B)\backslash\langle v_k \rangle_B$ where $k \neq i, j$. This implies that none of the vertices $\langle v_i \rangle_B,\ i = 1, 2, 3$ are cut-vertices of G/B. According to Claim 8, B must be good. □

7. Good separations of type 1

We suppose that $\{v_1, v_2, v_3\}$ is a minimal good separation which has type 1. We have that $dist_{G_1}(v_i, v_j) = 2$ for some $i \neq j$. We can assume without loss of generality that $dist_{G_1}(v_1, v_3) = 2$, $w_{13}^i \in V(G_i')$, and $v_1 v_2, v_2 v_3 \in E(G_i')$ for $i = 1, 2$. This we assume for the remainder of this section.

Claim 10. *Given* $\{v_1, v_2, v_3\}$ *is a good separation of type* 1 *and B is a cross-bond, we have that* $\langle v_1 \rangle_B \neq \langle v_3 \rangle_B$, *and* $\langle v_1 \rangle_B$ *and* $\langle v_3 \rangle_B$ *belong to the same block of* G/B.

Proof. Let B be a cross-bond. We may assume that $B \subseteq B_1' \cup B_2'$ where B_i' is contractible in G_i' for $i = 1, 2$. We have that $\langle v_1 \rangle_{B_i'} \neq \langle v_3 \rangle_{B_i'},\ i = 1, 2$, since B_i' is contractible in

G'_i. Thus $\langle v_1\rangle_{B_i} \neq \langle v_3\rangle_{B_i}$, $i = 1, 2$, and consequently, $\langle v_1\rangle_B \neq \langle v_3\rangle_B$. The bond B'_1 contains exactly 2 edges of the cycle $v_1v_2v_3w'_{13}v_1$ and exactly one of the edges $v_1w'_{13}$ or $v_3w'_{13}$. As such, there is an edge in $G_1/(B \cap B_1)$ between $\langle v_1\rangle_{B\cap B_1}$ and $\langle v_3\rangle_{B\cap B_1}$. Since G_2 is connected there is a path in $G_2/(B \cap B_2)$ from $\langle v_1\rangle_{B\cap B_2}$ to $\langle v_3\rangle_{B\cap B_2}$. Thus there is a cycle in G/B containing $\langle v_1\rangle_B$ and $\langle v_3\rangle_B$. This implies that $\langle v_1\rangle_B$ and $\langle v_3\rangle_B$ belong to the same block of G/B. □

Claim 11. *Given $\{v_1, v_2, v_3\}$ is a good separation of type 1 and B is a cross-bond, if $v_1v_2 \in B$ or $v_2v_3 \in B$, then B is contractible.*

Proof. If $v_1v_2 \in B$, then $\langle v_1\rangle_B = \langle v_2\rangle_B$. By Claim 8, B is contractible. A similar conclusion holds if $v_2v_3 \in B$. □

Claim 12. *Given $\{v_1, v_2, v_3\}$ is a good separation of type 1 and B is a cross-bond, if there is a path from $\langle v_1\rangle_B$ to $\langle v_2\rangle_B$ in $(G/B)\backslash\langle v_3\rangle_B$ and a path from $\langle v_2\rangle_B$ to $\langle v_3\rangle_B$ in $(G/B)\backslash\langle v_1\rangle_B$, then B is good.*

Proof. Let B be a cross-bond. Suppose that there is a path $\langle v_1\rangle_B$ to $\langle v_2\rangle_B$ in $(G/B)\backslash\langle v_3\rangle_B$ and a path from $\langle v_2\rangle_B$ to $\langle v_3\rangle_B$ in $(G/B)\backslash\langle v_1\rangle_B$. By Claim 10, $\langle v_1\rangle_B$ and $\langle v_3\rangle_B$ belong to the same block of G/B. Thus there is a path from $\langle v_1\rangle_B$ to $\langle v_3\rangle_B$ in $(G/B)\backslash\langle v_2\rangle_B$. It now follows by Claim 9 that B is good. □

7.1 Lemma. *Let H be a 2-connected planar graph with girth at least 4. If $E(H)$ is the edge-disjoint of two bonds $A_i = [X_i, Y_i]$, $i = 1, 2$ then for $i = 1, 2$ the induced subgraph $G(A_i)$ is a forest with two components $G(X_{3-i})$ and $G(Y_{3-i})$.*

Proof. We assume H has a plane embedding with f faces. Let $\varepsilon = |E(H)|$ and $v = |V(H)|$. Given that $E(H)$ is the disjoint union of two bonds $A_i = [X_i, Y_i]$ $i = 1, 2$ we see that $A_i = E(G(X_{3-i}) \cup G(Y_{3-i}))$ $i = 1, 2$. For $i = 1, 2$ we have that $G(X_i)$ and $G(Y_i)$ are connected and thus $|E(G(X_i) \cup G(Y_i))| \geqslant v - 2$, $i = 1, 2$. Thus $\varepsilon = |A_1| + |A_2| \geqslant 2v - 4$. Let H^* be the geometric dual of H. The bonds A_1 and A_2 correspond to two cycles C_1 and C_2 in H^* which partition $E(H^*)$. Thus the maximum degree in H^* is at most 4. However, since the girth of H is at least 4, each face of H is bounded by a cycle of length at least 4. Thus the minimum degree in H^* is at least 4. It follows that H^* must be 4-regular. Thus $\varepsilon = |E(H^*)| = 2|V(H^*)| = 2f$. Using Eulers formula , we have $v - \varepsilon + f = 2$. Substituting $f = \frac{\varepsilon}{2}$ we obtain $\varepsilon = 2v - 4$. Thus equality holds in the previous inequality, and this occurs only if for $i = 1, 2$, $G(A_i)$ is a forest with two components $G(X_{3-i})$ and $G(Y_{3-i})$. □

7.1. The bonds B'_{ij}

Lemma 2.3 implies that the dual H'_1 of G'_1 only has vertices of degree 2 or 4. This means that G'_1 only has faces of size 2 or 4. Since no multiple edges occur in G (by Claim 2), all faces of G_1' have size 4. By Lemma 5.2, G_1' has a G_1-good decomposition $\{\mathbf{B}_{11}', \mathbf{B}_{12}'\}$ where we may assume that $v_1v_2 \in B_{11}'$ and $v_2v_3 \in B_{12}'$. Let $\mathbf{B}_{1j}' = [\mathbf{P}_{1j}', \mathbf{Q}_{1j}'],\ \ j = 1, 2$ where $v_1 \in P_{11}'$ (and $v_2, v_3 \in Q_{11}'$) and $v_3 \in P_{12}'$ (and $v_1, v_2 \in Q_{12}'$). Since the edges of G_1' are partitioned by B_{11}' and B_{12}' we have that for $j = 1, 2$ G_1'/B_{1j}' is a multiple edge with endvertices $\langle v_1\rangle_{B_{1j}'}$ and $\langle v_3\rangle_{B_{1j}'}$. We note also that since G_1' is planar, Lemma 7.1 implies that each of the components $G(P_{1j}')$ and $G(Q_{1j}'),\ \ j = 1, 2$ are trees.

The graph G_2' has a good pair of bonds $\mathbf{B}_{21}' = [\mathbf{P}_{21}', \mathbf{Q}_{21}']$ and $\mathbf{B}_{22}' = [\mathbf{P}_{22}', \mathbf{Q}_{22}']$. For $i, j = 1, 2$ let

$$\mathbf{P}_{ij} = \mathbf{P}_{ij}' \cap \mathbf{V}(\mathbf{G}_i), \quad \mathbf{Q}_{ij} = \mathbf{Q}_{ij}' \cap \mathbf{V}(\mathbf{G}_i), \quad \mathbf{B}_{ij} = \mathbf{B}_{ij}' \cap \mathbf{E}(\mathbf{G}_i).$$

7.2. *Finding two good bonds*

We shall show that G contains a good pair of bonds. If $P_{2j}' \subseteq V(G_2)\backslash\{v_1, v_2, v_3\},\ \ j = 1, 2$, then B_{21} and B_{22} are seen to be a good pair of bonds in G. So we may assume without loss of generality that $P_{21}' \cap \{v_1, v_2, v_3\} \neq \emptyset$. We shall also assume that $P_{22}' \cap \{v_1, v_2, v_3\} \neq \emptyset$. The case where the intersection is empty, B_{22}' is a good bond of G, and this case is easier. We may assume that $v_1 \in P_{21}'$ (and $v_2, v_3 \in Q_{21}'$) and $v_3 \in P_{22}'$ (and $v_1, v_2 \in Q_{22}'$). We note that since $\{B_{11}', B_{12}'\}$ is a G_1-good decomposition, it holds that $P_{1j}\backslash V(K) \neq \emptyset,\ \ j = 1, 2$.

By Lemma 7.1 we have that $G_1'(Q_{1j}')$ is a tree for $j = 1, 2$ (since G_1' is planar). So for $j = 1, 2;\ G(Q_{1j})\backslash\{v_2v_{5-2j}\}$ is a forest with 2 components. Let Q_{1j}^2 and Q_{1j}^{5-2j} be sets of vertices of these components where $v_2 \in Q_{1j}^2$ and $v_{5-2j} \in Q_{1j}^{5-2j},\ \ j = 1, 2$. We define two cutsets

$$\mathbf{C}_{21} = [\mathbf{P}_{21} \cup \mathbf{Q}_{12}^1, \overline{\mathbf{P}_{21} \cup \mathbf{Q}_{12}^1}]$$

and

$$\mathbf{C}_{22} = [\mathbf{P}_{22} \cup \mathbf{Q}_{11}^3, \overline{\mathbf{P}_{22} \cup \mathbf{Q}_{11}^3}].$$

Claim 13. *If $P_{21} \neq \{v_1\}$, then the cutset C_{21} is a good bond in G.*

Proof. Suppose $P_{21} \neq \{v_1\}$. We will first show that C_{21} is non-trivial. Clearly $P_{21} \cup Q_{12}' \neq \{v_1\}$, and $G(P_{21} \cup Q_{12}')$ is connected. To show that $G(\overline{P_{21} \cup Q_{12}'})$ is connected, we note that $Q_{12}^2 \cup P_{12} \subseteq \overline{P_{21} \cup Q_{12}'}$, and hence it suffices to show that $G(Q_{12}^2 \cup P_{12})$ is connected. Let $v_2' \in N_{G_1}(v_2)\backslash\{v_1, v_3\}$. Then $v_2' \in Q_{12}^2 \cup P_{12}$. If $v_{12}' \in Q_{12}^2$, then $\langle v_2'\rangle_{B_{12}'} = \langle v_1\rangle_{B_{12}'}$, and consequently v_2' is adjacent to at least one vertex of P_{12}, implying that $G(Q_{12}^2 \cup P_{12})$ is connected. If $v_2' \in P_{12}$, then it is clear that $G(Q_{12}^2 \cup P_{12})$ is connected. This shows that $G(\overline{P_{21} \cup Q_{12}'})$ is connected, and C_{21} is a non-trivial bond. It is also a cross-bond since $C_{21} \subseteq B_{12}' \cup B_{21}'$. We will now show that C_{21} is good in G.

If $v_1v_2 \in E(G)$, then $v_1v_2 \in C_{21}$ and hence by Claim 11 C_{21} would be good. We may therefore assume that $v_1v_2 \notin E(G)$. To show that C_{21} is good, Claim 12 implies that it suffices to show that there is a path from $\langle v_1\rangle_{C_{21}}$ to $\langle v_2\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_3\rangle_{C_{21}}$ and a path from $\langle v_2\rangle_{C_{21}}$ to $\langle v_3\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_1\rangle_{C_{21}}$.

We shall first show that there is a path from $\langle v_1\rangle_{C_{21}}$ to $\langle v_2\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_3\rangle_{C_{21}}$. Let $v_2'\in N_{G_1}(v_2)\backslash\{v_1,v_2\}$. It holds that $v_2'\in Q_{12}^2\cup P_{12}$. Suppose first that $v_2'\in Q_{12}^2$. Then $\langle v_2'\rangle_{B_{12}'}=\langle v_1\rangle_{B_{12}'}$, and hence there is a path from $\langle v_2'\rangle_{C_{21}}$ to $\langle v_1\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_3\rangle_{C_{21}}$. Suppose now that $v_2'\in P_{21}$. Then $v_2v_2'\in B_{12}$, and hence $v_2'\in Q_{11}$. We have that $\langle v_2'\rangle_{B_{11}'}=\langle v_3\rangle_{B_{11}'}$, and consequently v_2' is adjacent to at least one vertex of P_{11}, say v_2''. Then $\langle v_2''\rangle_{B_{12}'}=\langle v_1\rangle_{B_{12}'}$, and thus $\langle v_2''\rangle_{B_{12}}=\langle v_1\rangle_{B_{12}}$. Consequently, there is a path from $\langle v_2''\rangle_{C_{21}}$ to $\langle v_1\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_3\rangle_{C_{21}}$. Since no edges of C_21 are incident with v_2', it follows that $\langle v_2'\rangle_{C_{21}}\neq\langle v_3\rangle_{C_{21}}$. Thus we can find a path from $\langle v_2\rangle_{C_{21}}$ to $\langle v_1\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_3\rangle_{C_{21}}$ via $\langle v_2'\rangle_{C_{21}}$ and $\langle v_2''\rangle_{C_{21}}$. In both cases there is a path from $\langle v_1\rangle_{C_{21}}$ to $\langle v_2\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_3\rangle_{C_{21}}$.

We shall now show that there is a path from $\langle v_2\rangle_{C_{21}}$ to $\langle v_3\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_1\rangle_{C_{21}}$. Let $v_2'\in N_{G_2}(v_2)\backslash\{v_1,v_3\}$. Then $v_2'\in P_{21}\cup Q_{21}$. Suppose first that $v_2'\in Q_{21}$. Then $\langle v_2'\rangle_{B_{21}'}\neq\langle v_1\rangle_{B_{21}'}$; for otherwise, the edge v_2v_2' would become a loop in G_2'/B_{21}'. If $\langle v_2'\rangle_{B_{21}'}=\langle v_3\rangle_{B_{21}'}$, then there is a path from $\langle v_2\rangle_{C_{21}}$ to $\langle v_3\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_1\rangle_{C_{21}}$. Otherwise, if $\langle v_2'\rangle_{B_{21}'}\neq\langle v_3\rangle_{B_{21}'}$, then since G_2'/B_{21}' is 2-connected, there is a path from $\langle v_2'\rangle_{B_{21}'}$ to $\langle v_3\rangle_{B_{21}'}$ in $(G_2'/B_{21}')\backslash\langle v_1\rangle_{B_{21}'}$. In this case there is a path from $\langle v_2\rangle_{C_{21}}$ to $\langle v_3\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_1\rangle_{C_{21}}$. Suppose now that $v_2'\in P_{21}$. If $\langle v_2'\rangle_{B_{21}}=\langle v_1\rangle_{B_{21}}$, then $\langle v_2\rangle_{C_{21}}=\langle v_2'\rangle_{C_{21}}=\langle v_1\rangle_{C_{21}}$. In this case, Claim 8 implies that C_{21} is good. We may therefore assume that $\langle v_2'\rangle_{B_{21}}\neq\langle v_1\rangle_{B_{21}}$. Since $G_2(P_{21})$ is connected, there is a vertex $v_2''\in N_{G_2}(v_2')\cap P_{21}$. Since G_2' contains no triangles, it holds that $v_2''\neq v_1$. We also have that $\langle v_2''\rangle_{B_{21}'}\neq\langle v_2'\rangle_{B_{21}'}$. Since $\langle v_1\rangle_{B_{21}'}=\langle v_2\rangle_{B_{21}'}=\langle v_2'\rangle_{B_{21}'}$, we have that $\langle v_2''\rangle_{B_{21}'}\neq\langle v_1\rangle_{B_{21}'}$. If $\langle v_2''\rangle_{B_{21}'}=\langle v_3\rangle_{B_{21}'}$, then $\langle v_2''\rangle_{C_{21}}=\langle v_3\rangle_{C_{21}}$, and hence there is a path from $\langle v_2\rangle_{C_{21}}$ to $\langle v_3\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_1\rangle_{C_{21}}$. If $\langle v_2''\rangle_{B_{21}'}\neq\langle v_3\rangle_{B_{21}'}$, then since G_2'/B_{21}' is 2-connected, there is a path in $(G_2'/B_{21}')\backslash\langle v_1\rangle_{B_{21}'}$ from $\langle v_2''\rangle_{B_{21}'}$ to $\langle v_3\rangle_{B_{21}'}$. Thus there would be a path from $\langle v_2\rangle_{C_{21}}$ to $\langle v_3\rangle_{C_{21}}$ in $(G/C_{21})\backslash\langle v_1\rangle_{C_{21}}$ (given that $\langle v_2\rangle_{C_{21}}\neq\langle v_1\rangle_{C_{21}}$). The proof of the claim now follows by Claim 12. $\square$

In the same way, one can show the following:

Claim 14. *If $P_{22}\neq\{v_3\}$, then C_{22} is a good bond in G.*

Let $\mathbf{B}_1=[\mathbf{P}_{11}\cup\mathbf{P}_{21},\overline{\mathbf{P}_{11}\cup\mathbf{P}_{21}}]$, and $\mathbf{B}_2=[\mathbf{P}_{12}\cup\mathbf{P}_{22},\overline{\mathbf{P}_{12}\cup\mathbf{P}_{22}}]$.

