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Symmetric multisplitting of a symmetric positive definite matrix

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Abstract

A parallel symmetric multisplitting method for solving a symmetric positive system $Ax = b$ is presented. Here the s.p.d. (symmetric positive definite) matrix A need not be assumed in a special form (e.g. the dissection form (R.E. White, SIAM J. Matrix Anal. Appl. 11 (1990) 69–82). The main tool for deriving our method is the diagonally compensated reduction (cf. (O. Axelsson, L. Kolotilina, Numer. Linear Algebra Appl. 1 (1994) 155–177)). The convergence of the presented parallel symmetric multisplitting method is also discussed by using this tool. © 1998 Elsevier Science Inc. All rights reserved.

Keywords: Multisplitting; Diagonally compensated reduction; Symmetric positive definite matrix

1. Introduction

Consider the solution of a large linear system of equations

$$Ax = b, \tag{1.1}$$

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where A is symmetric positive definite (s.p.d.). The multisplitting method was introduced by O'Leary and White [1] and was further studied by many others, see e.g. [2–5]. But the attention was mainly paid to monotone matrix (in particular M-matrix) and H-matrix. Only a few attention was paid to an s.p.d. matrix (cf. [6,1,5]) and the practically feasible parallel multisplitting method presented in [5] set strict demand on the coefficient matrix A being in the dissection form and required the weighting matrices $E_k, k = 1, \ldots, K$, being in some special forms. In this paper, by using the diagonally compensated reduction [7] we derive a symmetric multisplitting of a general s.p.d. matrix and discuss the convergence of the resulting parallel symmetric multisplitting method.

2. Preliminaries

We begin with some basic notation. For a matrix $A = (a_{ij}) \in \mathscr{R}^{n,n}, A \geqslant 0$ always means $a_{ij} \geqslant 0, i = 1, \ldots, n$.

A matrix $A = (a_{ij}) \in \mathscr{R}^{n,n}$ is called a Z-matrix if $a_{ij} \leqslant 0$ for $i \neq j$. If A is a nonsingular Z-matrix and $A^{-1} \geqslant 0$, then A is called an M-matrix. A is to be said an H-matrix if the comparison matrix $\langle A \rangle = (\alpha_{ij})$ with

$$\alpha_{ij} = \begin{cases} |a_{ii}| & \text{for } i = j \\ -|a_{ij}| & \text{for } i \neq j \end{cases}$$

is an M-matrix.

A splitting $A = M - N$ of A is called regular if $M^{-1} \geqslant 0$ and $N \geqslant 0$; (left) weak regular if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$; two-side weak regular if it is (left) weak regular and $NM^{-1} \geqslant 0$; convergent if $\rho(M^{-1}N) < 1$. Here $\rho(C)$ denotes the spectral radius of a matrix C.

We point out that in some literature, what we called a two-side regular splitting is known as weak regular splitting, see, e.g. Lanzkron et al. [8], Ortega et al. [9] and what we called (left) weak regular splitting is commonly used form, see, e.g. Berman and Plemmons [10].

As defined in [1] a multisplitting of A is a collection of triples of matrices $(M_k, N_k, E_k), k = 1, \ldots, K$, satisfying:

(i) $A = M_k - N_k, k = 1, \ldots, K$;

(ii) M_k is nonsingular for $k = 1, \ldots, K$;

(iii) $E_k, k = 1, \ldots, K$, are diagonal matrices with nonnegative entries such that $\sum_{k=1}^{K} E_k = I$.

Now let us turn to the conception of the diagonal compensation reduction, the description of the method of diagonal compensation reduction is word by word taken from [7]. Let A be an s.p.d. matrix, let R be symmetric and nonnegative. Consider the reduced matrix

$$B = A - R \tag{2.1}$$

and (arbitrarily) select a positive (weighting) vector v, then define the diagonally compensated reduced matrix of A

$$\widehat{A} = D + B, \tag{2.2}$$

where D is a diagonal matrix defined by

$$Dv = Rv, \tag{2.3}$$

D is the diagonal compensation matrix for the reduced entry matrix R. Note that $D \geqslant 0$ and $\widehat{A}v = Av$, then A becomes split as

$$A = \widehat{A} - (D - R). \tag{2.4}$$

Since $D \geqslant 0, R \geqslant 0$ and $(D - R)v = 0, D - R$ is positive semidefinite. Hence $\widehat{A} - A$ is positive semidefinite, and for any eigenvalue $\lambda_j(\widehat{A}^{-1}A)$ we have

$$\lambda_j(\widehat{A}^{-1}A) \leqslant 1. \tag{2.5}$$

Moreover, let

$$\widehat{A} = M - \widehat{N} \tag{2.6}$$

be a symmetric splitting with a positive definite matrix M. Then it holds the following convergent splitting result.

