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Universal acyclic resolutions for finitely generated coefficient groups

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Abstract

We prove that for every compactum X and every integer $n \geqslant 2$ there are a compactum Z of $\dim \leqslant n$ and a surjective UV^{n-1}-map $r: Z \to X$ having the property that:

for every finitely generated Abelian group G and every integer $k \geqslant 2$ such that $\dim_G X \leqslant k \leqslant n$ we have $\dim_G Z \leqslant k$ and r is G-acyclic, or equivalently:

for every simply connected CW-complex K with finitely generated homotopy groups such that e-dim $X \leqslant K$ we have e-dim $Z \leqslant K$ and r is K-acyclic. (A space is K-acyclic if every map from the space to K is null-homotopic. A map is K-acyclic if every fiber is K-acyclic.)

MSC: 55M10; 54F45

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1. Introduction

A space X is always assumed to be separable metrizable. The cohomological dimension $\dim_G X$ of X with respect to an Abelian group G is the least number n such that $\check{H}^{n+1}(X, A; G) = 0$ for every closed subset A of X. The covering dimension $\dim X$ coincides with the integral cohomological dimension $\dim_{\mathbb{Z}} X$ if X is finite dimensional. In 1987 Dranishnikov [1] showed that there is an infinite dimensional compactum (= compact metric space) with finite integral cohomological dimension. In 1978 Edwards discovered his resolution theorem [6,10] which shows that despite Dranishnikov's example the dimension functions dim and $\dim_{\mathbb{Z}}$ are closely related even for infinite dimensional

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compacta. The Edwards resolution theorem says that a compactum of $\dim_{\mathbb{Z}} \leqslant n$ can be obtained as the image of a cell-like map defined on a compactum of $\dim \leqslant n$. A compactum X is cell-like if any map $f: X \to K$ from X to a CW-complex K is null-homotopic. A map is cell-like if its fibers are cell-like. The reduced Čech cohomology groups of a cell-like compactum are trivial with respect to any group G.

Below are stated a few theorems representing main directions of generalizing the Edwards resolution theorem. We recall some notions used in the theorems.

A space is G-acyclic if its reduced Čech cohomology groups modulo G are trivial, a map is G-acyclic if every fiber is G-acyclic. By the Vietoris–Begle theorem a G-acyclic map of compacta cannot raise the cohomological dimension $\dim_G$.

A compactum X is approximately n-connected if any embedding of X into an ANR has the UV^n-property, i.e., for every neighborhood U of X there is a smaller neighborhood $X \subset V \subset U$ such that the inclusion $V \subset U$ induces the zero homomorphism of the homotopy groups in $\dim \leqslant n$. An approximately n-connected compactum has trivial reduced Čech cohomology groups in $\dim \leqslant n$ with respect to any group G. A map is called a UV^n-map if every fiber is approximately n-connected.

Let K be a CW-complex. A space is said to be K-acyclic if every map from the space to K is null-homotopic and a map is said to be K-acyclic if every fiber is K-acyclic.

Theorem 1.1 [2]. *Let p be a prime number and let X be a compactum with $\dim_{\mathbb{Z}_p} X \leqslant n$. Then there are a compactum Z with $\dim Z \leqslant n$ and a $\mathbb{Z}_p$-acyclic UV^{n-1}-map $r: Z \to X$ from Z onto X.*

Theorem 1.2 [8]. *Let G be an Abelian group and let X be a compactum with $\dim_G X \leqslant n$, $n \geqslant 2$. Then there are a compactum Z with $\dim_G Z \leqslant n$ and $\dim Z \leqslant n+1$ and a G-acyclic map $r: Z \to X$ from Z onto X.*

Theorem 1.3 [9]. *Let X be a compactum with $\dim_{\mathbb{Z}} X \leqslant n \geqslant 2$. Then there exist a compactum Z with $\dim Z \leqslant n$ and a cell-like map $r: Z \to X$ from Z onto X such for every integer $k \geqslant 2$ and every group G such that $\dim_G X \leqslant k$ we have $\dim_G Z \leqslant k$.*

Each of these theorems emphasizes different aspects of the Edwards resolution. The special feature of Theorem 1.3 is that it takes care of all possible cohomological dimensions bigger than 1. It is very natural to try to generalize Theorems 1.1 and 1.2 in the spirit of Theorem 1.3. As a part of this project this paper is mainly devoted to proving the following version of Theorem 1.1.