Claim 15. *If B_1 is a bond which is not good in G, then C_{21} and C_{22} are a good pair of bonds in G.*

Proof. We suppose that B_1 is a bond which is not good in G. The bond B_1 is non-trivial since $P_{11}\backslash\{v_1\}\neq\emptyset$, and it is also a cross-bond. According to Claims 8 and 10, G/B_1 consists of two blocks where one block contains $\langle v_1\rangle_{B_1}$ and $\langle v_3\rangle_{B_1}$. If $v_1v_2\in E(G)$, then $v_1v_2\in B_1$ and B_1 would be contractible by Claim 11 . So $v_1v_2\notin E(G)$. Since B_1 is a bond, $G(Q_{11}\cup Q_{21})$ is connected and consequently there is vertex $v_2'\in N_G(v_2)\cap(Q_{11}\cup Q_{21})$. Since $\langle v_2\rangle_{B_{i1}'}=\langle v_1\rangle_{B_{i1}'}$, $i=1,2$ we have that $\langle v_2'\rangle_{B_{i1}'}=\langle v_3\rangle_{B_{i1}'}$, $i=1,2$ and consequently $\langle v_2'\rangle_{B_1}=\langle v_3\rangle_{B_1}$. We deduce that there would be a path in $(G/B_1)\backslash\langle v_1\rangle_{B_1}$ from $\langle v_2'\rangle_{B_1}$ to $\langle v_3\rangle_{B_1}$. Now Claim 8 implies that $\langle v_2\rangle_{B_1}$ and $\langle v_3\rangle_{B_1}$ belong to the same block of G/B_1.

Arguing in a similar way with v_1 in place of v_2, we also deduce that $\langle v_1\rangle_{B_1}$ and $\langle v_3\rangle_{B_1}$ belong to the same block. Thus $\langle v_3\rangle_{B_1}$ is a cut-vertex of G/B_1 which separates $\langle v_1\rangle_{B_1}$ and $\langle v_2\rangle_{B_1}$.

We wish to show that $P_{21} \neq \{v_1\}$. Since $hom(G)$ is 3-connected, $hom(G_2')$ is 3-connected, and there is a path P from v_2 to a vertex of $N_{G_2}(v_1)$ which avoids v_1 and v_3. We have that $\langle v_3\rangle_{B_1} \in V(\langle P\rangle_{B_1})$ as $\langle v_3\rangle_{B_1}$ is a cut-vertex in G/B_1. So for some vertex $z \in V(P)$ we have $\langle z\rangle_{B_1} = \langle v_3\rangle_{B_1}$. If $z \in P_{21}$, then $z \neq v_1$ and hence $P_{21} \neq \{v_1\}$. So we can assume that $z \notin P_{21}$. If $z \in N_{G_2}(v_1)$, then $zv_1 \in B_1$ and hence $\langle v_1\rangle_{B_1} = \langle z\rangle_{B_1} = \langle v_3\rangle_{B_1}$. This gives a contradiction since $\langle v_1\rangle_{B_1} \neq \langle v_3\rangle_{B_1}$. On the other hand, if $z \notin N_{G_2}(v_1)$, then z is adjacent to some vertex in P_{21} since $\langle z\rangle_{B_1} = \langle v_3\rangle_{B_1}$. This means that $P_{21} \neq \{v_1\}$.

Since $P_{21} \neq \{v_1\}$, Claim 13 implies that C_{21} is a good bond. We now wish to show that $C_{22} = [P_{22} \cup Q^3_{11}, \overline{P_{22} \cup Q^3_{11}}]$ is a good bond. By Claim 14, it suffices to show that $P_{22} \neq \{v_3\}$. Since $hom(G_2')$ is 3-connected, there is a path in $G_2\backslash\{v_3\}$ from v_2 to v_1. Since $\langle v_3\rangle_{B_1}$ is a cut-vertex of G/B_1 separating $\langle v_1\rangle_{B_1}$ and $\langle v_2\rangle_{B_1}$, it follows that $\langle v_3\rangle_{B_1} \in V(\langle P\rangle_{B_1})$. Thus there must be edges of B_{21} incident with v_3, and such edges belong to $G_2(P_{22})$. We conclude that $P_{22} \neq \{v_3\}$ and thus C_{22} is good. This completes the proof of the claim. □

We have a similar result for B_2, namely:

Claim 16. *If B_2 is a bond which is not good in G, then C_{21} and C_{22} are a good pair of bonds.*

Claim 17. *If B_1 is not a bond, then C_{21} is good.*

Proof. Suppose B_1 is not a bond. Then $G(Q_{11} \cup Q_{21})$ consists of two components; one containing v_2 and the other v_3. Since $hom(G_2')$ is 3-connected, there is a path in $G_2\backslash\{v_1\}$ from v_2 to v_3. Such a path must contain vertices of $P_{21}\backslash\{v_1\}$ since $G_2(Q_{21})$ is disconnected. This means that $P_{21} \neq \{v_1\}$, and consequently, C_{21} is a good bond by Claim 13. □

In a similar fashion, one can show:

Claim 18. *If B_2 is not a bond, then C_{22} is good.*

Claim 19. *Given $\{v_1, v_2, v_3\}$ is a minimal good separation which is of type 1, G has a pair of good bonds.*

Proof. By Claims 15–18, if both B_1 and B_2 are bonds, then either B_1 and B_2 are a good pair of bonds, or C_{21} and C_{22} are a good pair of bonds. We can thus assume without loss of generality that B_1 is not a bond and thus by Claim 17, C_{21} is a good bond. If B_2 is not a bond, then Claim 18 implies that C_{22} is a good bond, in which case C_{21} and C_{22} are a good pair of bonds. We may thus assume that B_2 is a bond, and B_2 is good (otherwise, C_{12} and C_{22} are a good pair by Claims 16 and 17). Moreover, we may assume that $P_{22} = \{v_3\}$ for otherwise, C_{22} is good by Claim 14.

Since $B_1 = [P_{11} \cup P_{21}, Q_{11} \cup Q_{21}]$ is not a bond, $G(Q_{11} \cup Q_{21})$ consists of two components. We let Q^2 and Q^3 be the sets of vertices in the components containing v_2

and v_3, respectively. Since $P_{22} = \{v_3\}$ all edges incident with v_3 in G_2 belong to B_{22} and hence also to Q^3. It follows that $N_{G_2}(v_3) \subseteq Q^3$ and consequently $Q^3\backslash\{v_3\} \neq \emptyset$. Now $C = [Q^3, \overline{Q^3}]$ is clearly a non-trivial bond which is also a subset of B_1 (and hence is also a cross-bond). To show that C is contractible, it suffices to show that there are paths from $\langle v_2\rangle_C$ to $\langle v_1\rangle_C$ in $(G/C)\backslash\langle v_3\rangle_C$ and from $\langle v_2\rangle_C$ to $\langle v_3\rangle_C$ in $(G/C)\backslash\langle v_1\rangle_C$. Let $v_2' \in N_G(v_2)\backslash\{v_1, v_3\}$. If $v_2' \in Q_{11}$, then $\langle v_2'\rangle_{B_{11}} = \langle v_3\rangle_{B_{11}}$. In this case, we can find a path from $\langle v_2\rangle_C$ to $\langle v_3\rangle_C$ in $(G/C)\backslash\langle v_1\rangle_C$. If $v_2' \in P_{11}$, then v_2' is adjacent to a vertex $v_2'' \in P_{11}$, where $v_2'' \neq v_1$ (since $G_1(P_{11})$ is connected and G_1' contains no triangles). We have that $\langle v_2'\rangle_{B_{11}} = \langle v_2\rangle_{B_{11}}$ and hence $\langle v_2''\rangle_{B_{11}} = \langle v_3\rangle_{B_{11}}$. In this case, we can also find a path from $\langle v_2''\rangle_C$ to $\langle v_3\rangle_C$ in $(G/C)\backslash\langle v_1\rangle_C$ and hence there is a path from $\langle v_2\rangle_C$ to $\langle v_3\rangle_C$ in $(G/C)\backslash\langle v_1\rangle_C$. To prove that there is a path from $\langle v_2\rangle_C$ to $\langle v_1\rangle_C$ in $(G/C)\backslash\langle v_3\rangle_C$, we first observe that $hom(G_2')$ is 3-connected, and thus there is a path P from v_2 to v_1 in $G_2\backslash\{v_3, \}$. It follows that $\langle P\rangle_C$ does not contain $\langle v_3\rangle_C$, since no edges of B_{21} are incident with v_3 (as $P_{22} = \{v_3\}$). Consequently, $\langle P\rangle_C$ contains a path from $\langle v_2\rangle_C$ to $\langle v_1\rangle_C$ in $(G/C)\backslash\langle v_3\rangle_C$. This shows that C is good, and we conclude that C and B_2 are a good pair of bonds. □

8. Good separations of type 3: part I

In this section, we shall assume that $\{v_1, v_2, v_3\}$ is a minimal good separation which has type 3. G_1' has a plane representation where the cycle $K = v_1w_{12}^1v_2w_{23}^1v_3w_{13}^1v_1$ bounds the face F. By Lemma 5.2, the graph G_1' has a G_1-good decomposition. There are two possibilities: either the decomposition consists of three G_1-good bonds, or it consists of three G_1-good bonds and a contractible semi-bond. We shall assume in this section that the former holds; that is, G_1' has an G_1-good decomposition consisting of three G_1-good bonds $\mathbf{B}_{1j}' = [P_{1j}', Q_{1j}']$, $j = 1, 2, 3$ where for $i = 1, 2, 3$ we have $v_i \in P_{1j}'$ if and only if $i = j$. For $j = 1, 2, 3$ we let $\mathbf{P}_{1j} = P_{1j}' \cap V(G_1)$ and $\mathbf{Q}_{1j} = \mathbf{Q}_{1j}' \cap \mathbf{V}(\mathbf{G}_1)$. According to Lemma 2.5, we may assume that every face of G_1' is a 4-face apart from the 6-face bounded by K and possibly one other 6-face. The graph G_2' has a good pair of bonds which we denote by $\mathbf{B}_{2j}' = [\mathbf{P}_{2j}', \mathbf{Q}_{2j}']$, $j = 1, 2$. We let $\mathbf{P}_{2j} = \mathbf{P}_{2j}' \cap \mathbf{V}(\mathbf{G}_2)$ and $\mathbf{Q}_{2j} = \mathbf{Q}_{2j}' \cap \mathbf{V}(\mathbf{G}_2)$ for $j = 1, 2$. We can assume that $|P_{2j}' \cap \{v_1, v_2, v_3\}| \leqslant 1$, $j = 1, 2$. Since $\{B_{1j} : j = 1, 2, 3\}$ is a G_1-good decomposition, we have $P_{1i}\backslash V(K) \neq \emptyset$, $i = 1, 2, 3$. We may assume that for at least one of the bonds $B_{2j}' = [P_{2j}', Q_{2j}']$, $j = 1, 2$ that $P_{2j}' \cap \{v_1, v_2, v_3\} \neq \emptyset$. For otherwise, $B_{2j} = B_{2j}'$, $j = 1, 2$, would be a good pair of bonds of G. We may assume without loss of generality that $v_1 \in P_{21}'$ and $v_2, v_3 \in Q_{21}'$. Let $\mathbf{B}_1 = [\mathbf{P}_{11} \cup \mathbf{P}_{21}, \mathbf{Q}_{11} \cup \mathbf{Q}_{21}]$.

The cutset B_1 is a non-trivial bond; to see this, we have that $dist_G(v_2, v_3) = 2$, and as such there is a 2-path v_2zv_3 from v_2 to v_3. If $z \in P_{11} \cup P_{21}$, then either $\langle v_2\rangle_{B_{11}'} = \langle v_3\rangle_{B_{11}'}$ or $\langle v_2\rangle_{B_{21}'} = \langle v_3\rangle_{B_{21}'}$, depending on whether $z \in P_{11}$ or $z \in P_{21}$. However, neither the former nor the latter can occur since B_{11}' and B_{21}' are good bonds in G_1' and G_2', respectively. Thus $z \in Q_{11} \cup Q_{21}$, and this means that $G(Q_{11} \cup Q_{21})$ is connected and B_1 is a bond. The bond B_1 is non-trivial since $P_{11}\backslash V(K) \neq \emptyset$. Let $\mathbf{G}_2'' = \mathbf{G}_2'\backslash\{\mathbf{w}_{23}^2\}$. We have that G_2'' is 2-connected and therefore has a good pair of bonds $\mathbf{B}_{21}'' = [\mathbf{P}_{21}'', \mathbf{Q}_{21}'']$ and $\mathbf{B}_{22}'' = [\mathbf{P}_{22}'', \mathbf{Q}_{22}'']$.

Let

$$\mathbf{V}_i = \{\mathbf{v} \in \mathbf{V}(\mathbf{G}_1) : \langle \mathbf{v} \rangle_{\mathbf{B}_{11}} = \langle \mathbf{v}_i \rangle_{\mathbf{B}_{11}}\}, \quad i = 1, 2, 3.$$

Claim 20. *If B_1 is not a good bond, then there is a good pair of bonds in G.*

Proof. We suppose that B_1 is not good. B_1 is a cross-bond since $B_1 \subseteq B'_{11} \cup B'_{21}$. Clearly $\langle v_i \rangle_{B_1} \neq \langle v_j \rangle_{B_1}$, $i \neq j$ since B'_{11} is good in G'_1 and B'_{21} is good in G'_2. By Claim 8, B_1 would be good. Therefore, we can assume that $\langle v_1 \rangle_{B_1} \neq \langle v_2 \rangle_{B_1}, \langle v_3 \rangle_{B_1}$. We have that $dist_G(v_1, v_j) = 2$, $j = 1, 2$ and in fact $d_{G_1}(v_1, v_j) = 2$, $j = 1, 2$ since v_1 and v_j belong to a 4-face in G'_1. Let $v_1 x v_2$ be a path of length 2 from v_1 to v_2 in G_1. Then B_{11} and B_{12} each contain one of the edges $v_1 x$ and $x v_2$, and consequently $\langle v_1 \rangle_{B_1}$ and $\langle v_2 \rangle_{B_1}$ are adjacent vertices in G/B_1. Similarly, $\langle v_1 \rangle_{B_1}$ and $\langle v_3 \rangle_{B_1}$ are adjacent vertices in G/B_1. Since B_1 is not good, Claim 8 implies that G/B_1 consists of two blocks; a block K'_1 containing $\langle v_1 \rangle_{B_1}$ and $\langle v_2 \rangle_{B_1}$ and a block K'_2 containing $\langle v_1 \rangle_{B_1}$ and $\langle v_3 \rangle_{B_1}$. The set of edges $\langle B'_{12} \rangle_{B'_{11}}$ is a bond in G'_1/B'_{11}. Thus $\langle B_{12} \rangle_{B_1} \subseteq E(K'_1)$ or $\langle B_{12} \rangle_{B_1} \subseteq E(K'_2)$. Since $\langle B_{12} \rangle_{B_1}$ contains an edge between $\langle v_1 \rangle_{B_1}$ and $\langle v_2 \rangle_{B_1}$, it must hold that $\langle B_{12} \rangle_{B_1} \subseteq E(K'_1)$. Similarly, $\langle B_{13} \rangle_{B_1} \subseteq E(K'_2)$. Since $E(G'_1) = B'_{11} \cup B'_{12} \cup B'_{13}$, it holds that $G'_1/(B'_{11} \cup B'_{12})$ and $G'_1/(B'_{13} \cup B'_{12})$ are multiple edges. Consequently, G/B_{11} consists of two multiple, one between $\langle v_1 \rangle_{B_{11}}$ and $\langle v_2 \rangle_{B_{11}}$, and the other between $\langle v_1 \rangle_{B_{11}}$ and $\langle v_3 \rangle_{B_{11}}$, each representing the portions of K'_1 and K'_2 in G_1/B_{11}, respectively. In particular, this means that there is no vertex $w_{23} \in V(G)$; that is, a vertex in G having exactly v_2 and v_3 as its neighbors. Consider G''_2. If $P''_{2i} \cap \{v_1, v_2, v_3\} = \emptyset$, $i = 1, 2$, then B''_{2i}, $i = 1, 2$ is seen to be a good pair of bonds in G (since $w_{23} \notin V(G)$). We may therefore assume that $|P''_{21} \cap \{v_1, v_2, v_3\}| = 1$. We shall also assume that $|P''_{22} \cap \{v_1, v_2, v_3\}| = 1$, as the easier case when $P''_{22} \cap \{v_1, v_2, v_3\} = \emptyset$ can be dealt with by similar arguments.