Lemma 1 (cf. [7]). *Let A be an $n \times n$ s.p.d. matrix and let $\widehat{A} = D + B$ be a diagonally compensated reduced matrix of A, then*

$$0 < \lambda_j(M^{-1}A) \leqslant \lambda_j(M^{-1}\widehat{A}), \quad j = 1, \ldots, n$$

and thus, the splitting

$$A = M - N$$

(where $N = M - A$) is convergent if the splitting $\widehat{A} = M - \widehat{N}$ is convergent.

We note that if the reduced matrix B is a Z-matrix (e.g. all positive off-diagonal entries of A are reduced by the reduced entry matrix R), then $\widehat{A} = D + B$ is a Z-matrix too. $\widehat{A} - A$ is positive semidefinite, $\widehat{A}$ is a Stieltjes matrix (i.e. $\widehat{A}$ is an s.p.d. Z-matrix). Hence, $\widehat{A}$ is an s.p.d. M-matrix (cf. [11]). Thus the problem of constructing a convergent splitting for an s.p.d. matrix, owing to the Lemma 1, can always be reduced to that for a Stieltjes matrix which is an s.p.d. M-matrix. Henceforth, when we construct a diagonally compensated reduced matrix $\widehat{A}$ from an s.p.d. matrix A we will always make $\widehat{A}$ a Stieltjes matrix.

3. Main results

Let A be an $n \times n$ s.p.d. matrix and let $(M_k, N_k, E_k), k = 1, \ldots, K$, be a multisplitting of A. Then we construct a parallel symmetric multisplitting algorithm as follows (cf. [4]).

$$x^{m+1} = x^m + Gr^m, \tag{3.1}$$

where

$$G = \frac{1}{2}\sum_{k=1}^{K}(E_k M_k^{-1} + M_k^{-T} E_k) \quad \text{and} \quad r^m = b - Ax^m, \tag{3.2}$$

the iteration matrix of Eq. (3.1) is

$$H = I - GA.$$

If G is nonsingular, the iteration can be induced by a single splitting of A

$$A = G^{-1} - G^{-1}H.$$

Now let us determine the multisplitting $(M_k, N_k, E_k), k = 1, \dots, K$, by using the diagonally compensated reduction. Let a diagonally compensated (Stieltjes) matrix $\widehat{A}$ of A have been constructed (see Section 2). Let $\widehat{A}$ be partitioned into $K \times K$ blocks.

$$\widehat{A} = \begin{pmatrix} \widehat{A}_1 & -C_{12} & \dots & -C_{1K} \\ -C_{21} & \widehat{A}_2 & \dots & -C_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ -C_{K1} & -C_{K2} & \dots & \widehat{A}_K \end{pmatrix}, \tag{3.3}$$

where $\widehat{A}_i, i = 1, \dots, K$, are $n_i \times n_i$ matrices with $\sum_{i=1}^{K} n_i = n$. Since $\widehat{A}$ is a Stieltjes matrix, $\widehat{A}_i, i = 1, \dots, K$, are Stieltjes matrices too, and $C_{ij} = C_{ji}^{\mathrm{T}} \geqslant 0$, for $i \neq j$ and $i, j = 1, \dots, K$.