Theorem 1.4. *Let X be a compactum. Then for every integer $n \geqslant 2$ there are a compactum Z of $\dim \leqslant n$ and a surjective UV^{n-1}-map $r: Z \to X$ having the property that for every finitely generated Abelian group G and every integer $k \geqslant 2$ such that $\dim_G X \leqslant k \leqslant n$ we have that $\dim_G Z \leqslant k$ and r is G-acyclic.*

Theorem 1.4 can be reformulated in terms of extensional dimension [4,5]. The extensional dimension of X is said not to exceed a CW-complex K, written $\text{e-dim}\, X \leqslant K$, if for every closed subset A of X and every map $f: A \to K$ there is an extension of f

over X. It is well known that $\dim X \leqslant n$ is equivalent to e-dim $X \leqslant \mathbb{S}^n$ and $\dim_G X \leqslant n$ is equivalent to e-dim $X \leqslant K(G,n)$ where $K(G,n)$ is an Eilenberg–Mac Lane complex of type (G,n).

The following theorem shows a close connection between cohomological and extensional dimensions.

Theorem 1.5 [3]. *Let X be a compactum and let K be a simply connected* CW*-complex. Consider the following conditions*:

(1) e-dim $X \leqslant K$;
(2) $\dim_{H_i(K)} X \leqslant i$ *for every* $i > 1$;
(3) $\dim_{\pi_i(K)} X \leqslant i$ *for every* $i > 1$.

Then (2) *and* (3) *are equivalent and* (1) *implies both* (2) *and* (3). *If X is finite dimensional then all the conditions are equivalent.*

Theorems 1.4 and 1.5 imply

Theorem 1.6. *Let X be a compactum. Then for every integer $n \geqslant 2$ there exist a compactum Z with* $\dim Z \leqslant n$ *and a surjective UV^{n-1}-map $r: Z \to X$ such that for every simply connected* CW*-complex K with finitely generated homotopy groups such that* e-dim $X \leqslant K$ *we have* e-dim $Z \leqslant K$ *and r is K-acyclic.*

Proof. Let Z and $r: Z \to X$ be as in Theorem 1.4. Let a simply connected CW-complex K with finitely generated homotopy groups be such that e-dim $X \leqslant K$. Then $H_*(K)$ are finitely generated and by Theorem 1.5, $\dim_{H_i(K)} X \leqslant i$ and $\dim_{\pi_i(K)} X \leqslant i$ for every $i > 1$. Hence by Theorem 1.4, $\dim_{H_i(K)} Z \leqslant i$ for every $1 < i \leqslant n$. Since $\dim Z \leqslant n$, $\dim_{H_i(K)} Z \leqslant i$ for $i > n$ and it follows from Theorem 1.5 that e-dim $Z \leqslant K$.

Let $x \in X$ and let $Cr^{-1}(x)$ and $\Sigma r^{-1}(x)$ be the cone and the suspension of $r^{-1}(x)$, respectively. Then $\check{H}^{i+1}(Cr^{-1}(x), r^{-1}(x); \pi_i(K)) = \check{H}^{i+1}(Cr^{-1}(x)/r^{-1}(x); \pi_i(K)) = \check{H}^{i+1}(\Sigma r^{-1}(x); \pi_i(K)) = \widetilde{\check{H}}^i(r^{-1}(x); \pi_i(K))$. Recall that $\dim_{\pi_i(K)} X \leqslant i$ for every $i > 1$ and hence, by Theorem 1.4, r is $\pi_i(K)$-acyclic for $1 < i \leqslant n$. Then, since $\dim Z \leqslant n$, $\check{H}^i(r^{-1}(x); \pi_i(K)) = 0$ for every $i > 1$ and therefore $\check{H}^{i+1}(Cr^{-1}(x), r^{-1}(x); \pi_i(K)) = 0$ for $i > 1$. Since $Cr^{-1}(x)$ is finite dimensional it follows from Obstruction Theory that every map from $r^{-1}(x)$ to K extends over $Cr^{-1}(x)$ and hence is null-homotopic. Thus we showed that r is K-acyclic. □