Since G_1/B_{11} consists of two multiple edges, it only has vertices $\langle v_i \rangle_{B_{11}}$, $i = 1, 2, 3$. If $v \in Q_{13}$, then $\langle v \rangle_{B_{11}} \neq \langle v_3 \rangle_{B_{11}}$, since v and v_3 are separated by the edges of B_{13} in G_1. Thus $v \notin V_3$ and hence $v \in V_1 \cup V_2$. This means that $Q_{13} \subseteq V_1 \cup V_2$. On the other hand, if $v \in P_{13}$, then $\langle v \rangle_{B_{11}} \neq \langle v_1 \rangle_{B_{11}}, \langle v_2 \rangle_{B_{11}}$. Thus $v \notin V_1 \cup V_2$, and hence $v \in V_3$. Since $P_{13} \cup Q_{13} = V_1 \cup V_2 \cup V_3$, it follows that $Q_{13} = V_1 \cup V_3$ and $P_{13} = V_3$. By the same token, $Q_{12} = V_1 \cup V_3$, and $P_{12} = V_2$.

Since the edges of $\langle B_{12} \rangle_{B_{11}}$ form a multiple edge between vertices $\langle v_1 \rangle_{B_{11}}$ and $\langle v_2 \rangle_{B_{11}}$, it follows that every edge of B_{12} has one endvertex in V_1 and the other in V_2. Similarly, every edge of B_{13} has one endvertex in V_1 and the other in V_3 (Fig. 5).

Case 1: Suppose $v_1 \in P''_{21} \cap P''_{22}$. Since $v_1 \in P''_{21} \cap P''_{22}$, it must hold that for $i = 1$ or $i = 2$ that $w_2 \in P''_{2i}$ (recall from the definition of G'_2 that w_2 is a vertex in G'_2 with neighbours v_1, v_2, and v_3). We may assume without loss of generality that $w_2 \in P''_{21}$. Since $\langle v_1 \rangle_{B_1}$ is a cut-vertex of G/B_1, it is clear that $V_1 \neq \{v_1\}$. Let

$$\mathbf{C}_1 = [(\mathbf{P}'_{11} \cup \mathbf{P}''_{22}) \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{P}'_{11} \cup \mathbf{P}''_{22}) \cap \mathbf{V}(\mathbf{G})}]$$

and

$$\mathbf{C}_2 = [(\mathbf{V}_1 \cup \mathbf{P}''_{21}) \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{V}_1 \cup \mathbf{P}''_{21}) \cap \mathbf{V}(\mathbf{G})}].$$

We shall consider two subcases:

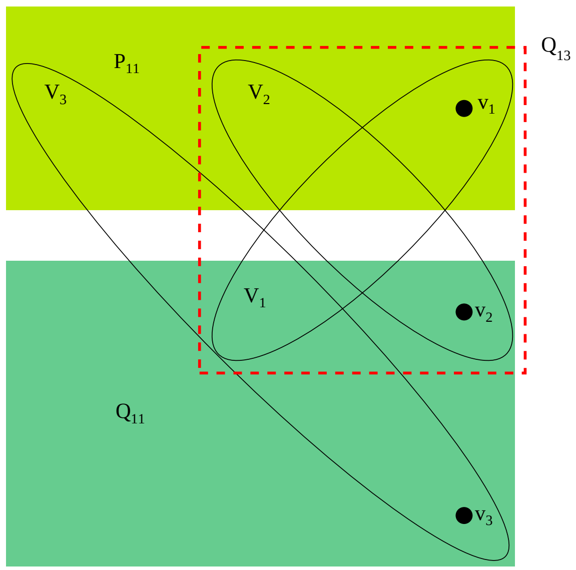

Fig. 5.

Case 1.1: Suppose $G(Q_{11})$ is connected. We wish to show that C_1 and C_2 is a good pair of bonds of G. Since $P_{11} \neq \{v_1\}$ and $G(Q_{11})$ is connected (and hence $G(\overline{(P'_{11} \cup P''_{22}) \cap V(G)})$ is connected), we have that C_1 is a non-trivial bond. Since B''_{21} is a bond in G''_2, we have that $G''_2(Q''_{21})$ is connected and hence $G_2(Q''_{21} \cap V(G))$ is connected (because $w_2, w^2_{23} \notin Q''_{21}$). Thus C_2 is a bond, and it is non-trivial since $V_1 \neq \{v_1\}$.

(i) C_1 is good. We will now show that C_1 is good. If $\langle v_2 \rangle_{C_1} = \langle v_3 \rangle_{C_1}$, then C_1 is clearly contractible since G_1/B_{11} consists of two multiple edges, one containing $\langle v_1 \rangle_{B_{11}}$ and $\langle v_2 \rangle_{B_{11}}$ and the other containing $\langle v_1 \rangle_{B_{11}}$ and $\langle v_3 \rangle_{B_{11}}$. We suppose therefore that $\langle v_2 \rangle_{C_1} \neq \langle v_3 \rangle_{C_1}$. Since B''_{22} is good in G''_2, it follows that $G''_2 \backslash B''_2$ is connected and there is a path in $(G''_2/B''_{22}) \backslash \langle v_1 \rangle_{B''_{22}}$ from $\langle v_2 \rangle_{B''_{22}}$ to $\langle v_3 \rangle_{B''_{22}}$. This means that there is a path in $(G/C_1) \backslash \langle v_1 \rangle_{C_1}$ from $\langle v_2 \rangle_{C_1}$ to $\langle v_3 \rangle_{C_1}$. Thus C_1 is good, since $\langle v_i \rangle_{C_1}$, $i = 1, 2, 3$ are all seen to belong to the same block.

(ii) C_2 is good. We will now show that C_2 is good. Since all the edges of $B_{12} \cup B_{13}$ are incident with V_1, we have $C_2 \cap E(G_1) = B_{12} \cup B_{13}$. Since $G(Q_{11})$ is connected and contains only edges of $B_{12} \cup B_{13}$, it follows that $G_1/(B_{12} \cup B_{13})$ is a multiple edge between $\langle v_1 \rangle_{B_{12} \cup B_{13}}$ and $\langle v_2 \rangle_{B_{12} \cup B_{13}}$. This together with the fact that B''_{21} is contractible in G_2 (where $\langle v_2 \rangle_{B''_{21}} = \langle v_3 \rangle_{B''_{21}}$) implies that C_2 is contractible. This completes Case 1.1.

Case 1.2: Suppose that $G(Q_{11})$ is not connected.

(i) C_1 is good or there is a good pair of bonds. If $G(Q''_{22} \cap V(G))$ is connected, then C_1 is a non-trivial bond, and it can be shown to be contractible in the same way as in Case 1.1. If on the other hand $G(Q''_{22} \cap V(G))$ is not connected, then it has two components, say Q^j_{22}, $j = 2, 3$ where $v_j \in Q^j_{22}$, $j = 2, 3$. Then $C^j_2 = [P_{1j} \cup Q^j_{22}, \overline{P_{1j} \cup Q^j_{22}}]$, $j = 2, 3$ is seen to be a pair of bonds in G. Since $dist_{G_1}(v_1, v_3) = 2$, there is a path $v_1 z v_3$ in G_1.

We have that $z \notin P_{12}$; for otherwise, $\langle v_1\rangle_{B'_{12}} = \langle v_2\rangle_{B'_{12}}$ and G/B'_{12} would have a cut-vertex $\langle v_1\rangle_{B'_{12}}$. If $z \in P_{11}$, then $\langle z\rangle_{C_2^2} \neq \langle v_2\rangle_{C_2^2}$, and hence there is a path from $\langle v_1\rangle_{C_2^2}$ to $\langle v_3\rangle_{C_2^2}$ in $(G/C_2^2)\backslash\langle v_2\rangle_{C_2^2}$.

Suppose $z \in Q_{11}$. If $\langle z\rangle_{C_2^2} = \langle v_2\rangle_{C_2^2}$, then there is a path P in $G(C_2^2)$ from z to v_2. Since P cannot cross B_{11}, we have that $P \subseteq G(Q_{11})$. We see that $P \cup zv_3$ is a path in $G(Q_{11})$ from v_2 to v_3. However, $G(Q_{11})$ is assumed to be disconnected, and therefore no such path exists. In this case, we conclude that if $z \in Q_{11}$, then $\langle z\rangle_{C_2^2} \neq \langle v_2\rangle_{C_2^2}$. Thus there is a path from $\langle v_1\rangle_{C_2^2}$ to $\langle v_3\rangle_{C_2^2}$ in $(G/C_2^2)\backslash\langle v_2\rangle_{C_2^2}$. One sees that C_2^2 is contractible, and the same holds for C_2^3. In this case, we have a good pair of bonds. Thus we may assume that $G(Q''_{22} \cap V(G))$ is connected and C_1 is a good bond.

(ii) C_2 is good. We have that C_2 is a non-trivial bond of G (as in Case 1.1). If $\langle v_2\rangle_{C_2} = \langle v_3\rangle_{C_2}$, then, as in Case 1.1, C_2 is contractible. Suppose instead that $\langle v_2\rangle_{C_2} \neq \langle v_3\rangle_{C_2}$. Since $G(Q''_{22} \cap V(G))$ is assumed to be connected, it contains a path P from v_2 to v_3. Since the vertices of $Q''_{22} \cap V(G)$ are separated from v_1 by the edges of $(B''_{22} \cup B'_{11}) \cap E(G)$, any path from P to v_1 must contain at least one edge from this set. Since C_2 contains no such edges, we conclude that no path in $G(C_2)$ from P to v_1 can exist. Consequently, $\langle v_1\rangle_{C_2} \notin \langle P\rangle_{C_2}$. This means that $\langle P\rangle_{C_2}$ contains a path from $\langle v_2\rangle_{C_2}$ to $\langle v_3\rangle_{C_2}$ in $(G/C_2)\backslash\langle v_1\rangle_{C_2}$. Thus C_2 is good in G, and C_1 and C_2 is a good pair of bonds. This completes Case 1.2.

Case 2: Suppose $v_1 \in P''_{21}$, and $v_2 \in P''_{22}$. Let

$$\mathbf{C}_1 = [(\mathbf{P}'_{11} \cup \mathbf{P}''_{21}) \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{P}'_{11} \cup \mathbf{P}''_{21}) \cap \mathbf{V}(\mathbf{G})}]$$

and

$$\mathbf{C}_2 = [(\mathbf{P}'_{12} \cup \mathbf{P}''_{22}) \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{P}'_{12} \cup \mathbf{P}''_{22}) \cap \mathbf{V}(\mathbf{G})}].$$

We note first that $w_2 \notin P''_{21}$ since $v_2 \in P''_{22}$ (and likewise, $w_2 \notin P''_{22}$. Similar to Case 1, we can show that either C_1 is a good bond, or we can find a good pair of bonds. We can therefore assume that C_1 is a good bond, and it remains show that C_2 is a good bond.

Since the edges of B_{13} are incident with V_1 and V_3, and $P_{12} = V_2$, there is a path in $G_1\backslash P_{12}$ from v_1 to v_3. We conclude that $G_1\backslash P_{12}$ is connected, and hence C_2 is a bond. Moreover, C_2 is non-trivial since $P_{12} \neq \{v_2\}$. We have that C_2 is a cross-bond, and $\langle v_i\rangle_{C_2} \neq \langle v_j\rangle_{C_2}$, $i \neq j$. Since $dist_G(v_1, v_2) = dist_G(v_2, v_3) = 2$, we have that $\langle v_1\rangle_{C_2}\langle v_2\rangle_{C_2}$ and $\langle v_2\rangle_{C_2}\langle v_3\rangle_{C_2}$ are edges of G/C_2.

To show that C_2 is good, it suffices(by Claim 9) to show that there is a path in $(G/C_2)\backslash \langle v_2\rangle_{C_2}$ from $\langle v_1\rangle_{C_2}$ to $\langle v_3\rangle_{C_2}$ and since $P_{13}\backslash V(K) \neq \emptyset$. Since $G_1(P_{13})$ is connected and contains only edges of B_{11},(because $P_{13} = V_3$) there is an edge in $G_1(P_{13})$ from v_3 to a vertex $z \in P_{11}$. Since $G(P_{11})$ is connected, it contains a path from z to v. Thus there is a path P from v_1 to v_3 in $G(P_{13} \cup P_{11})$. Since any path from P to v in G_1 must contain edges of $B_{11} \cup B_{13}$ there is no path in $G(C_2)$ from P to v_2. Thus $\langle v_2\rangle_{C_2} \notin \langle P\rangle_{C_2}$, we have that $\langle P\rangle_{C_2}$ contains the desired path from $\langle v_1\rangle_{C_2}$ to $\langle v_3\rangle_{C_2}$. This completes Case 2.

By similar arguments, one may deal with the case where $v_1 \in P''_{21}$, and $v_3 \in P''_{22}$. We have one remaining case:

Case 3: Suppose $v_2 \in P''_{21}$, and $v_3 \in P''_{22}$. Let

$$\mathbf{C}_2 = [(\mathbf{P}'_{12} \cup \mathbf{P}''_{21}) \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{P}'_{12} \cup \mathbf{P}''_{21}) \cap \mathbf{V}(\mathbf{G})}]$$

$$\mathbf{C}_3 = [(\mathbf{P}'_{13} \cup \mathbf{P}''_{22}) \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{P}'_{13} \cup \mathbf{P}''_{21}) \cap \mathbf{V}(\mathbf{G})}].$$

As in Case 2, we can show that C_2 is a good bond, and in the same way, we can show that C_3 is a good bond. Thus C_2 and C_3 is a good pair of bonds.

The proof of the claim follows from the consideration of Cases 1–3. $\square$

Remark. We observe that in the proof of the above claim, for each good bond C constructed, we have that $\langle v_1\rangle_C \neq \langle v_2\rangle_C, \langle v_3\rangle_C$.