Let $\widehat{A}_i = B_i - C_i$ be a splitting of $\widehat{A}_i$ for $i = 1, \dots, K$. We construct a multisplitting $(M_k, \widehat{N}_k, E_k), k = 1, \dots, K$, of $\widehat{A}$ in the form:

$$M_k = \begin{pmatrix} B_1 & & & & & \\ & \ddots & & & & \\ & & B_k & & & \\ & & -C_{k+1,k} & B_{k+1} & & \\ & & \vdots & & \ddots & \\ & & -C_{Kk} & & & B_K \end{pmatrix}, \tag{3.4}$$

$$\widehat{N}_k = M_k - \widehat{A},$$

$$E_k = \begin{pmatrix} d_1^{(k)} I_1 & & & & \\ & \ddots & & & \\ & & d_k^{(k)} I_k & & \\ & & & \ddots & \\ & & & & d_K^{(k)} I_K \end{pmatrix}, \tag{3.5}$$

where $0 \leqslant d_i^{(k)} \leqslant 1, \sum_{k=1}^{K} d_i^{(k)} = 1$, and I_i, $i = 1, \dots, K$, are $n_i \times n_i$ identity matrices.

Obviously, when a multisplitting $(M_k, \widehat{N}_k, E_k), k = 1, \dots, K$, of $\widehat{A}$ is defined, a multisplitting $(M_k, N_k, E_k), k = 1, \dots, K$, of A is also defined. The difference between these two multisplittings is that $\widehat{N}_k = M_k - \widehat{A}$, while $N_k = M_k - A$, $k = 1, \dots, K$. From Eq. (3.4) we have

$$M_k^{-1} = \begin{pmatrix} B_1^{-1} & & & & \\ & \ddots & & & \\ & & B_k^{-1} & & \\ & & B_{k+1}^{-1} C_{k+1,k} B_k^{-1} & B_{k+1}^{-1} & \\ & & \vdots & & \ddots \\ & & B_K^{-1} C_{Kk} B_k^{-1} & & & B_K^{-1} \end{pmatrix}. \tag{3.6}$$

Some simple but tedious matrix manipulations show

$$\begin{aligned} G &= \frac{1}{2} \sum_{k=1}^{K} (E_k M_k^{-1} + M_k^{-\mathrm{T}} E_k) \\ &= \begin{pmatrix} \frac{1}{2}(B_1^{-1} + B_1^{-\mathrm{T}}) & \frac{d_2^{(1)}}{2} B_1^{-\mathrm{T}} C_{12} B_2^{-\mathrm{T}} & \dots & \frac{d_K^{(1)}}{2} B_1^{-\mathrm{T}} C_{1K} B_K^{-\mathrm{T}} \\ \frac{d_2^{(1)}}{2} B_2^{-1} C_{21} B_1^{-1} & \frac{1}{2}(B_2^{-1} + B_2^{-\mathrm{T}}) & \dots & \frac{d_K^{(2)}}{2} B_2^{-\mathrm{T}} C_{2K} B_K^{-\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_K^{(1)}}{2} B_K^{-1} C_{K1} B_1^{-1} & \frac{d_K^{(2)}}{2} B_K^{-1} C_{K2} B_2^{-1} & \dots & \frac{1}{2}(B_K^{-1} + B_K^{-\mathrm{T}}) \end{pmatrix}, \end{aligned} \tag{3.7}$$

$$\widehat{H} = I - G\widehat{A} = \begin{pmatrix} \hat{h}_{11} & \hat{h}_{12} & \dots & \hat{h}_{1K} \\ \hat{h}_{21} & \hat{h}_{22} & \dots & \hat{h}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{h}_{K1} & \hat{h}_{K2} & \dots & \hat{h}_{KK} \end{pmatrix}, \tag{3.8}$$

where

$$\hat{h}_{11} = \frac{1}{2}(B_1^{-1}C_1 + B_1^{-\mathrm{T}}C_1^{\mathrm{T}}) + \frac{1}{2}\sum_{j=2}^{K} d_j^{(1)} B_1^{-\mathrm{T}} C_{1j} B_j^{-\mathrm{T}} C_{j1},$$

$$\hat{h}_{12} = \frac{1}{2}(1 - d_2^{(1)})B_1^{-\mathrm{T}}C_{12} + \frac{1}{2}B_1^{-1}C_{12} + \frac{d_2^{(1)}}{2}B_1^{-\mathrm{T}}C_{12}B_2^{-\mathrm{T}}C_2^{\mathrm{T}} + \frac{1}{2}\sum_{j=3}^{K} d_j^{(1)} B_1^{-\mathrm{T}} C_{1j} B_j^{-\mathrm{T}} C_{j2},$$