Theorem 1.6 is equivalent to Theorem 1.4. To show this we need to show that Theorem 1.6 implies Theorem 1.4. Let X be a compactum, let $n \geqslant 2$ and let $r: Z \to X$ satisfy the conclusions of Theorem 1.6. Then for every finitely generated Abelian group G with $\dim_G X \leqslant k \leqslant n$, $k \geqslant 2$ we have that $\dim_G Z \leqslant k$. Let $x \in X$. The $K(G,k)$-acyclicity of r implies that $\check{H}^k(r^{-1}(x); G) = 0$, the fact that r is UV^{n-1} implies that $\widetilde{\check{H}}^i(r^{-1}(x); G) = 0$ for $i \leqslant k-1$ and finally $\dim_G Z \leqslant k$ implies that $\check{H}^k(r^{-1}(x); G) = 0$ for $i > k$. All this together implies that $r^{-1}(x)$ is G-acyclic and hence Theorems 1.6 and 1.4 are equivalent.

It is known that Theorem 1.4 does not hold for arbitrary groups G even if we do not require that r is UV^{n-1}, see [7]. However allowing in Theorem 1.4 for the dimension of Z to be raised to $n+1$ one can drop the restrictions on G and obtain the following theorem which we will state without a proof.

Theorem 1.7. *Let X be a compactum. Then for every integer $n \geqslant 2$ there are a compactum Z of $\dim \leqslant n+1$ and a surjective UV^{n-1}-map $r: Z \to X$ having the property that for every Abelian group G and every integer $k \geqslant 2$ such that $\dim_G X \leqslant k \leqslant n$ we have that $\dim_G Z \leqslant k$ and r is G-acyclic.*

Finally let us note that it would be interesting to know if the restriction $k \geqslant 2$ in Theorem 1.4 can be omitted.

2. Preliminaries

A map between CW-complexes is said to be combinatorial if the preimage of every subcomplex of the range is a subcomplex of the domain.

Let M be a simplicial complex and let $M^{[k]}$ be the k-skeleton of M (= the union of all simplexes of M of $\dim \leqslant k$). By a resolution $EW(M,k)$ of M we mean a CW-complex $EW(M,k)$ and a combinatorial map $\omega: EW(M,k) \to M$ such that ω is 1-to-1 over $M^{[k]}$. Let $f: N \to K$ be a map of a subcomplex N of M into a CW-complex K. The resolution is said to be suitable for f if the map $f \circ \omega|_{\omega^{-1}(N)}$ extends to a map $f': EW(M,k) \to K$. We call f' a resolving map for f. The resolution is said to be suitable for a compactum X if for every simplex Δ of M, e-$\dim X \leqslant \omega^{-1}(\Delta)$. Note that if $\omega: EW(M,k) \to M$ is a resolution suitable for X then for every map $\phi: X \to M$ there is a map $\psi: X \to EW(M,k)$ such that for every simplex Δ of M, $(\omega \circ \psi)(\phi^{-1}(\Delta)) \subset \Delta$. We call ψ a combinatorial lifting of ϕ.

Let M be a finite simplicial complex and let $f: N \to K$ be a cellular map from a subcomplex N of M to a CW-complex K such that $M^{[k]} \subset N$. A standard way of constructing a resolution suitable for f is described in [9]. Such a resolution $\omega: EW(M,k) \to M$ is called the standard resolution of M for f and it has the following properties:

ω is a map onto and for every simplex Δ of M, $\omega^{-1}(\Delta)$ is either contractible or homotopy equivalent to K;

the (integral) homology groups of $EW(M,k)$ are finitely generated if so are the homology groups of K. This property can be derived from the previous one using the Mayer–Vietoris sequence and induction on the number of simplexes of M;

$EW(M,k)$ is $(k-1)$-connected if so are M and K;

for every subcomplex T of M, $\omega|_{\omega^{-1}(T)}: EW(T,k) = \omega^{-1}(T) \to T$ is the standard resolution of T for $f|_{N\cap T}: N \cap T \to K$.