Claim 21. *If $\{v_1, v_2, v_3\}$ is a minimal good separation which is of type 3 where G'_1 is the edge disjoint union of three good bonds, then G has a good pair of bonds.*

Proof. From Claim 20, we may assume that B_1 is a good bond. We may also assume that $P'_{22} \cap \{v_1, v_2, v_3\} \neq \emptyset$, for otherwise $B_{22} = B'_{22}$ and B_{22} and B_1 is a good pair of bonds. We may assume without loss of generality that $v_2 \in P'_{22}$ (and $v_1, v_3 \in Q'_{22}$). Let $\mathbf{B}_2 = [\mathbf{P}_{12} \cup \mathbf{P}_{22}, \overline{\mathbf{P}_{12} \cup \mathbf{P}_{22}}]$. Similar to B_1, one can show that B_2 is non-trivial, and if B_2 is not good, then G has a good pair of bonds. So either B_1 and B_2 are a good pair of bonds, or we can find 2 other bonds which are a good pair. $\square$

9. Good separations of type 3: part II

In this section, we shall assume that $\{v_1, v_2, v_3\}$ is a minimal good separation which is of type 3 where G'_1 has a G_1-good decomposition consisting of three G_1-good bonds and a contractible semi-bond S. According to Lemma 2.5, we can assume that G'_1 has only 4-faces, with the exception of one 6-face F (bounded by K) and two 5-faces. Let

$$\mathbf{G}^* = \mathbf{G}/\mathbf{S}, \quad \mathbf{G}^*_i = \mathbf{G}_i/\mathbf{S}, \quad \mathbf{G}'^*_i = \mathbf{G}'_i/\mathbf{S}, \quad i = 1, 2,$$

$$\mathbf{v}^*_i = \langle \mathbf{v}_i\rangle_S, \quad i = 1, 2, 3.$$

Claim 22. *Suppose B^* is a contractible bond of G^*. Then $B = > B^* <_S$ is seen to be a bond of G. If B is non-contractible, then for some $i \neq j$, $\langle v^*_i\rangle_{B^*} = \langle v^*_j\rangle_{B^*}$ and for $k = 1, 2$, the graph G^*_k contains a path $P^*_k \subset G^*(B^*)$ from v^*_i to v^*_j. In particular, $> P^*_1 <_S$ contains a path $P_1 \subset K_{ij}$ of length three between v_i and v_j.*

Proof. Suppose B^* is a contractible bond of G^*, and let $B = > B^* <_S$. Then B is a bond, and we suppose that B is non-contractible. Since S is a contractible semi-bond, we have that $G \backslash S$ is connected and G/S is 2-connected. Thus Lemma 2.6 implies that G/B contains loops(and is 2-connected apart from these loops). Such loops belong to $\langle S\rangle_B$ since $G/B/S = G/S/B = G^*/B^*$ is 2-connected. Thus there is an edge $e = xy \in S$ and a path $P \subseteq G(B)$ from x to y. We shall choose e and P such that $|P|$ is minimum. This means that $P \cup \{e\}$ is a cycle and $C^* = \langle P\rangle_S$ is a cycle containing $\langle X\rangle_S = \langle y\rangle_S$. Suppose $C^* \subset G_1^*$. If the regions inside and outside C^* contain vertices, then $\langle C^*\rangle_{B^*}$ is a cut-vertex of G^*/B^* which contradicts the contractibility of B^* in G^*. Thus C^* bounds a face of G_1^*. Lemma 2.7 implies that $|E(C^*) \cap B^*| \leqslant 2$. This means that $|E(C^*)| = 2$, as $C^* \subseteq B^*$. Thus $|P| = 2$ and $P \cup \{e\}$ is a triangle, contradicting the fact that G is triangle-free. We conclude that $C^* \not\subset G_1^*$. Thus for some $i \neq j$, C^* contains a path $P_1^* \subset G_1^*$ from v_i^* to v_j^* and a path $P_2^* \subset G_2^*$ from v_i^* to v_j^*. Consider the cycle $P_1^* \cup \{w_{ij}^1,\ w_{ij}^1 v_i^*,\ w_{ij}^1 v_j^*\}$. Similar to the previous arguments, one deduces that the cycle bounds a face of $G_1'^*$ and $|P_1^*| \leqslant 2$. Thus $> P_1^* <_S$ contains a path P_1 of length at most 3 from v_i to v_j and $P_1 \subset K_{ij}$. This path contains exactly one edge of S, namely e. Thus K_{ij} contains exactly one edge of S (which is e) and this means that $|K_{ij}| = 5$, since S corresponds to a removable path P in H_1' between two vertices of degree 5. Consequently, $|P_1| = 3$, and $|P_1^*| = 2$. $\square$

Claim 23. *Let B be a cross-bond of G not containing edges of S. If $B^* = \langle B\rangle_S$ is a contractible bond of G^*, then B is contractible in G.*

Proof. Let B be a cross-bond of G not containing edges of S and let $B^* = \langle B\rangle_S$. Then B^* is a bond of G^*. Suppose that B^* is a contractible bond of G^*. If B is non-contractible in G, then Claim 22 implies that G_2^* contains a path with edges in B^* from v_i^* to v_j^* for some $i \neq j$. Since G_2^* contains no edges of S, such a path has only edges in B. Thus $\langle v_i\rangle_B = \langle v_j\rangle_B$ for some $i \neq j$. By Claim 8 and consequently, B is contractible in G. $\square$

The graph G_1' has a G_1-good decomposition consisting of three good bonds, denoted by $\mathbf{B}_{1j}' = [\mathbf{P}_{1j}', \mathbf{Q}_{1j}'],\ j = 1, 2, 3$, and a contractible semi-bond $\mathbf{S}$. The graph G_2' has a good pair of bonds $\mathbf{B}_{2j}' = [\mathbf{P}_{2j}', \mathbf{Q}_{2j}'],\ j = 1, 2$. For all $i \neq j$ let

$$\mathbf{B}_{ij} = \mathbf{B}_{ij}' \cap \mathbf{E}(\mathbf{G}), \quad \mathbf{P}_{ij} = \mathbf{P}_{ij}' \cap \mathbf{V}(\mathbf{G}), \quad \mathbf{Q}_{ij} = \mathbf{Q}_{ij}' \cap \mathbf{V}(\mathbf{G}),$$

$$\mathbf{B}_{ij}'^* = \langle \mathbf{B}_{ij}'\rangle_S, \quad \mathbf{P}_{ij}'^* = \langle \mathbf{P}_{ij}'\rangle_S, \quad \mathbf{Q}_{ij}'^* = \langle \mathbf{Q}_{ij}'\rangle_S,$$

$$\mathbf{B}_{ij}^* = \langle \mathbf{B}_{ij}\rangle_S, \quad \mathbf{P}_{ij}^* = \langle \mathbf{P}_{ij}\rangle_S, \quad \mathbf{Q}_{ij}^* = \langle \mathbf{Q}_{ij}\rangle_S.$$

Since the decomposition $B_{ij}',\ j = 1, 2, 3$ and S is G_1-good, we have that $P_{1j} \backslash V(K) \neq \emptyset,\ j = 1, 2, 3$. We may assume that for some $j \in \{1, 2\}$ it holds that $|P_{2j}' \cap \{v_1, v_2, v_3\}| \leqslant 1$. If $P_{2j} \cap \{v_1, v_2, v_3\} = \emptyset,\ j = 1, 2$, then $B_{2j}' = B_{2j},\ j = 1, 2$ and these are a good pair of bonds of G. Consequently, we can assume that $P_{21} \cap \{v_1, v_2, v_3\} \neq \emptyset$, and $v_1 \in P_{21}$. We shall also assume that $P_{22} \cap \{v_1, v_2, v_3\} \neq \emptyset$ as the case where $P_{22} \cap \{v_1, v_2, v_3\} = \emptyset$ is easier and can be dealt with using the same arguments. We may assume without loss of generality that $P_{22} \cap \{v_1, v_2, v_3\} = \{v_3\}$.

Let

$$\mathbf{V}_i^* = \{\mathbf{v}^* \in \mathbf{V}(\mathbf{G}_1^*) : \langle \mathbf{v}^*\rangle_{B_{11}^*} = \langle v_i^*\rangle_{B_{11}^*}\}, \quad \mathbf{V}_i = > \mathbf{V}_i^* <_S,\ \ i = 1, 2, 3.$$

For $i = 1, 2, 3$ let $\mathbf{Y}_i$ (resp. $\mathbf{Y}'_i$) be the vertices of the component in $G_1(B_{12} \cup B_{13})$ (resp. $G'_1(B'_{12} \cup B'_{13})$) containing v_i. Let

$$\mathbf{B}_1 = [\mathbf{P}_{11} \cup \mathbf{P}_{21}, \mathbf{Q}_{11} \cup \mathbf{Q}_{21}] \quad \text{and} \quad \mathbf{B}_2 = [\mathbf{P}_{13} \cup \mathbf{P}_{22}, \mathbf{Q}_{13} \cup \mathbf{Q}_{22}].$$

We shall first show that the bonds $B_i,\ i = 1, 2$ are cross-bonds of G. We have that $|K_{23}| = 4$, or 5. If $|K_{23}| = 4$, then $E(K_{23}) \subset B_{12} \cup B_{13}$. Otherwise, if $|K_{23}| = 5$, then $E(K_{23}) \subset B_{12} \cup B_{13} \cup S$. This means that K_{23} contains no edges of B'_{11} and hence $V(K_{23}) \subset Q'_{11}$. This implies that $G(Q_{11} \cup Q_{21})$ is connected and B_1 is a bond. Furthermore, B_1 is non-trivial since $P_{11} \backslash V(K) \neq \emptyset$. Hence B_1 is a cross-bond, and the same applies to B_2.

Claim 24. *If* $|K_{12}| = |K_{23}| = 5$, *then G has a good pair of bonds.*

Proof. Suppose that $|K_{12}| = |K_{23}| = 5$. Let $\mathbf{G}''_1$ be the graph obtained from G'_1 by deleting w^1_{12} and w^1_{23} and adding edges v_1v_2 and v_2v_3. Note that there is no 2-path v_1wv_2 in G''_1, for then $\{v_1, w, v_2\}$ would be a good separation, contradicting the minimality of $\{v_1, v_2, v_3\}$. Similarly, there is no 2-path between v_2 and v_3 in G''_1. Thus G''_1 is triangle-free.

As in Section 7, G''_1 has a good pair of bonds $\mathbf{B}''_{1j} = [\mathbf{P}''_{1j}, \mathbf{Q}''_{1j}],\ j = 1, 2$ where $E(G''_1) = B_{11} \cup B''_{12}$ and $v_1 \in P''_{11},\ v_3 \in P''_{12}$. Let $\mathbf{D}_j = [(\mathbf{P}''_{1j} \cup \mathbf{P}_{2j}) \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{P}''_{1j} \cup \mathbf{P}_{2j}) \cap \mathbf{V}(\mathbf{G})}],\ j = 1, 2$. Since $dist_G(v_2, v_3) = 2$, there is a 2-path v_2wv_3 in G_2. Since B'_{21} is good in G'_2, we have $w \notin P'_{21}$. Thus $w \in Q'_{21}$, and D_1 is seen to be a non-trivial bond, in fact a cross-bond. If D_1 is not good, then as was shown in the proof of Claim 15, G/D_1 would consist of two blocks; one containing $\langle v_1 \rangle_{D_1}$ and $\langle v_2 \rangle_{D_1}$ and the other containing $\langle v_2 \rangle_{D_1}$ and $\langle v_3 \rangle_{D_1}$. However, since $dist_G(v_1, v_2) = 2$, there is an edge between $\langle v_1 \rangle_{D_1}$ and $\langle v_2 \rangle_{D_1}$ in G/D_1. This would imply that $\langle v_1 \rangle_{D_1},\ \langle v_2 \rangle_{D_1},\ \langle v_3 \rangle_{D_1}$ all belong to the same block in G/D_1—a contradiction. Thus D_1 is good in G, and following similar reasoning, D_2 is also good. □

9.1. *The case where B_1 is non-contractible*

If $|K_{23}| = 5$, then we may assume that $|K_{12}| = 4$ (by Claim 24). In this case, we shall assume (as guaranteed by Lemma 5.3) that the bonds $B'_{1i},\ i = 1, 2, 3$ and semi-bond S are chosen so that $yv_3 \notin S$, given $K_{23} = v_2xyv_3w^1_{23}v_2$. On the other hand, if $|K_{12}| = 5$, and $|K_{23}| = 4$, then we shall choose the bonds $B'_{1i},\ i = 1, 2, 3$ and semi-bond S so that $yv_1 \notin S$ where $K_{12} = v_2xyv_1w^1_{12}v_2$.

Suppose that B_1 is non-contractible. As in Part I, Claim 8 implies that G/B_1 consists of two blocks, one containing $\langle v_1 \rangle_{B_1}$ and $\langle v_2 \rangle_{B_1}$ and the other containing $\langle v_1 \rangle_{B_1}$ and $\langle v_3 \rangle_{B_1}$. This means that $\langle v_1 \rangle_{B_1}$ is a cut-vertex of G/B_1 and hence $w_{23} \notin V(G)$. Since B_1 is not contractible and is a cross-bond, Claim 23 implies that $B^*_1 = \langle B_1 \rangle_S$ is a non-contractible bond of G^*. This in turn implies that G^*_1/B^*_{11} consists of two multiple edges; one between $\langle v^*_1 \rangle_{B^*_{11}}$ and $\langle v^*_2 \rangle_{B^*_{11}}$, and another between $\langle v^*_1 \rangle_{B^*_{11}}$ and $\langle v^*_3 \rangle_{B^*_{11}}$. Thus G^*_1/B^*_{11} has exactly 3 vertices $\langle v^*_i \rangle_{B^*_{11}},\ i = 1, 2, 3$. As in Part I, we have that $V^*_1 \cup V^*_2 = Q^*_{13},\ V^*_2 = P^*_{12}$, and $B^*_{12} \cup B^*_{13} = [V^*_1, V(G^*_1) \backslash V^*_1]$. Clearly $V_1 \neq \{v_1\}$, as $\langle v_1 \rangle_{B_1}$ is a cut-vertex of G/B_1.

As was done in the proof of Claim 20, we define the graph $\mathbf{G}_2'' = \mathbf{G}_2' \backslash \mathbf{w}_{23}^2$. The graph G_2'' has a good pair of bonds $\mathbf{B}_{21}'' = [\mathbf{P}_{2j}'', \mathbf{Q}_{2j}''],\ j = 1, 2$, where $|P_{2j}'' \cap \{v_1, v_2, v_3\}| \leqslant 1$. We may assume that for some $j = 1, 2$ it holds that $|P_{2j}'' \cap \{v_1, v_2, v_3\}| = 1$, for otherwise $B_{2j}'',\ j = 1, 2$ would be a good pair of bonds of G (since $w_{23} \notin V(G)$). We shall assume that $|P_{2j}'' \cap \{v_1, v_2, v_3\}| = 1$, for both $j = 1, 2$; the case where it holds for only one of $j = 1$ or $j = 2$ is easily handled by the same arguments.

Claim 25. *If B_1 is non-contractible and $|K_{23}| = 5$, then G contains a good pair of bonds.*

Proof. Suppose B_1 is non-contractible and $|K_{23}| = 5$. Then there is no path from v_2 to v_3 in Q_{11}. Let $K_{23} = v_2xyv_3w_{23}^1v_2$ and $P_1 = K_{23} \backslash w_{23}^1$. By assumption, the bonds $B_{1i}',\ i = 1, 2, 3$ and the semi-bond S are chosen so that $yv_3 \notin S$.

Recall the definition of $Y_i,\ i = 1, 2, 3$. We shall first show that $Y_2 \neq Y_3$. Suppose on the contrary that $Y_2 = Y_3$. Then there is a path Q in $G(B_{12} \cup B_{13})$ connecting v_2 and v_3. We may assume that v_1 lies outside the region R bounded by the cycle $Q \cup v_2w_{23}^1v_3$. For any vertex v lying in the interior of R, it holds that any path from v to v_1 must intersect Q, and hence it must intersect vertices of Q_{11}. Thus $v \notin P_{11}$, for otherwise there would be a path in $G_1(P_{11})$ from v to v_1 which does not intersect Q_{11}. Consequently, R contains no vertices of P_{11} and hence no edges of B_1.