$$,\ldots,$$

$$\hat{h}_{1K} = \frac{1}{2}(1 - d_K^{(1)})B_1^{-\mathrm{T}}C_{1K} + \frac{1}{2}B^{-1}C_{1K} + \frac{d_K^{(1)}}{2}B_1^{-\mathrm{T}}C_{1K}B_K^{-\mathrm{T}}C_K^{\mathrm{T}} + \frac{1}{2}\sum_{j=2}^{K-1} d_j^{(1)} B_1^{-\mathrm{T}} C_{jK},$$

$$\hat{h}_{21} = \frac{1}{2}(1 - d_2^{(1)})B_2^{-1}C_{21} + \frac{1}{2}B_2^{-\mathrm{T}}C_{21} + \frac{d_2^{(1)}}{2}B_2^{-1}C_{21}B_1^{-1}C_1 + \frac{1}{2}\sum_{j=3}^{K} d_j^{(2)} B_j^{-\mathrm{T}} C_{2j} B_j^{-\mathrm{T}} C_{j1},$$

$$\hat{h}_{22} = \frac{1}{2}(B_2^{-1}C_2 + B_2^{-\mathrm{T}}C_2^{\mathrm{T}}) + \frac{d_2^{(1)}}{2}B_2^{(1)}C_{21}B_1^{-1}C_{12} + \frac{1}{2}\sum_{j=3}^{K} d_j^{(2)} B_2^{-\mathrm{T}} C_{2j} B_j^{-\mathrm{T}} C_{j2},$$

$$,\ldots,$$

$$\hat{h}_{2K} = \frac{d_2^{(1)}}{2}B_2^{-1}C_{21}B_1^{-1}C_{1K} + \frac{1}{2}(1 - d_K^{(2)})B_2^{-\mathrm{T}}C_{2K} + \frac{1}{2}B_2^{-1}C_{2K} + \frac{d_K^{(2)}}{2}B_2^{-\mathrm{T}}C_{2K}B_K^{-\mathrm{T}}C_K^{\mathrm{T}} + \frac{1}{2}\sum_{j=3}^{K-1} d_j^{(2)} B_2^{-\mathrm{T}} C_{2j} B_j^{-\mathrm{T}} C_{jK},$$

$$,\ldots,\ldots,$$

$$\hat{h}_{K1} = \frac{1}{2}(1 - d_K^{(1)})B_K^{-1}C_{K1} + \frac{1}{2}B_K^{-\mathrm{T}}C_{K1} + \frac{d_K^{(1)}}{2}B_K^{-1}C_{K1}B_1^{-1}C_1 + \frac{1}{2}\sum_{j=2}^{K-1} d_K^{(j)} B_K^{-1} C_{Kj} B_j^{-1} C_{j1},$$

$$\hat{h}_{K2} = \frac{1}{2}(1 - d_K^{(2)})B_K^{-1}C_{K2} + \frac{1}{2}B_K^{-\mathrm{T}}C_{K2} + \frac{d_K^{(2)}}{2}B_K^{-1}C_{K2}B_2^{-1}C_2$$

$$+ \frac{1}{2}\sum_{\substack{j=1\\ j\neq 2}}^{K-1} d_K^{(j)}B_K^{-1}C_{Kj}B_j^{-1}C_{j2},$$

$$, \dots ,$$

$$\hat{h}_{KK} = \frac{1}{2}(B_K^{-1}C_k + B_K^{-\mathrm{T}}C_K^{\mathrm{T}}) + \frac{1}{2}\sum_{j=1}^{K-1} d_K^{(j)}B_K^{-1}C_{Kj}B_j^{-1}C_{jK}.$$

Theorem 1. *Let A be s.p.d., let $\widehat{A}$ be its Stieltjes diagonally compensated reduced matrix, its multisplitting $(M_k, \widehat{N}_k, E_k), k = 1, \dots, K$, is in the form* (3.4) *and* (3.5) *with $\widehat{A}_i = B_i - C_i$ being two-side weak regular for $i = 1, \dots, K$. Then G is nonsingular and $\rho(\widehat{H}) < 1$.*