All groups are assumed to be Abelian and functions between groups are homomorphisms. $\mathcal{P}$ stands for the set of primes. Let G be a group and $p \in \mathcal{P}$. We say that $g \in G$ is p-torsion if $p^k g = 0$ for some integer $k \geqslant 1$. $\mathrm{Tor}_p G$ stands for the subgroup of the p-torsion elements of G. G is p-torsion if $\mathrm{Tor}_p G = G$, G is p-torsion free if $\mathrm{Tor}_p G = 0$

and G is p-divisible if for every $g \in G$ there is $h \in G$ such that $ph = g$. G is p-local if it is q-divisible and q-torsion free for every $q \in \mathcal{P}$, $q \neq p$.

Let G be a group, let $\alpha : L \to M$ be a surjective combinatorial map of a CW-complex L and a finite simplicial complex M and let n be a positive integer such that $\widetilde{H}_i(\alpha^{-1}(\Delta); G) = 0$ for every $i < n$ and every simplex Δ of M. One can show by induction on the number of simplexes of M using the Mayer–Vietoris sequence and the Five Lemma that $\alpha_* : \widetilde{H}_i(L; G) \to \widetilde{H}_i(M; G)$ is an isomorphism for $i < n$. We will refer to this fact as the combinatorial Vietoris–Begle theorem.

We need the following slightly more precise version of [9, Proposition 2.1(i)].

Proposition 2.1. *Let $2 \leqslant k \leqslant n$, $p \in \mathcal{P}$ and let M be an $(n-1)$-connected finite simplicial complex. Let $\omega : EW(M, k) \to M$ be the standard resolution for a cellular map $f : N \to K(\mathbb{Z}_p, k)$ from a subcomplex N of M containing $M^{[k]}$. Then $\pi_i(EW(M, k))$ is p-torsion for every $1 \leqslant i \leqslant n-1$.*

Proof. Recall that ω is a combinatorial surjective map and for every simplex Δ of M, $\omega^{-1}(\Delta)$ is either contractible or homotopy equivalent to $K(\mathbb{Z}_p, k)$. $EW(M, k)$ is $(k-1)$-connected since so are M and $K(\mathbb{Z}_p, k)$, and $H_*(EW(M, k))$ is finitely generated since so is $H_*(K(\mathbb{Z}_p, k))$.

By the generalized Hurewicz theorem the groups $H_i(K(\mathbb{Z}_p, k))$, $i \geqslant 1$ are p-torsion and therefore $H_i(K(\mathbb{Z}_p, k))$, $i \geqslant 1$ are p-local. Let $q \in \mathcal{P}$ and $q \neq p$. From the p-locality of $H_i(K(\mathbb{Z}_p, k))$, $i \geqslant 1$ it follows that $H_i(K(\mathbb{Z}_p, k); \mathbb{Z}_q) = 0$, $i \geqslant 1$. Since M is $(n-1)$-connected, the combinatorial Vietoris–Begle theorem implies that $H_i(EW(M, k); \mathbb{Z}_q) = 0$, $1 \leqslant i \leqslant n-1$. Then from the universal coefficient theorem it follows that $H_i(EW(M, k)) \otimes \mathbb{Z}_q = 0$ for $1 \leqslant i \leqslant n-1$ and every $q \in \mathcal{P}$, $q \neq p$. Since $H_*(EW(M, k))$ is finitely generated, the last property implies that $H_i(EW(M, k))$, $1 \leqslant i \leqslant n-1$ is p-torsion and by the generalized Hurewicz theorem $\pi_i(EW(M, k))$, $1 \leqslant i \leqslant n-1$ is p-torsion. □

In the proof of Theorem 1.4 we will also use the following facts.