Since the cycle $Q \cup v_2w_{23}^1v_3$ contains no edges of S, R must contain the other 5-face which is bounded by a 5-cycle, say $x_1x_2x_3x_4x_5x_1$ where $x_1x_2 \in S$. For $i = 1, \ldots, 5$ we have that $\langle x_i \rangle_{B_1 \cup S}$ is one of the vertices $\langle v_i \rangle_{B_1 \cup S},\ i = 1, 2, 3$. The cycle $x_1x_2x_3x_4x_5x_1$ contains no edges of B_1 since R contains no edges of B_1. We have that two of the vertices x_1, x_3, x_4, x_5 contract to the same vertex in $G_1/B_1 \cup S$. Suppose $\langle x_1 \rangle_{B_1 \cup S} = \langle x_4 \rangle_{B_1 \cup S}$. Then there is a path Q_1 in $G(B_{11} \cup S)$ from x_1 to x_4. Now any path in $G(B_1 \cup S)$ from x_3 to v_1, v_2, or v_3 must intersect Q_1, in which case $\langle x_3 \rangle_{B_1 \cup S} = \langle x_1 \rangle_{B_1 \cup S} = \langle x_4 \rangle_{B_1 \cup S}$, yielding a contradiction. Thus $\langle x_1 \rangle_{B_1 \cup S} \neq \langle x_4 \rangle_{B_1 \cup S}$, and by similar reasoning $\langle x_3 \rangle_{B_1 \cup S} \neq \langle x_5 \rangle_{B_1 \cup S}$. Thus the vertices $\langle x_1 \rangle_{B_1 \cup S}, \langle x_3 \rangle_{B_1 \cup S}, \langle x_4 \rangle_{B_1 \cup S}, \langle x_5 \rangle_{B_1 \cup S}$ are all different, yielding a contradiction. Thus no such path Q exists , and $Y_2 \neq Y_3$.

We define C_1 and C_2 as follows (see Fig. 6): let

$$\mathbf{C}_1 = [(\mathbf{V}_1 \cup \mathbf{P}_{21}), \overline{\mathbf{V}_1 \cup \mathbf{P}_{21}}]$$

and

$$\mathbf{C}_2 = [\mathbf{Y}_3 \cup \mathbf{P}_{22}, \overline{\mathbf{Y}_3 \cup \mathbf{P}_{22}}].$$

9.1.1. C_1 is good

We will first show that C_1 is a bond by showing that $G(\overline{V_1 \cup P_{21}})$ is connected. Since $dist_G(v_2, v_3) = 2$, there is a 2-path v_2wv_3 in G. This 2-path does not belong to G_1, for otherwise $\{v_2, w, v_3\}$ would be a good separation of G, contradicting the minimality of $\{v_1, v_2, v_3\}$. Thus the 2-path belongs to G_2. We have that $w \notin P_{21}$, for otherwise $\langle v_2 \rangle_{B_{21}'} = \langle v_3 \rangle_{B_{21}'}$, contradicting the fact that B_{21}' is good. So $w \in Q_{21}$ and consequently, $G(Q_{21})$ is connected and C_1 is a non-trivial bond.

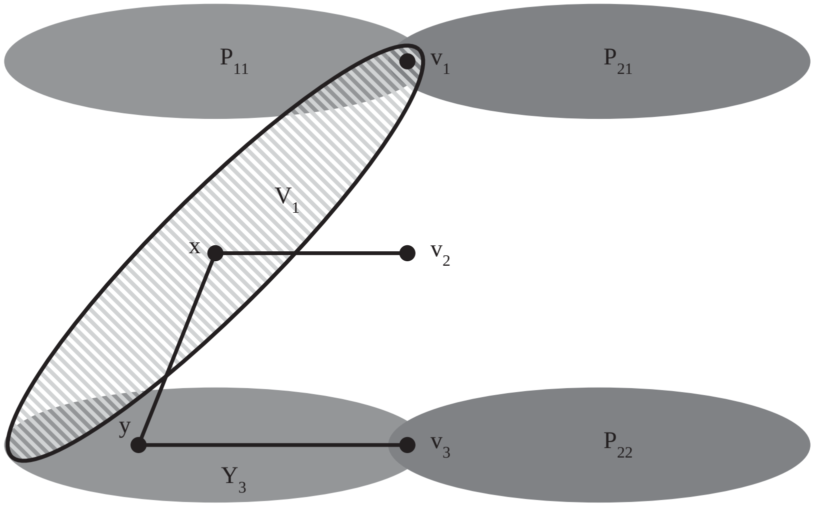

Fig. 6.

Let $C_1^* = \langle C_1 \rangle_S$. We have that $C_1^* \cap E(G_1^*) = B_{12}^* \cup B_{13}^*$, and seeing as $G_1^*/(B_{12}^* \cup B_{13}^*)$ is a multiple edge with vertices $\langle v_1^* \rangle_{B_{12}^* \cup B_{13}^*}$ and $\langle v_2^* \rangle_{B_{12}^* \cup B_{13}^*}$, it follows that C_1^* is a contractible bond of G^*. Since $C_1^* \cap G_2^* = B_{21}^*$, and $\langle v_i^* \rangle_{B_{21}^*} \neq \langle v_j^* \rangle_{B_{21}^*}$, $\forall i \neq j$, it follows that for $i \neq j$, G_2^* contains no path in $G^*(C_1^*)$ from v_i^* to v_j^*. Thus Claim 22 implies that C_1 must be contractible in G and hence is a good bond.

9.1.2. C_2 is good

We shall now show that C_2 is a good bond. To show that C_2 is a non-trivial bond, we note first that $dist_G(v_1, v_2) = 2$, and there is a path $v_1 z v_2$ between v_1 and v_2. We have that $Y_3 \cap P_{11} = \emptyset$ since every path from v_3 to P_{11} in G_1 contains an edge of B_{11}. Suppose $z \in Y_3$. Then $z \notin P_{11}$ and thus $zv_2 \in B_{12} \cup B_{13} \cup S$. Clearly $zv_2 \notin S$, for otherwise $v_1^* v_2^*$ would be an edge of G_1^*. Thus $zv_2 \in B_{12} \cup B_{13}$ and this implies $v_2 \in Y_2$, which is impossible since $Y_2 \cap Y_3 = \emptyset$. We conclude that $z \notin Y_3$. If $z \in P_{22}$, then $\langle v_1 \rangle_{B_{22}} = \langle v_2 \rangle_{B_{22}}$, which is impossible since $\langle v_i \rangle_{B'_{22}} \neq \langle v_j \rangle_{B'_{22}}$, $\forall i \neq j$. From this and the above, we conclude that $z \in \overline{Y_3 \cup P_{22}}$ and thus $G(\overline{Y_2 \cup P_{22}})$ is connected, and C_2 is a bond of G. Furthermore, since S was chosen so that $v_3 y \notin S$, it holds that $v_3 y \in B_{12} \cup B_{13}$. Thus $y \in Y_3$, and C_2 is non-trivial.

To show that C_2 is contractible, we will first show that it is a cross-bond. Let

$$\mathbf{C}'_{12} = [\mathbf{Y}'_3, \mathbf{V}(\mathbf{G}'_1) \backslash \mathbf{Y}'_2], \quad \mathbf{C}'_{22} = \mathbf{B}'_{22}, \quad \mathbf{C}_2^* = \langle \mathbf{C}_2 \rangle_S.$$

For $i = 1, 2$ let

$$\mathbf{C}_{i2} = \mathbf{C}_2 \cap \mathbf{E}(\mathbf{G}_i), \quad \mathbf{C}'^*_{i2} = \langle \mathbf{C}'_{i2} \rangle_S, \quad \mathbf{C}^*_{i2} = \langle \mathbf{C}_{i2} \rangle_S.$$

To show C_2 is a cross-bond, it suffices to show that C'_{i2}, $i = 1, 2$ is contractible in G'_i. We have that $C'_{22} = B'_{22}$ is a contractible bond of G'_2. It remains to show that C'_{12} is contractible in G_1'. Since $C_{12}^* \subseteq B_{11}^*$, and $B_{11}'^*$ is contractible in $G_1'^*$, it follows that $C_{12}'^*$ is contractible in $G_1'^*$. Let $T = S\backslash C_{12}$. Let $H = > G_1'^* <_T$ and let $C = > C_{12}'^* <_T$. We have that $H\backslash T$ is connected and $(H/C)/T = (H/T)/C = G_1'^*/C_{12}'^*$. Thus $(H/C)/T$ is 2-connected, and according to Lemma 2.6, either H/C is 2-connected or it contains loops. If H/C is 2-connected, then G_1'/C_{12}' is 2-connected since $H/C = G_1'/C_{12}'$. We suppose therefore that H/C contains loops. Then there is an edge $f \in T,\ f = wz$, and a path Q in H from w to z with $E(Q) \subseteq C$. Choose f and Q such that the region bounded by $Q \cup f$ is minimal. Then $Q \cup f$ is a cycle. Since H/C is 2-connected apart from loops, it follows that $Q \cup f$ bounds a face of H. By Lemma 2.7, Q has at most two edges. If $|Q| = 2$, then $Q \cup \{f\}$ is a triangle. Since G_1' is triangle-free, the edges of $> E(Q) \cup \{f\} <_{(S\backslash T)}$ belong to a cycle D in G_1' where $|D| \geqslant 4$ and C_{12}' contains all the edges of D except $\{f\}$. By Lemma 2.7, D cannot bound a face of G_1' since it contains at least three edges of a bond of G (i.e. C_2). Thus D contains vertices in both its interior and exterior. Since the vertices of $D^* = \langle D\rangle_S$ are contracted together in $G_1'^*/C_{12}'^*$, it follows that $G_1'^*/C_{12}'^*$ would have a cut-vertex. This contradicts the fact that $C_{12}'^*$ is contractible in $G_1'^*$. We conclude that such a path Q cannot exist, and consequently H/C has no loops. This in turn implies that C_{12}' is contractible in G_1' and C_2 is a cross-bond of G.

To show that C_2 is contractible in G_1, it suffices to show (by Claim 9) that for all $i \neq j$, there is a path from $\langle v_i\rangle_{C_2}$ to $\langle v_j\rangle_{C_2}$ in $(G/C_2)\backslash\langle v_k\rangle_{C_2}$ where $k \neq i, j$. Given that $C_{12} \subset B_{11} \cup S$, there are paths from $\langle v_1\rangle_{C_2}$ to $\langle v_2\rangle_{C_2}$ in $(G/C_2)\backslash\langle v_3\rangle_{C_2}$ and from $\langle v_1\rangle_{C_2}$ to $\langle v_3\rangle_{C_2}$ in $(G/C_2)\backslash\langle v_2\rangle_{C_2}$. It remains to show that there is a path from $\langle v_2\rangle_{C_2}$ to $\langle v_3\rangle_{C_2}$ in $(G/C_2)\backslash\langle v_1\rangle_{C_2}$. Recall that C_1 is assumed to be a non-trivial (contractible) bond. This means that $G_2(Q_{21})$ is connected and there is a path Q in $G_2(Q_{21})$ from v_2 to v_3. No vertex of Q contracts to v_1 in G_2/B_{22} as every path from Q to v_1 must contain an edge from B_{21}. Thus $\langle Q\rangle_{C_2}$ contains a path from $\langle v_2\rangle_{C_2}$ to $\langle v_3\rangle_{C_2}$ in $(G/C_2)\backslash\langle v_1\rangle_{C_2}$. This shows that C_2 is contractible in G.

From the above, we have that C_1 and C_2 are good pair of bonds. This completes the proof of the claim. □

Claim 26. *If B_1 is not contractible, then G contains a good pair of bonds.*

Proof. Suppose that B_1 is non-contractible. By the previous claim, we may assume that $|K_{23}| = 4$. As was done in Section 7, define $G_2'' = G_2'\backslash\{w_{23}^2\}$, and let $B_{21}'' = [P_{21}'', Q_{21}'']$ and $B_{22}'' = [P_{22}'', Q_{22}'']$ be a good pair of bonds for G_2''. We may assume that $|P_{21}'' \cap \{v_1, v_2, v_3\}| = 1$ and $|P_{22}'' \cap \{v_1, v_2, v_3\}| = 1$ (the easier case where $P_{21}'' \cap \{v_1, v_2, v_3\} = \emptyset$ can be dealt with by similar arguments). We shall examine a few cases.

Case 1: Suppose $v_1 \in P_{21}''$ and $v_1 \in P_{22}''$. By definition, G_2' has a vertex w_2 whose neighbours are v_1, v_2, and v_3. Thus $w_2 \in V(G_2'')$ and we may assume that $w_2 \in P_{21}''$. Let

$$\mathbf{C}_1 = [(\mathbf{P}_{11}' \cup \mathbf{P}_{22}'') \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{P}_{11}' \cup \mathbf{P}_{22}'') \cap \mathbf{V}(\mathbf{G})}]$$

and

$$\mathbf{C}_2 = [(\mathbf{V}_1 \cup \mathbf{P}_{21}'') \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{V}_1 \cup \mathbf{P}_{21}'') \cap \mathbf{V}(\mathbf{G})}].$$

Let $\mathbf{C}_i^* = \langle \mathbf{C}_i \rangle_S$, $i = 1, 2$. Using the same arguments in the proof of Claim 20 (Case 1), one can show that C_i^*, $i = 1, 2$ are contractible in G^*. We have that $B_{11}'^*$ is a contractible bond in $G_1'^*$ and thus $\langle v_i \rangle_{B_{11}'^*} \neq \langle v_j \rangle_{B_{11}'^*}$, $\forall i \neq j$. Consequently, $\langle v_i^* \rangle_{B_{11}^*} \neq \langle v_j^* \rangle_{B_{11}^*}$, $\forall i \neq j$. Since $C_1^* \cap E(G_1^*) = B_{11}^*$, we have that for all $i \neq j$ there is no path in $G_1^*(C_1^*)$ from v_i^* to v_j^*. It follows by Claim 22, that C_1 is contractible in G. We may therefore assume that C_2 is not contractible in G.

Now Claim 22 implies that for some $i \neq j$ it holds that $\langle v_i^* \rangle_{C_2^*} = \langle v_j^* \rangle_{C_2^*}$. Since $\langle v_1^* \rangle_{C_2^*} \neq \langle v_2^* \rangle_{C_2^*}$, $\langle v_3^* \rangle_{C_2^*}$, it follows that $\langle v_2^* \rangle_{C_2^*} = \langle v_3^* \rangle_{C_2^*}$, and there is a path $P_1^* = v_2^* u^* v_3^*$ in $G_1^*(C_2^*)$. According to Claim 22, there is a path $P_1 \subset\!> P_1^* <_S$ having length 3 where $P_1 \subset K_{23}$ and thus $|K_{23}| = 5$. However, we are assuming that $|K_{23}| = 4$, and we have a contradiction. Thus C_2 is contractible and C_1 and C_2 are a good pair of bonds. This completes the proof of Case 1.

Case 2: Suppose $v_1 \in P_{21}''$ and $v_2 \in P_{22}''$. Let

$$\mathbf{C}_i = [(\mathbf{P}_{1i}' \cup \mathbf{P}_{2i}'') \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{P}_{1i}' \cup \mathbf{P}_{2i}'') \cap \mathbf{V}(\mathbf{G})}], \quad \mathbf{C}_i^* = \langle \mathbf{C}_i \rangle_S, \ \ i = 1, 2.$$

(i) C_1 is good. One can show that $G_1^*(Q_{11}^*)$ is connected, and hence C_1^* is a bond. Using the same arguments as given in the proof of Claim 20 (Case 1.1), one can show that C_1^* is a contractible bond of G^*. Since $C_1^* \cap E(G_1^*) = B_{11}^*$ and $\langle v_i^* \rangle_{B_{11}^*} \neq \langle v_j^* \rangle_{B_{11}^*}$, $\forall i \neq j$, it follows by Claim 22 that C_1 is contractible in G.

(ii) C_2 is good. The bond C_2 is seen to be a cross-bond of G. We shall now show that C_2 is contractible in G. If $P_{13}^* \backslash \langle V(K) \rangle_S \neq \emptyset$, then it follows from the arguments in the proof of Claim 20 (Case 2) that C_2^* is contractible in G^*. In this case, Claim 23 implies that C_2 is contractible.