Proof. Since $\widehat{A}$ is a Stieltjes matrix and the splittings $\widehat{A}_i = B_i - C_i$ of $\widehat{A}_i, i = 1, \dots, K$, are two-side weak regular, we have $B_i^{-1} \geqslant 0, B_i^{-1}C_i \geqslant 0$, $B_i^{-\mathrm{T}}C_i^{\mathrm{T}} \geqslant 0$ and $C_{ij} \geqslant 0$ $(i \neq j)$ for $i, j = 1, \dots, K$. It follows from Eqs. (3.7) and (3.8) that

$$G \geqslant 0 \quad \text{and} \quad \widehat{H} \geqslant 0. \tag{3.9}$$

It is also easy to know from Eq. (3.7) that each row of G has at least one nonzero component. Noting Eq. (3.9), $\widehat{A}^{-1} \geqslant 0$ and $\widehat{H} = I - G\widehat{A}$, it follows from Theorem 1 in [4] that G is nonsingular and $\rho(\widehat{H}) < 1$. Thus the proof is finished. □

We now have two single splittings

$$A = G^{-1} - G^{-1}H \quad \text{and} \quad \widehat{A} = G^{-1} - G^{-1}\widehat{H} \tag{3.10}$$

of A and $\widehat{A}$, respectively. Moreover, the latter is convergent, i.e. $\rho(\widehat{H}) < 1$. On the convergence of our parallel symmetric multisplitting method (3.1) we have the following result.

Theorem 2. *Let A be s.p.d., let $\widehat{A}$ be a Stieltjes diagonally compensated reduced matrix of A and be partitioned into the form* (3.3). *Assume that A and $\widehat{A}$ have multisplittings (M_k, N_k, E_k) and $(M_k, \widehat{N}_k, E_k), k = 1, \dots, K$, respectively, where M_k and E_k are in Eqs.* (3.4) *and* (3.5), *respectively, with splittings $\widehat{A}_i = B_i - C_i$ of $\widehat{A}_i$ being two-side weak regular for $i = 1, \dots, K$. If the matrix G in Eq.* (3.7) *is s.p.d., then the parallel symmetric multisplitting method* (3.1) *converges.*

Proof. Theorem 1 implies $\rho(\widehat{H}) < 1$ i.e. the splitting $\widehat{A} = G^{-1} - G^{-1}\widehat{H}$ of $\widehat{A}$ is convergent. It follows from Lemma 1 that $A = G^{-1} - G^{-1}H$ a convergent splitting of A, i.e. $\rho(H) < 1$. Thus, the proof is finished. □

If we restrict the splittings $\widehat{A}_i = B_i - C_i$ of $\widehat{A}_i, i = 1, \ldots, K$, being symmetric, then the matrix G in Eq. (3.7) has the form

$$G = \operatorname{diag}(B_1^{-1}, \ldots, B_K^{-1})\widetilde{G}\operatorname{diag}(B_1^{-1}, \ldots, B_K^{-1}),$$

where

$$\tilde{G} = \begin{pmatrix} B_1 & \frac{d_2^{(1)}}{2}C_{12} & \cdots & \frac{d_K^{(1)}}{2}C_{1K} \\ \frac{d_2^{(1)}}{2}C_{21} & B_2 & \cdots & \frac{d_K^{(2)}}{2}C_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_K^{(1)}}{2}C_{K1} & \frac{d_K^{(2)}}{2}C_{K2} & \cdots & B_K \end{pmatrix} \tag{3.11}$$

In this case Theorem 2 can be rewritten as follows.

Theorem 2′. *Let A be s.p.d.,* let *$\widehat{A}$ be a Stieltjes diagonally compensated reduced matrix of A and be partitioned into the form* (3.3). *Assume that A and $\widehat{A}$ have multisplittings (M_k, N_k, E_k) and $(M_k, \widehat{N}_k, E_k), k = 1, \ldots, K$, respectively, where M_k and E_k are in Eqs.* (3.4) *and* (3.5), *respectively, with splittings $\widehat{A}_i = B_i - C_i$ of $\widehat{A}_i$ being symmetric and weak regular for $i = 1, \ldots, K$. If the matrix $\tilde{G}$ in Eq.* (3.11) *is s.p.d., then the parallel symmetric multisplitting method* (3.1) *converges.*