We may therefore assume that $P_{13}^* \backslash \langle V(K) \rangle_S = \emptyset$. This means that all edges incident with v_3 in $G_1 \backslash E(K)$ belong to $S \cup B_{13}$. We have for $j = 1, 3$ that $dist_{G_1^*}(v_2^*, v_j^*) = 2$ and $\langle v_2^* \rangle_{C_2^*} \langle v_j^* \rangle_{C_2^*}$ is an edge of G^*/C_2^* for $j = 1, 3$. Thus there are paths from $\langle v_2 \rangle_{C_2}$ to $\langle v_1 \rangle_{C_2}$ in $(G/C_2) \backslash \langle v_3 \rangle_{C_2}$ and from $\langle v_2 \rangle_{C_2}$ to $\langle v_3 \rangle_{C_2}$ in $(G/C_2) \backslash \langle v_1 \rangle_{C_2}$. Since C_2 is a cross-bond, to show that C_2 is contractible it suffices to show that there is a path from $\langle v_1 \rangle_{C_2}$ to $\langle v_3 \rangle_{C_2}$ in $(G/C_2) \backslash \langle v_2 \rangle_{C_2}$. We suppose that no such path exists. This means that G_1/B_{12} consists of two blocks between $\langle v_2 \rangle_{B_{12}}$ and $\langle v_j \rangle_{B_{12}}$, for $j = 1, 3$, the corresponding blocks in G_1^*/B_{12}^* being multiple edges. This means that for each vertex $v^* \in Q_{13}^*$ either $\langle v^* \rangle_{B_{12}^*} = \langle v_1^* \rangle_{B_{12}^*}$ or $\langle v^* \rangle_{B_{12}^*} = \langle v_2^* \rangle_{B_{12}^*}$. We shall show that this cannot happen. Since $|K_{23}| = 4$, there is a path $P_1 = v_2 z_1 v_3 \subset K_{23}$. Since all edges incident with v_3 in $G \backslash E(K)$ belong to $S \cup B_{13}$, we have that $v_3 z_1 \in B_{13}$, and hence $v_2 z_1 \in B_{12}$. Thus $\langle z_1 \rangle_{B_{11}} = \langle v_1 \rangle_{B_{11}}$ (since B_1 is not contractible).

Suppose $|K_{13}| = 4$, then there is a path $P_2 = v_1 z_2 v_3 \subset K_{13}$ where $z_2 v_3 \notin B_{11}$ (since $P_{13}^* \backslash \langle V(K) \rangle_S = \emptyset$). Then $v_1 z_2 \in B_{11}$, $v_3 z_2 \in B_{13}$, and $\langle z_2 \rangle_{B_{11}} = \langle v_1 \rangle_{B_{11}}$. We have that $\langle z_2 \rangle_{B_{12}} = \langle v_2 \rangle_{B_{12}}$; otherwise there would be a path from $\langle v_1 \rangle_{B_2}$ to $\langle v_3 \rangle_{B_2}$ in $(G/B_2) \backslash \langle v_2 \rangle_{B_2}$ in which case we are done. Since $\langle z_i \rangle_{B_{11}} = \langle v_1 \rangle_{B_{11}}$ for $i = 1, 2$ there is a path $L_1 \subset G_1(B_{11})$ from z_1 to z_2. Let R_1 be the region of G_1' bounded by $L_1 \cup \{v_3, z_1 v_3, z_2 v_3\}$ which does not contain v_2. Similarly, since $\langle z_i \rangle_{B_{12}} = \langle v_2 \rangle_{B_{12}}$, $i = 1, 2$, there is a path $L_2 \subset G_1(B_{12})$ from z_1 to z_2. Since for each vertex $v^* \in Q_{13}^*$ we have that $v^* \in V_1^* \cup V_2^*$, it follows that for each $v \in V(L_2)$ which lies inside R_1 or on L_1, $\langle v \rangle_S \in V_1^*$. This holds since any path from v to v_2 must contain vertices of L_1 (and $V(L_1) \subset V_1$) and consequently $\langle v \rangle_S \notin V_2^*$

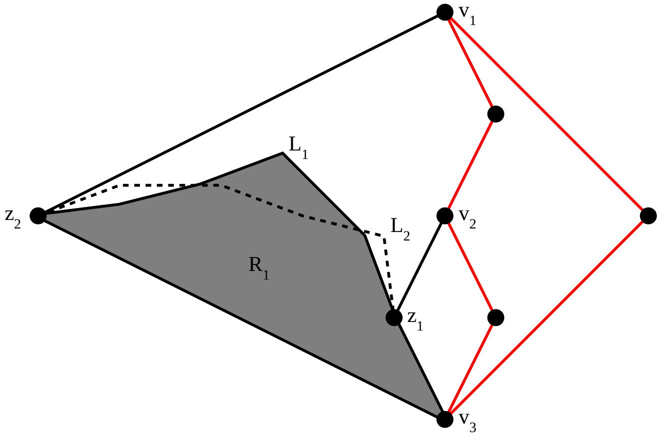

Fig. 7.

(see Fig. 7). The above implies that R_1 contains no edges of L_2, for both endvertices of such edges would contract to $\langle v_1^* \rangle_{B_{11}^*}$ in G_1^*/B_{11}^*, producing a loop. We now define R_2 to be the region bounded by $L_2 \cup \{v_3, z_1v_3, z_2v_3\}$ which does not contain v_1. Similar to R_1, the region R_2 contains no edges of L_1. However, since G_1' is planar, we cannot meet both of the requirements that R_1 contains no edges of L_2, and R_2 contains no edges of L_1. So in this case, C_2 must be contractible.

Suppose $|K_{13}| = 5$. Let $K_{13} = v_1wz_2v_3w_{13}^1v_1$. We have that either $\langle w \rangle_{B_{12}} = \langle v_2 \rangle_{B_{12}}$ or $\langle z_2 \rangle_{B_{12}} = \langle v_2 \rangle_{B_{12}}$. We have that $v_1w \in B_{11} \cup S$ (since $P_{13}^* \backslash \langle V(K) \rangle_S = \emptyset$). Suppose $v_1w \in S$. Then $\langle w \rangle_{B_{12}} \neq \langle v_2 \rangle_{B_{12}}$ (otherwise $\langle v_1^* \rangle_{B_{12}^*} = \langle v_2^* \rangle_{B_{12}^*}$). Thus we have that $\langle z_2 \rangle_{B_{12}} = \langle v_2 \rangle_{B_{12}}$, $z_2v_3 \in B_{13}$, and hence $z_2w \in B_{11}$. Then there is a path $L_1 \subset G_1(B_{11} \cup S)$ from z_1 to z_2. Let R_1 be the region bounded by $L_1 \cup \{v_3, z_1v_3, z_2v_3\}$ which does not contain v_2. Since $\langle z_1 \rangle_{B_{12}} = \langle z_2 \rangle_{B_{12}} = \langle v_2 \rangle_{B_{12}}$, there is a path $L_2 \subset G_1(B_{12})$ from z_1 to z_2. Let R_2 be the region bounded by $L_2 \cup \{v_3, z_1v_3, z_2v_3\}$ which does not contain v_1. As before, R_1 cannot contains edges of L_2, and R_2 cannot contain edges of $L_1 \cup B_{11}$. However, since G_1' is planar, both of these requirements cannot be met simultaneously. In this case, C_2 must be contractible.

If $v_1w \in B_{11}$, then one can argue in a similar fashion as in the above. Having considered all cases, we conclude that C_2 must be contractible, and hence good. This completes Case 2.

If $v_1 \in P_{21}''$ and $v_3 \in P_{22}''$, then we can find two contractible bonds via similar arguments as used in Case 2. There is one remaining case:

Case 3: Suppose $v_2 \in P_{21}''$ and $v_3 \in P_{22}''$. Let

$$\mathbf{C}_1 = [(\mathbf{P}_{12}' \cup \mathbf{P}_{21}'') \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{P}_{12}' \cup \mathbf{P}_{21}'') \cap \mathbf{V}(\mathbf{G})}]$$

and

$$\mathbf{C}_2 = [(\mathbf{P}_{13}' \cup \mathbf{P}_{22}'') \cap \mathbf{V}(\mathbf{G}), \overline{(\mathbf{P}_{13}' \cup \mathbf{P}_{22}'') \cap \mathbf{V}(\mathbf{G})}].$$

The sets C_1 and C_2 are seen to be cross-bonds of G. One can show that C_1 and C_2 are contractible bonds of G using the same arguments as given in Case 2. Consequently, C_1 and C_2 is a good pair of bonds. This completes Case 3.

The proof of the claim now follows from Cases 1–3. □

Similar to the above we have:

Claim 27. *If B_2 is non-contractible, then G contains a good pair of bonds.*

To conclude this section, we have

Claim 28. *If $\{v_1, v_2, v_3\}$ is a minimal good separation which is of type 3 where G'_1 is the edge-disjoint union of three good bonds and a contractible semi-bond, then G has a good pair of bonds.*

Proof. By Claims 26 and 27, either B_1 and B_2 are a good pair of bonds, or we can find another good pair of bonds. □

10. Separating sets of type 2

In this section, we shall assume that $\{v_1, v_2, v_3\}$ is a minimal good separation which has type 2. We shall assume that $dist_G(v_1, v_j) = 2,\ j = 2, 3$ and $dist_G(v_2, v_3) \neq 2$.

10.1. The case $v_2v_3 \in E(G)$

Claim 29. *If $\{v_1, v_2, v_3\}$ is a minimal good separation which is of type 2, and $v_2v_3 \in E(G)$, then G has a good pair of bonds.*

Proof. We suppose that $\{v_1, v_2, v_3\}$ is a separating set of type 2 where $v_2v_3 \in E(G)$. The graph G'_2 has a good pair of bonds $B'_{2j} = [P'_{2j}, Q'_{2j}],\ j = 1, 2$. If $P'_{2j} \cap \{v_1, v_2, v_3\} = \emptyset,\ j = 1, 2$, then $B_{2j} = B'_{2j},\ j = 1, 2$ is a good pair bonds of G. We may therefore assume that $P'_{21} \cap \{v_1, v_2, v_3\} \neq \emptyset$. We shall also assume that $P'_{22} \cap \{v_1, v_2, v_3\} \neq \emptyset$, as the case where the intersection is empty is easier and follows from the same arguments. By Lemma 5.2, $E(G'_1)$ is the edge-disjoint union of two G_1-good bonds $B'_{1j} = [P'_{1j}, Q'_{1j}],\ j = 1, 2$ and a contractible semi-bond S.

We consider two cases:

Case 1: Suppose for $j = 1, 2$ that $v_1 \in P'_{2j}$, and $v_2, v_3 \in Q'_{2j}$. We have that the dual H'_1 contains no good cycle which avoids u (corresponding to the face F in G'_1). Lemma 2.4 implies that H'_1 has a decomposition consisting of two good cycles C'_1 and C'_2, and a removable path P'. The vertex u is incident with two digons and an edge e, where e corresponds to the edge v_2v_3. By Lemma 2.4, P' can be chosen so that it contains e, and consequently, $e \notin E(C'_i),\ i = 1, 2$. The cycles $C'_i,\ i = 1, 2$ correspond to good bonds $B'_i = [P'_{1i}, Q'_{1i}]$ in $G'_1,\ i = 1, 2$. Since $e \notin E(C'_i),\ i = 1, 2$ we have that $v_2v_3 \notin B'_i,\ i = 1, 2$. Thus we may assume that $v_1 \in P'_{1i}$, (and $v_2, v_3 \in Q'_{1i}$,) for $i = 1, 2$, and $P_{1i} \neq$

$\{v_1\},\ i = 1, 2$. Let $B_1 = [P_{11} \cup P_{22}, Q_{11} \cup Q_{22}]$ and $B_2 = [P_{12} \cup P_{21}, Q_{12} \cup Q_{21}]$. Since $v_2v_3 \in E(G)$, one sees that $G(Q_{11} \cup Q_{22})$ and $G(Q_{12} \cup Q_{21})$ are connected. Thus B_1 and B_2 are non-trivial bonds, which are also cross-bonds. Since $dist_G(v_1, v_2) = dist_G(v_1, v_3) = 2$, and $v_2v_3 \in E(G)$, one sees that $\langle v_i \rangle_{B_1} \langle v_j \rangle_{B_1} \in E(G/B_1),\ \forall i \neq j$, and the same holds for B_2 as well. It now follows by Claim 9, that $B_i,\ i = 1, 2$ is a good pair of bonds in G.

Case 2: Suppose $v_1 \in P'_{21}$, (and $v_2, v_3 \in Q'_{21}$), and $v_2 \notin P'_{21}$. We can assume without loss of generality that $v_2 \in P'_{22}$ and $v_1, v_3 \in Q'_{22}$. We can, according to Lemma 2.4, choose a decomposition of H'_1 consisting of two good cycles C'_1 and C'_2, and a removable path P' such that the corresponding good bonds and contractible semi-bond, which we can assume are $B'_{1i},\ i = 1, 2$, and S, are such that $v_1 \in P'_{11}$ (and $v_2, v_3 \in Q'_{11}$) and $v_2 \in P'_{12}$ (and $v_1, v_3 \in Q'_{12}$). We may assume that the decomposition $\{C'_1, C'_2, P'\}$ is H_1-good, since if it is not, then we can swap pairs of members to achieve one which is. This means that we can assume that $\{B'_1, B'_2, S\}$ is a G_1-good decomposition, and hence $P_{1i} \backslash V(K) \neq \emptyset,\ i = 1, 2$. Let $B_1 = [P_{11} \cup P_{21}, Q_{11} \cup Q_{21}]$ and $B_2 = [P_{21} \cup P_{12}, Q_{12} \cup Q_{22}]$. One sees that B_1 is a cross-bond of G (since $v_2v_3 \in E(G)$). To show that B_2 is a cross-bond, we note that $dist_{G_1}(v_1, v_3) = 2$, and hence there is a path v_1zv_3 in G_1. If $z \in P_{12}$, then $zv_1, zv_3 \in B'_{12}$. However, B'_{12} is contractible in G'_1, and hence this is impossible. Thus $z \in Q_{12}$, and $G(Q_{12} \cup Q_{22})$ is connected. This shows that B_2 is a non-trivial bond of G, which is also seen to be a cross-bond.

As in the previous case, one can show that B_1 is contractible. To show that B_2 is contractible, we note that $v_2v_3 \in B_2$. Thus $\langle v_2 \rangle_{B_2} = \langle v_3 \rangle_{B_2}$, and by Claim 8, B_2 is contractible. We conclude that B_1 and B_2 are a good pair of bonds. This completes Case 2.

The proof of the claim now follows from Cases 1 and 2. □

10.2. *The case $v_2v_3 \notin E(G)$*

In the rest of this section, we may assume that $v_2v_3 \notin E(G)$. We define the triangle-free graphs

$$\mathbf{G}''_1 = (\mathbf{G}'_1 \backslash \{\mathbf{v}_2\mathbf{v}_3\}) \cup \{\mathbf{w}^1_{23}, \mathbf{w}^1_{23}\mathbf{v}_2, \mathbf{w}^1_{23}\mathbf{v}_3\},$$

$$\mathbf{G}''_2 = (\mathbf{G}'_2 \backslash \{\mathbf{v}_2\mathbf{v}_3\}) \cup \{\mathbf{w}_2, \mathbf{w}^2_{23}, \mathbf{w}_2\mathbf{v}_1, \mathbf{w}_2\mathbf{v}_2, \mathbf{w}_2\mathbf{v}_3, \mathbf{w}^2_{23}\mathbf{v}_2, \mathbf{w}^2_{23}\mathbf{v}_3\}.$$

The graph G has no good bond contained in $E(G''_1)$ for such bonds are good in $E(G_1)$, violating the fact that $\{v_1, v_2, v_3\}$ is a good separation. The graph G''_2 has a good pair of bonds $B''_{2j} = [P''_{2j}, Q''_{2j}],\ j = 1, 2$. We shall assume that $|P''_{2j} \cap \{v_1, v_2, v_3\}| = 1,\ j = 1, 2$; the other cases where $P''_{2j} \cap \{v_1, v_2, v_3\} = \emptyset$ for some $j \in \{1, 2\}$ are easier and can be dealt with using similar arguments.

Claim 30. *If $|K_{23}| = 5$, in G'_1, then G has a good pair of bonds.*

Proof. We assume that $|K_{23}| = 5$ where $K_{23} = v_2xyzv_3v_2$. Thus all faces of G''_1 are 4-faces apart from the faces $v_2xyzv_3w^1_{23}v_2$ and $v_1w^1_{12}v_2w^1_{23}v_3w^1_{13}v_1$. Thus G''_1 has a G_1-good decomposition consisting of three G_1-good bonds $B''_{1j} = [P''_{1j}, Q''_{1j}],\ j = 1, 2, 3$ where we may assume that $v_i \in P''_{1j}$ iff $i = j$. For $i, j = 1, 2$ we shall write $\langle G_i \rangle_{ij}$ to mean

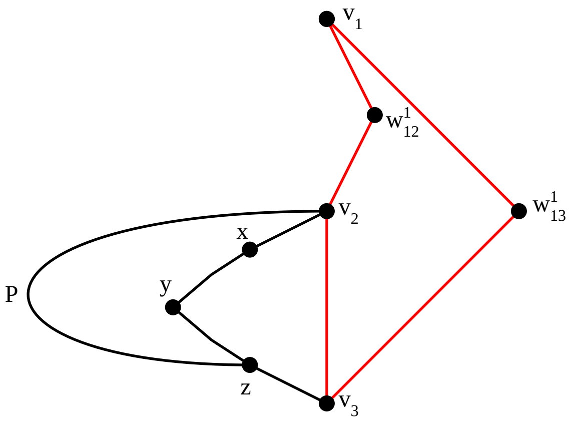

Fig. 8.

$G_i/(B''_{ij} \cap E(G_i))$. Similarly, for $k = 1, 2, 3$ and $i, j = 1, 2$ we shall write $\langle v_k \rangle_{ij}$ to mean the vertex $\langle v_k \rangle_{B''_{ij} \cap E(G_i)}$ in $\langle G_i \rangle_{ij}$. We shall consider two cases:

Case 1: Suppose there is a path from $\langle v_2 \rangle_{11}$ to $\langle v_3 \rangle_{11}$ in $\langle G_1 \rangle_{11} \backslash \langle v_1 \rangle_{11}$.

We shall consider two subcases:

Case 1.1: Suppose $v_1 \in P''_{21}$ and $v_2 \in P''_{22}$. Let $\mathbf{B}_1 = [(\mathbf{P}''_{11} \cup \mathbf{P}''_{21}) \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}''_{11} \cup \mathbf{Q}''_{21}) \cap \mathbf{V}(\mathbf{G})]$.

(i) Suppose that B_1 is not a bond. Then $(Q''_{11} \cup Q''_{21}) \cap V(G)$ induces a subgraph with two components. Let Q^j, $j = 2, 3$ be the vertices in the component containing v_j. Let $C_2 = [Q^2, V(G) \backslash Q^2]$. Suppose $Q^2 \backslash \{v_2\} \neq \emptyset$. Then C_2 is a non-trivial bond. Since $dist_G(v_1, v_i) = 2$, $i = 2, 3$ we have that $\langle v_1 \rangle_{B_1} \langle v_i \rangle_{B_1}$ is an edge of G/B_1 for $i = 2, 3$. Thus there is a path from $\langle v_1 \rangle_{C_2}$ to $\langle v_2 \rangle_{C_2}$ in $(G/C_2) \backslash \langle v_3 \rangle_{C_2}$ and from $\langle v_1 \rangle_{C_2}$ to $\langle v_3 \rangle_{C_2}$ in $(G/C_2) \backslash \langle v_2 \rangle_{C_2}$. By assumption, we have $\langle G_1 \rangle_{11}$ contains a path from $\langle v_2 \rangle_{11}$ to $\langle v_3 \rangle_{11}$ in $\langle G_1 \rangle_{11} \backslash \langle v_1 \rangle_{11}$. Thus there is a path from $\langle v_2 \rangle_{C_2}$ to $\langle v_3 \rangle_{C_2}$ in $(G/C_2) \backslash \langle v_1 \rangle_{C_2}$. One sees that C_2 is a good bond of G.

Suppose that $Q^2 \backslash \{v_2\} = \emptyset$. We redefine C_2 as $C_2 = [P''_{12} \cap V(G), \overline{P''_{12} \cap V(G)}]$. One sees that C_2 is a non-trivial bond. We shall show that C_2 is good. If C_2 is non-contractible, then G/C_2 consists of 2 blocks, one containing $\langle v_1 \rangle_{C_2}, \langle v_2 \rangle_{C_2}$ and another containing $\langle v_2 \rangle_{C_2}, \langle v_3 \rangle_{C_2}$. Note that the blocks restricted to $\langle G_1 \rangle_{12}$ are both multiple edges. We have that C_2 contains exactly one edge of the path $v_2xyzv_3 \subset K_{23}$ since it contains exactly two edges of the cycle $v_2xyzv_3w^1_{23}$, one of which is one of the edges $v_2w^1_{23}$ or $v_3w^1_{23}$. Suppose $v_3z \notin C_2$. Then $\langle z \rangle_{C_2} = \langle v_2 \rangle_{C_2}$ and there is a path P in $G_1(C_2 \cap E(G_1))$ from z to v_2. Since $Q^2 \backslash \{v_2\} = \emptyset$, it follows that $xv_2 \in B''_{11}$ and thus $\langle X \rangle_{C_2} = \langle v_1 \rangle_{C_2}$. However, considering the planarity of G''_1, any path from x to v_1 or v_3 must intersect a vertex of P (see Fig. 8). This implies that $\langle X \rangle_{C_2} = \langle v_2 \rangle_{C_2}$, yielding a contradiction. Suppose instead that $v_3z \in C_2$. Then $\langle y \rangle_{C_2} = \langle v_2 \rangle_{C_2}$. There is a path P in $G_1(C_2 \cap E(G_1))$ from y to v_2. By planarity, any path from x to v_1 must intersect a vertex of P. This means that $\langle X \rangle_{C_2} = \langle v_2 \rangle_{C_2}$, yielding a contradiction. We conclude that C_2 is contractible and hence good.

In the same way, we can define a bond C_3 where $C_3 = [Q^3, \overline{V(G)\backslash Q^3}]$ if $Q^3\backslash\{v_3\} \neq \emptyset$, and $C_3 = [P''_{13} \cap V(G), \overline{P''_{13} \cap V(G)}]$, otherwise. One can show that C_3 is good in the same way as was done for C_2, and it follows that C_2 and C_3 are a good pair of bonds. Thus we may assume that B_1 is a bond, and B_1 is seen to be good.

(ii) Suppose B_1 is a bond. Let $\mathbf{B}_2 = [(\mathbf{P}''_{12} \cup \mathbf{P}''_{22}) \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}''_{12} \cup \mathbf{Q}''_{22}) \cap \mathbf{V}(\mathbf{G})]$. Then B_2 is a non-trivial bond (since $dist_G(v_1, v_3) = 2$). We may assume that B_2 is non-contractible. Then G/B_2 consists of two blocks, one of which contains $\langle v_1\rangle_{B_2}$ and $\langle v_2\rangle_{B_2}$. Since B_1 is assumed to be a good bond, there is a path P in $G(Q''_{11} \cup Q''_{21}) \cap V(G))$ between v_2 and v_3. Since any path from P to v_1 must contain edges of B''_{11}, it follows that $\langle v_1\rangle_{B_2} \notin \langle P\rangle_{B_2}$ and consequently there is a path from $\langle v_2\rangle_{B_2}$ to $\langle v_3\rangle_{B_2}$ in $(G/B_2)\backslash\langle v_1\rangle_{B_2}$. Thus the second block of G/B_2 contains $\langle v_2\rangle_{B_2}$ and $\langle v_3\rangle_{B_2}$.

Applying the same reasoning as was used for C_2 in the previous paragraph, we deduce that G/B_2 cannot consist of two blocks, one containing $\langle v_1\rangle_{B_2},\ \langle v_2\rangle_{B_2}$, and another block containing $\langle v_2\rangle_{B_2},\ \langle v_3\rangle_{B_2}$. So it must be the case that B_2 is contractible, and hence B_1 and B_2 are a good pair of bonds. This completes Case 1.1.

If $v_1 \in P''_{21}$ and $v_3 \in P''_{22}$, then we can find a good pair of bonds in the same way as in the previous case. So essentially there is just one remaining subcase:

Case 1.2: Suppose $v_2 \in P''_{21}$ and $v_3 \in P''_{22}$. Let

$$\mathbf{B}_1 = [(\mathbf{P}''_{12} \cup \mathbf{P}''_{21}) \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}''_{12} \cup \mathbf{Q}''_{21}) \cap \mathbf{V}(\mathbf{G})],$$

$$\mathbf{B}_2 = [(\mathbf{P}''_{13} \cup \mathbf{P}''_{22}) \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}''_{13} \cup \mathbf{Q}''_{22}) \cap \mathbf{V}(\mathbf{G})].$$

Using the fact that $dist_{G_1}(v_1, v_j) = 2,\ j = 2, 3$, one can show that B_1 and B_2 are (non-trivial) bonds. Suppose B_1 is non-contractible. Then G/B_1 consists of two blocks; if these blocks contain $\langle v_1\rangle_{B_1}, \langle v_2\rangle_{B_1}$ and $\langle v_2\rangle_{B_1}, \langle v_3\rangle_{B_1}$, respectively, then by arguing in a manner similar to the above, we reach a contradiction. Thus we may assume that G/B_1 consists of two blocks, one containing $\langle v_1\rangle_{B_1}, \langle v_2\rangle_{B_1}$, and another containing $\langle v_1\rangle_{B_1},\ \langle v_3\rangle_{B_1}$. It follows that $G_1(Q''_{11} \cap V(G))$ is disconnected and has two components. Let $Q^j_1,\ j = 2, 3$ be the vertices in the component containing v_j. If $Q^j_1 \cup P''_{2(j-1)}\backslash\{v_j\} \neq \emptyset$, then let $C_j = [(Q^j_1 \cup P''_{2(j-1)}) \cap V(G), \overline{(Q^j_1 \cup P''_{2(j-1)}) \cap V(G)}]$; otherwise, for $j = 1, 2$ let $C_j = [P''_{1j} \cap V(G), \overline{P''_{1j} \cap V(G)}]$. One sees that $C_j,\ j = 2, 3$ are good bonds and hence form a good pair.

The same reasoning holds if B_2 is not good. Thus either B_1 and B_2 are a good pair of bonds, or we can find another good pair of bonds. This completes the proof of Case 1.2.

The proof of Case 1 follows from Cases 1.1 and 1.2.

Case 2: Suppose there is no path from $\langle v_2\rangle_{11}$ to $\langle v_3\rangle_{11}$ in $\langle G_1\rangle_{11}\backslash\langle v_1\rangle_{11}$. The graph $\langle G_1\rangle_{11}$ consists of two blocks, which are multiple edges, one containing $\langle v_1\rangle_{11},\ \langle v_2\rangle_{11}$ and another containing $\langle v_1\rangle_{11},\ \langle v_3\rangle_{11}$. For $i = 1, 2, 3$ let $\mathbf{V}_i = \{\mathbf{v} \in \mathbf{V}(\mathbf{G}_1) : \langle \mathbf{v}\rangle_{11} = \langle \mathbf{v}_i\rangle_{11}\}$. Since $\langle G_1\rangle_{11}$ consists of just three vertices $\langle v_i\rangle_{11},\ i = 1, 2, 3$, it follows that $V(G_1) = V_1 \cup V_2 \cup V_3$, $V_2 = P''_{12} \cap V(G)$, and $V_3 = P''_{13} \cap V(G)$. There are no edges from V_2 to V_3, for otherwise $\langle G_1\rangle_{11}$ would contain a path from $\langle v_2\rangle_{11}$ to $\langle v_3\rangle_{11}$ which avoids $\langle v_1\rangle_{11}$, contradicting our assumption. Thus $[V_1, V(G_1)\backslash V_1] = (B''_{12} \cup B''_{13}) \cap E(G_1)$. We also have that $Q''_{13} \cap V(G_1) = V_1 \cup V_2$ and $Q''_{12} \cap V(G_1) = V_1 \cup V_3$.

Let $\mathbf{G}_2''' = \mathbf{G}_2''\backslash\{\mathbf{w}_{23}^2\}$. The graph G_2''' has a good pair of bonds $\mathbf{B}_{2j}''' = [\mathbf{P}_{2j}''', \mathbf{Q}_{2j}''']$, $j = 1, 2$. We shall assume that $|P_{2j}''' \cap \{v_1, v_2, v_3\}| = 1$, $j = 1, 2$; the other cases, where $P_{2j}''' \cap \{v_1, v_2, v_3\} = \emptyset$ for some $j \in \{1, 2\}$, can be handled in the same way. We shall examine a few subcases:

Case 2.1: Suppose $v_1 \in P_{21}'''$, and $v_1 \in P_{22}'''$. We have that w_2 belongs to exactly one of P_{21}''' or P_{22}'''. We may assume that $w_2 \in P_{21}'''$. Let

$$\mathbf{B}_1 = [(\mathbf{P}_{11}'' \cup \mathbf{P}_{22}''') \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}_{11}'' \cup \mathbf{Q}_{22}''') \cap \mathbf{V}(\mathbf{G})],$$

$$\mathbf{B}_2 = [(\mathbf{V}_1 \cup \mathbf{P}_{21}''') \cap \mathbf{V}(\mathbf{G}), (\mathbf{V}_2 \cup \mathbf{V}_3 \cup \mathbf{Q}_{21}''') \cap \mathbf{V}(\mathbf{G})].$$

We have that $V_1\backslash\{v_1\} \neq \emptyset$ as $\langle v_1\rangle_{11}$ is a cut-vertex of $\langle G_1\rangle_{11}$. Since $w_2 \in P_{21}'''$, it follows that $G(Q_{21}''' \cap V(G))$ is connected (since B_{21}''' is a bond). Thus B_2 is a non-trivial bond. Given that $G(Q_{21}''' \cap V(G))$ is connected, it contains a path P from v_2 to v_3. Since any path from P to v_1 must contain edges of B_2, this implies that $\langle P\rangle_{B_1}$ contains a path in $(G/B_1)\backslash\langle v_1\rangle_{B_1}$ from $\langle v_2\rangle_{B_1}$ to $\langle v_3\rangle_{B_1}$. We conclude that B_1 is contractible, and if it is a bond, then it is good.

If B_1 is not a bond, then $G_2(Q_{22}''' \cap V(G))$ has 2 components. For $j = 2, 3$ let Q_2^j be the vertices in the component containing v_j. For $j = 2, 3$, let

$$C_j = [(P_{1j}'' \cup Q_2^j) \cap V(G), \overline{(P_{1j}'' \cup Q_2^j) \cap V(G)}].$$

Consider C_2. Suppose that C_2 is non-contractible. Then G/C_2 consists of two blocks where one block contains $\langle v_1\rangle_{C_2}$ and $\langle v_2\rangle_{C_2}$. Since B_{22}''' is good, G_2'''/B_{22}''' is 2-connected and there is a path in $(G_2/B_{22}''')\backslash\langle v_1\rangle_{B_{22}'''}$ from $\langle v_2\rangle_{B_{22}'''}$ to $\langle v_3\rangle_{B_{22}'''}$. Thus there is a path in $(G/C_2)\backslash\langle v_1\rangle_{C_2}$ from $\langle v_2\rangle_{C_2}$ to $\langle v_3\rangle_{C_2}$, and consequently the other block of G/C_2 contains $\langle v_2\rangle_{C_2}$ and $\langle v_3\rangle_{C_2}$. Now following the same arguments as in Case 1, one can show that this is impossible. Thus C_2 is contractible and hence good. In the same way, it can be shown that C_3 is also good and hence C_2 and C_3 are a good pair. We may therefore assume that B_1 is a good bond.

Consider B_2. Since B_1 is assumed to be a bond, it holds that $G((Q_{11}'' \cup Q_{22}''') \cap V(G))$ is connected and hence contains a path P from v_2 to v_3. Then $\langle v_1\rangle_{B_2} \notin \langle P\rangle_{B_2}$ and consequently there is a path in $(G/B_2)\backslash\langle v_1\rangle_{B_2}$ between $\langle v_2\rangle_{B_2}$ and $\langle v_3\rangle_{B_2}$. We deduce that B_2 is contractible and hence also good. In this case, B_1 and B_2 are a good pair of bonds. This completes Case 1.2.