In order to get a more practically feasible parallel multisplitting algorithm we further restrict the weighting matrices $E_k, k = 1, \ldots, K$, in Eq. (3.5) to satisfy the following conditions.

$$E_k = \begin{pmatrix} d_1^{(k)}I_1 & & & & \\ & \ddots & & & \\ & & d_k^{(k)}I_k & & \\ & & & \ddots & \\ & & & & d_K^{(k)}I_K \end{pmatrix},$$

where

$$0 \leqslant d_j^{(k)} \leqslant 1, \quad \sum_{k=1}^{K} d_j^{(k)} = 1 \quad \text{and} \quad d_j^{(k)} = 0 \quad \text{for } j > k,$$
$$k = 1, \ldots, K; \; j = 1, \ldots, K. \tag{3.12}$$

In this case, the matrix G in Eq. (3.7) has the following simple form

$$G = \frac{1}{2}\begin{pmatrix} B_1^{-1} + B_1^{-\mathrm{T}} & & \\ & \ddots & \\ & & B_K + B_K^{-\mathrm{T}} \end{pmatrix}.$$

If we restrict in addition the splittings $\widehat{A}_i = B_i - C_i$ of $\widehat{A}_i$, $i = 1, \ldots, K$, being symmetric, then

$$G = \begin{pmatrix} B_1^{-1} & & \\ & \ddots & \\ & & B_K^{-1} \end{pmatrix}. \tag{3.13}$$

We now give some examples of symmetric splittings such that $\tilde{G}$ in Eq. (3.11) or G in Eq. (3.13) are s.p.d., hence the resulting parallel symmetric multisplitting method (3.1) converges.

Let $\widehat{A}_i = \hat{D}_i - \hat{L}_i - \hat{L}_i^{\mathrm{T}}, i = 1, \ldots, K$, where $\hat{D}_i = \mathrm{diag}(\widehat{A}_i), \hat{L}_i$ is the strictly lower part of $\widehat{A}_i$.

(i) Band splitting: $\widehat{A}_i = \hat{B}_i - \hat{C}_i$, where $\mathrm{diag}(\hat{B}_i) = \mathrm{diag}(\widehat{A}_i)$ and $\hat{B}_i$ is a symmetric band matrix consisting of some (symmetric) diagonals of $\widehat{A}_i$

(ii) Jacobi splitting: $\widehat{A}_i = \hat{D}_i - \hat{B}_i$. Obviously, this can be regarded as a special case of the band splitting.

(iii) SSOR splitting:

$$\begin{aligned} \widehat{A}_i = &\frac{\omega}{2-\omega}\left\{\frac{1}{\omega}(\hat{D}_i - \omega\hat{L}_i)\hat{D}_i^{-1}\frac{1}{\omega}(\hat{D}_i - \omega\hat{L}_i^{\mathrm{T}})\right\} \\ &- \frac{\omega}{2-\omega}\left\{\frac{1}{\omega}[(1-\omega)\hat{D}_i + \omega\hat{L}_i]\hat{D}_i^{-1}\frac{1}{\omega}[(1-\omega)\hat{D}_i + \omega\hat{L}_i^{\mathrm{T}}]\right\} \end{aligned}$$

We now need a well-known lemma on symmetric H-matrix. We add a proof for completeness.

Lemma 2. *Let $Y \in \mathscr{R}^{n,n}$ be a symmetric H-matrix with positive diagonal entries, then Y is symmetric positive definite.*

Proof. It is well known (cf. [10]) that an H-matrix is generalized strictly diagonally dominant. Thus, for H-matrix Y it exists a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_i > 0, i = 1, \ldots, n$, such that matrix YD is strictly diagonally dominant. Obviously, matrix $D^{-1}YD$ is also strictly diagonally dominant. Since the diagonal entries of $D^{-1}YD$ are positive and $D^{-1}YD$ is similar to the symmetric matrix Y, all eigenvalues of $D^{-1}YD$ are positive. Thus, matrix Y is s.p.d. and the proof is completed. $\square$

Theorem 3. *Let A be s.p.d., let $\widehat{A}$ be any Stieltjes diagonally compensated reduced matrix of A. Assume $\widehat{A}$ is partitioned into Eq.* (3.3). *If a multisplitting $(M_k, \widehat{N}_k, E_k), k = 1, \ldots, K$, of $\widehat{A}$ is in the form* (3.4) *and* (3.5) *with the splittings of $\widehat{A}_i, i = 1, \ldots, K$, being* (i) *band splittings or* (ii) *Jacobi splittings. Then the resulting parallel symmetric multisplitting method* (3.1) *according to the multisplitting $(M_k, N_k, E_k), k = 1, \ldots, K$, of A converges.*

Proof. Since $\widehat{A}$ is a Stieltjes matrix, i.e. a s.p.d. Z-matrix, the main submatrices $A_i, i = 1, \ldots, K$, of $\widehat{A}$ are all s.p.d. Z-matrices and hence are all Stieltjes matrices. Obviously, in either case (i) or (ii) of splittings $\widehat{A}_i = B_i - C_i$ of $\widehat{A}_i$for $i = 1, \ldots, K$, the comparison matrix $\langle \tilde{G} \rangle$ of the corresponding matrix $\tilde{G}$ in Eq. (3.11) is a Z- matrix and satisfies $\langle \tilde{G} \rangle \geqslant \widehat{A}$. Hence $\langle \tilde{G} \rangle$ is a symmetric M-matrix, i.e. $\tilde{G}$ is a symmetric H-matrix. Since $\tilde{G}$ has positive diagonal entries, $\tilde{G}$ is s.p.d. by Lemma 2. Moreover, all the splittings $\widehat{A}_i = B_i - C_i$ of $\widehat{A}_i, i = 1, \ldots, K$, in case (i) or (ii) are symmetric and regular (note that in either case $B_i, i = 1, \ldots, K$, are Stieltjes matrices and hence are M-matrices). Then the conclusion of this theorem follows from Theorem 2′ and the proof is finished. □

Since strictly diagonally dominant or irreducibly diagonally dominant symmetric matrices with positive diagonal entries are positive definite, we have the following result.

Corollary 1. *Let A be symmetric with positive diagonal entries and be strictly diagonally dominant or irreducibly diagonally dominant. Let $\widehat{A}$ be any Stieltjes diagonally compensated reduced matrix of A. Assume $\widehat{A}$ is partitioned into form* (3.3). *If a multisplitting $(M_k, \widehat{N}_k, E_k), k = 1, \ldots, K$, of $\widehat{A}$ is in the form* (3.4) *and* (3.5) *with the splittings of $\widehat{A}_i, i = 1, \ldots, K$, being* (i) *band splittings or* (ii) *Jacobi splittings. Then the resulting parallel symmetric multisplitting method* (3.1) *according to the multisplitting $(M_k, N_k, E_k), k = 1, \ldots, K$, of A converges.*

Proof. With either assumption on matrix A, it is well known (cf. [11]) that A is s.p.d. Then the conclusion of the theorem follows from Theorem 3. □

If we restrict the weighting matrices $E_k, k = 1, \ldots, K$, in the form (3.12), then we have the following result.

Theorem 4. *Let A be s.p.d., let $\widehat{A}$ be any Stieltjes diagonally compensated reduced matrix of A. Assume $\widehat{A}$ is partitioned into form* (3.3). *If a multisplitting $(M_k, \widehat{N}_k, E_k), k = 1, \ldots, K$, of $\widehat{A}$ is such that E_k are in the form* (3.12), *M_k are in the form* (3.4) *with the splittings of $\widehat{A}_i, i = 1, \ldots, K$, being* (i) *band splittings or* (ii) *Jacobi splittings or* (iii) *SSOR splittings with $\omega \in (0, 1]$. Then the resulting parallel symmetric multisplitting method* (3.1) *according to the multisplitting $(M_k, N_k, E_k), k = 1, \ldots, K$, of A converges.*

Proof. In all the splittings of $\widehat{A}_i, i = 1, \ldots, K$, for either of the three cases (i), (ii) and (iii) (with $\omega \in (0, 1]$) are symmetric and regular and the corresponding matrix G in Eq. (3.13) is s.p.d., then by Theorem 2′ we have $\rho(H) < 1$. Thus the proof is completed. □

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