Case 2.2: Suppose $v_1 \in P_{21}'''$ and $v_2 \in P_{22}'''$. Let

$$\mathbf{B}_1 = [(\mathbf{P}_{11}'' \cup \mathbf{P}_{21}''') \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}_{11}'' \cup \mathbf{Q}_{21}''') \cap \mathbf{V}(\mathbf{G})],$$

$$\mathbf{B}_2 = [(\mathbf{P}_{12}'' \cup \mathbf{P}_{22}''') \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}_{12}'' \cup \mathbf{Q}_{22}''') \cap \mathbf{V}(\mathbf{G})].$$

We first note that $w_2 \notin P_{21}'''$ as $v_2 \in P_{22}'''$. Suppose that B_1 is not a bond. As in Case 2.1, we define C_2 and C_3. Since C_2 is a bond and G_2'''/B_{21}''' is 2-connected, we can find a path from $\langle v_2\rangle_{C_2}$ to $\langle v_3\rangle_{C_3}$ in $(G_2'''/C_2)\backslash\langle v_1\rangle_{C_2}$ (via the same arguments in the previous case) and this implies that C_2 is good. We can argue the same for C_3, and hence C_2 and C_3 are a good pair of bonds. We may thus assume that B_1 is a bond, and it is seen to be good.

We suppose therefore that B_2 is non-contractible (noting that B_2 is a non-trivial bond). Similar to Case 1, one can show that G/B_2 consists of 2 blocks, one containing $\langle v_1\rangle_{B_2}$, $\langle v_2\rangle_{B_2}$

and another containing $\langle v_1\rangle_{B_2}, \langle v_3\rangle_{B_2}$. Since B_1 is assumed to be a bond, we have that $G((Q''_{11} \cup Q'''_{21}) \cap V(G))$ is connected and contains a path P from v_2 to v_3. We have that $\langle v_1\rangle_{B_2} \notin \langle P\rangle_{B_2}$. Thus there is a path in $(G/B_2)\backslash\langle v_1\rangle_{B_2}$ from $\langle v_2\rangle_{B_2}$ to $\langle v_3\rangle_{B_2}$. This contradicts the fact that $\langle v_1\rangle_{B_2}$ is a cut-vertex of G/B_2. Thus B_2 is contractible, and B_1 and B_2 are a good pair of bonds. This completes Case 2.2.

If $v_1 \in P'''_{21}$ and $v_3 \in P'''_{22}$, then one can find a good pair of bonds in exactly the same way as in Case 2.2. There is just one case remaining:

Case 2.3: Suppose $v_2 \in P'''_{21}$ and $v_3 \in P'''_{22}$. Let

$$\mathbf{B}_1 = [(\mathbf{P}''_{12} \cup \mathbf{P}'''_{21}) \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}''_{12} \cup \mathbf{Q}'''_{21}) \cap \mathbf{V}(\mathbf{G})],$$

$$\mathbf{B}_2 = [(\mathbf{P}''_{13} \cup \mathbf{P}'''_{22}) \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}''_{13} \cup \mathbf{Q}'''_{22}) \cap \mathbf{V}(\mathbf{G})].$$

Both B_1 and B_2 are non-trivial bonds. Suppose B_1 is non-contractible.

Then G/B_1 consists of two blocks, one containing $\langle v_1\rangle_{B_1}, \langle v_2\rangle_{B_1}$. Following the reasoning as in Case 1.1, one can show that the other block does not contain $\langle v_2\rangle_{B_1}$ and $\langle v_3\rangle_{B_1}$. Thus we have that the other block contains $\langle v_1\rangle_{B_1}$ and $\langle v_3\rangle_{B_1}$. Moreover, the block containing $\langle v_1\rangle_{B_1}, \langle v_2\rangle_{B_1}$ is a multiple edge. Since there is no path from $\langle v_2\rangle_{11}$ to $\langle v_3\rangle_{11}$ in $\langle G_1\rangle_{11}\backslash\langle v_1\rangle_{11}$ it follows that $G_1(Q''_{11} \cap V(G))$ is disconnected and has two components. Let Q_1^j, $j = 2, 3$ be the vertices of the component containing v_j. Let $C_2 = [(Q_1^2 \cup P'''_{21}) \cap V(G), \overline{(Q_1^2 \cup P'''_{21}) \cap V(G)}]$. If $P'''_{21} \cap V(G) = \{v_2\}$, then there would be a path in $(G/B_1)\backslash\langle v_2\rangle_{B_1}$ from $\langle v_2\rangle_{B_1}$ to $\langle v_3\rangle_{B_1}$. This contradicts the fact that $\langle v_1\rangle_{B_1}$ is a cut-vertex in G/B_1. Thus $P'''_{21} \cap V(G) \neq \{v_2\}$, and C_2 is a non-trivial bond.

We shall show that C_2 is contractible.

(i) Suppose that $xv_2 \in B''_{12}$. Then $xy \in B''_{11}$. We have $\langle X\rangle_{B''_{11}} = \langle v_1\rangle_{B''_{11}}$, and there is a path L in $G_1(B''_{11} \cap E(G_1))$ from x to v_1. We can assume that L is chosen such that it contains no vertices of Q_1^3; for if no such path existed, then $\langle X\rangle_{C_2} \neq \langle v_1\rangle_{C_2}$, and C_2 would be contractible. Suppose $y \notin V(L)$. Let R be the region bounded by $L \cup \{xv_2w^1_{12}v_1\}$ where y does not lie in R. We have that the vertices of $V_2\backslash\{v_2\}$ lie in the interior of R. We have that $\langle y\rangle_{B_1} = \langle v_1\rangle_{B_1}$. Thus there is a path in $G_1(B''_{12} \cap E(G_1))$ from y to v_1, and y is adjacent to a vertex in $P''_{12} \cap V(G) = V_2$. However, this is impossible since y lies outside R.

Suppose $y \in V(L)$. Then y is adjacent to a vertex $y' \in V(L)\backslash\{x\}$. We have that $y' \in Q_1^2$. Again let R be the region bounded by $L \cup \{xv_2w^1_{12}v_1\}$, where z lies outside R. Since $x, y' \in Q_1^2$, there is a path P_1 from x to y' in $G_1(Q_1^2)$. Since $\langle y\rangle_{B_1} = \langle v_1\rangle_{B_1}$, there is a path P_2 from y to v_1 in $G_1(B''_{12} \cap E(G_1))$. Such a path lies in R since the vertices of $V_2\backslash\{v_2\}$ lie in R (see Fig. 9). We conclude that by planarity, the paths P_1 and P_2 must cross. However, this is impossible since $V(P_1) \subset V(Q''_{11})$ and $V(P_2) \subset V(P''_{11})$. In this case, C_2 must be contractible.

(ii) Suppose $xv_2 \notin B''_{12}$. Then $xv_2 \in B''_{11}$. If $xy \in B''_{11}$, then $y \in Q_1^3$ and C_2 is seen to be good since there would be a path between $\langle v_2\rangle_{C_2}$ and $\langle v_3\rangle_{C_2}$ in $(G/C_2)\backslash\langle v_1\rangle_{C_2}$. We may thus assume that $xy \notin B''_{11}$ and hence $xy \in B''_{12}$. Thus there is a path $L_1 \subset G(P''_{11})$ from y to v_1. We also have that y is adjacent to a vertex $y' \in Q_1^2$ and there is a path $L_2 \subset G(Q_1^2)$ from y' to v_2. Due to planarity considerations, the paths L_1 and L_2 must cross, which is impossible since $L_2 \subseteq P''_{11}$. We reach a contradiction, and we conclude that C_2 must be contractible in this case.

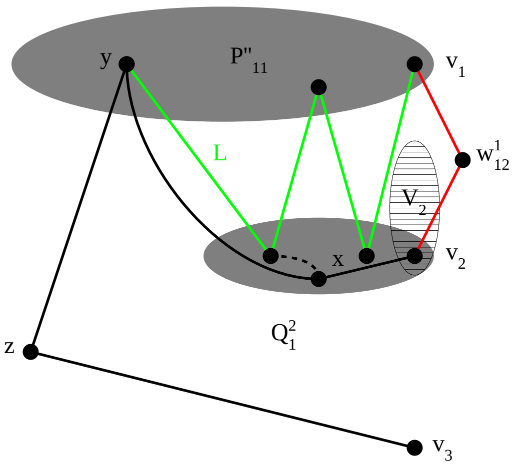

Fig. 9.

We have thus shown that if B_1 is non-contractible, then C_2 is good. If B_2 is good, then either B_1, B_2 or C_2, B_2 is a good pair of bonds. We suppose therefore that B_2 is non-contractible. Let $C_3 = [(Q_1^3 \cup P_{22}'') \cap V(G), \overline{(Q_1^3 \cup P_{22}'') \cap V(G)}]$. As with C_2, we have that C_3 is a good bond. Thus either B_1, C_3 or C_2, C_3 is a good pair of bonds. This completes the proof of Case 2.3. Case 2 now follows from Cases 2.1–2.3. This completes the proof of the claim. $\square$

Claim 31. *Suppose* $|K_{23}| = 4$ *in* G_1'. *Then* G *has a good pair of bonds.*

Proof. G_1'' contains exactly two 5-faces and has a G_1-good decomposition consisting of three G_1-good bonds $\mathbf{B}_{1j} = [\mathbf{P}_{1j}'', \mathbf{Q}_{1j}'']$, $j = 1, 2, 3$ and a contractible semi-bond $\mathbf{S}$. We may assume for $i, j = 1, 2, 3$ that $v_i \in P_{1j}''$ iff $i = j$.

Case 1. Suppose $v_1 \in P_{21}''$. Let

$$\mathbf{B}_1 = [(\mathbf{P}_{11}'' \cup \mathbf{P}_{21}'') \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}_{11}'' \cup \mathbf{Q}_{21}'') \cap \mathbf{V}(\mathbf{G})].$$

B_1 is seen to be a non-trivial bond. In the same way as was done in the proof of Claim 25, one can show that if B_1 is non-contractible, then it is possible to construct a good pair of bonds. Given this, we may assume that B_1 is a good bond.

Suppose $v_2 \in P_{22}''$. If $|K_{13}| = 5$, then let $\mathbf{G}_1''' = (\mathbf{G}_1' \backslash \{\mathbf{w}_{13}^1\}) \cup \{\mathbf{v}_1\mathbf{v}_3\}$. We have that G_1''' is triangle-free and has a G_1-good decomposition consisting of two G_1-good bonds $\mathbf{B}_{1j}''' = [\mathbf{P}_{1j}''', \mathbf{Q}_{1j}''']$, $j = 1, 2$ where $v_j \in P_{1j}'''$, $j = 1, 2$. We can now proceed in the same manner as in section 7 to show that G has a good pair of bonds. Consequently, we may assume that $|K_{13}| = 4$ and $dist_{G_1}(v_1, v_3) = 2$. Let

$$\mathbf{B}_2 = [(\mathbf{P}_{12}'' \cup \mathbf{P}_{22}'') \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}_{12}'' \cup \mathbf{Q}_{22}'') \cap \mathbf{V}(\mathbf{G})].$$

We see that B_2 is a non-trivial bond. Given that B_1 is assumed to be good, we may assume that B_2 is non-contractible. Since $dist_G(v_1, v_2) = 2$, we have that $\langle v_1\rangle_{B_2}\langle v_2\rangle_{B_2}$ is an edge of G/B_2. We have that $|K_{23}| = 4$, and consequently there is a path $P \subset K_{23}\backslash w_{23}^1$ from v_2 to v_3. We have that $V(P) \subset Q_{11}''$ and this implies that $\langle v_1\rangle_{B_2} \notin \langle P\rangle_{B_2}$, and there is a path in $\langle P\rangle_{B_2}$ from $\langle v_2\rangle_{B_2}$ to $\langle v_3\rangle_{B_2}$ which avoids $\langle v_1\rangle_{B_2}$. Thus G/B_2 consists of two blocks; one containing $\langle v_1\rangle_{B_2}, \langle v_2\rangle_{B_2}$ and another containing $\langle v_2\rangle_{B_2}, \langle v_3\rangle_{B_2}$.

Let $G^* = \langle G\rangle_S,\ B_2^* = \langle B_2\rangle_S,\ v_i^*,\ i = 1, 2, 3$. We have that G^*/B_2^* consists of two blocks; one containing $\langle v_1^*\rangle_{B_2^*}, \langle v_2^*\rangle_{B_2^*}$ and another containing $\langle v_2^*\rangle_{B_2^*}, \langle v_3^*\rangle_{B_2^*}$. Using the same methods as in the proof of Claim 20 (where B_2^* plays the role of B_1 and G^* plays the role of G) we can construct a good pair of bonds, say $C_i^*,\ i = 1, 2$ such that $C_i = > C_i^* <_S,\ i = 1, 2$, are non-trivial bonds. Suppose C_1 is non-contractible in G. Then Claim 22 implies that $\langle v_i^*\rangle_{C_1^*} = \langle v_j^*\rangle_{C_1^*}$ for some $i \neq j$ and there is a path of length 3 between v_i and v_j in K_{ij}. Since no such path exists other than for $i = 2$ and $j = 3$, we deduce that $\langle v_2^*\rangle_{C_1^*} = \langle v_3^*\rangle_{C_1^*}$ if C_1 is non-contractible. However, for the bonds $C_i^*,\ i = 1, 2$ constructed it holds that $\langle v_2^*\rangle_{C_1^*} \neq \langle v_3^*\rangle_{C_1^*}$ (see the remark following the proof of Claim 22). We conclude that C_1 is contractible, and the same applies to C_2. Thus C_1 and C_2 are a good pair of bonds.

If instead $v_3 \in P_{22}''$, then we let $B_2 = [(P_{13}'' \cup P_{22}'') \cap V(G), (Q_{13}'' \cup Q_{22}'') \cap V(G)]$. One can show in a similar manner as to the above that either B_2 is good (in which case B_1 and B_2 is a good pair), or one can construct another good pair of bonds. This completes the proof for Case 1.

Case 2: Suppose $v_2 \in P_{21}''$ and $v_3 \in P_{22}''$. Let

$$\mathbf{B}_1 = [(\mathbf{P}_{12}'' \cup \mathbf{P}_{21}'') \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}_{11}'' \cup \mathbf{Q}_{21}'') \cap \mathbf{V}(\mathbf{G})],$$

$$\mathbf{B}_2 = [(\mathbf{P}_{12}'' \cup \mathbf{P}_{22}'') \cap \mathbf{V}(\mathbf{G}), (\mathbf{Q}_{12}'' \cup \mathbf{Q}_{22}'') \cap \mathbf{V}(\mathbf{G})].$$

If $|K_{13}| = 4$, then using the same reasoning as in Case 1 with G^* etc., one can show that either B_1 and B_2 are a good pair of bonds or one can construct another such pair. We may therefore assume that $|K_{13}| = 5$. Again, using the same arguments as in Case 1 with G^* etc., one can show that either B_2 is good, or one can construct a good pair of bonds of G. We may therefore assume that B_2 is good and B_1 is not contractible. We have that $\langle v_1\rangle_{B_1}\langle v_2\rangle_{B_1}$ is an edge of G/B_1 and there is a path from $\langle v_2\rangle_{B_1}$ to $\langle v_3\rangle_{B_1}$ in $(G/B_1)\backslash\langle v_1\rangle_{B_1}$. Thus G/B_1 consists of two blocks; one containing $\langle v_1\rangle_{B_1},\ \langle v_2\rangle_{B_1}$ and another containing $\langle v_2\rangle_{B_1},\ \langle v_3\rangle_{B_1}$. Using the same technique as in the proof of Claim 25, we can construct a good pair of bonds. This completes Case 2.

The proof of the claim now follows from Cases 1 and 2 above. □

Claim 32. *If* $\{v_1, v_2, v_3\}$ *is a minimal good separation which is of type* 2, *then* G *has a good pair of bonds*.

Proof. The proof of the claim follows from Claims 29–31. □

11. Conclusion

In consideration of the results given in the previous sections, notably Claims 19, 21, 28, and 32, one deduces that no minimal counterexample H can exist, thereby concluding the proof of main theorem (Theorem 1.4). We venture the following conjecture for matroids:

Conjecture 11.1. *Let M be a connected binary matroid having cogirth at least* 4. *If M is not a circuit, and has no minor isomorphic to* $P_{10},\ M^*(K_5),\ F_7^*$, *or* R_{10}, *then M contains two disjoint circuits* C_1 *and* C_2 *such that* $M\backslash C_i,\ i=1,2$ *are connected, but* $M/C_i,\ i=1,2$ *are disconnected*.

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