Proposition 2.2 [9]. *Let K be a simply connected* CW*-complex such that K has only finitely many non-trivial homotopy groups. Let X be a compactum such that* $\dim_{\pi_i(K)} X \leqslant i$ *for $i > 1$. Then* e-dim $X \leqslant K$.

Let K' be a simplicial complex. We say that maps $h : K \to K'$, $g : L \to L'$, $\alpha : L \to K$ and $\alpha' : L' \to K'$

$$\begin{array}{ccc} L & \xrightarrow{\alpha} & K \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ L' & \xrightarrow{\alpha'} & K' \end{array}$$

combinatorially commute if for every simplex Δ of K' we have that $(\alpha' \circ g)((h \circ \alpha)^{-1}(\Delta)) \subset \Delta$. (The direction in which we want the maps h, g, α and α' to combinatorially commute is indicated by the first map in the list. Thus saying that α', h, g and α

combinatorially commute we would mean that $(h \circ \alpha)((\alpha' \circ g)^{-1}(\Delta)) \subset \Delta$ for every simplex Δ of K'.) Recall that a map $h' : K \to L'$ is a combinatorial lifting of h to L' if for every simplex Δ of K' we have that $(\alpha' \circ h')(h^{-1}(\Delta)) \subset \Delta$.

For a simplicial complex K and $a \in K$, $\mathrm{st}(a)$ denotes the union of all the simplexes of K containing a. The following proposition whose proof is left to the reader is a collection of simple combinatorial properties of maps.

Proposition 2.3.

(i) *Let a compactum X be represented as the inverse limit $X = \varprojlim K_i$ of finite simplicial complexes K_i with bonding maps $h^i_j : K_j \to K_i$. Fix i and let $\omega : EW(K_i, k) \to K_i$ be a resolution of K_i which is suitable for X. Then there is a sufficiently large j such that h^i_j admits a combinatorial lifting to $EW(K_i, k)$.*

(ii) *Let $h : K \to K'$, $h' : K \to L'$ and $\alpha' : L' \to K'$ be maps of a simplicial complex K' and* CW*-complexes K and L' such that h and α' are combinatorial and h' is a combinatorial lifting of h. Then there is a cellular approximation of h' which is also a combinatorial lifting of h.*

(iii) *Let K and K' be simplicial complexes, let maps $h : K \to K'$, $g : L \to L'$, $\alpha : L \to K$ and $\alpha' : L' \to K'$ combinatorially commute and let h be combinatorial.*

Then

$$g\big(\alpha^{-1}\big(\mathrm{st}(x)\big)\big) \subset \alpha'^{-1}\big(\mathrm{st}\big(h(x)\big)\big) \quad \textit{and} \quad h\big(\mathrm{st}\big(\alpha(z)\big)\big) \subset \mathrm{st}\big((\alpha' \circ g)(z)\big)$$

for every $x \in K$ and $z \in L$.

3. Proof of Theorem 1.4

If $n \geqslant \dim_{\mathbb{Z}} X$ then Theorem 1.4 follows from Theorem 1.3 (see also a remark at the end of this section). Hence we may assume that $n < \dim_{\mathbb{Z}} X$. Then for a finitely generated Abelian group G the condition $\dim_G X \leqslant n$ implies that G is torsion. Thus we may assume that the groups G considered in the theorem are torsion and therefore the Bockstein basis $\sigma(G)$ of G consists only of groups of type $\mathbb{Z}_p$ (p is always assumed to be a prime number).

Represent X as the inverse limit $X = \varprojlim (K_i, h_i)$ of simplicial complexes K_i with combinatorial bonding maps $h_{i+1} : K_{i+1} \to K_i$ and the projections $p_i : X \to K_i$ such that for every simplex Δ of K_i, $\mathrm{diam}(p_i^{-1}(\Delta)) \leqslant 1/i$. We will construct by induction simplicial complexes L_i and maps $g_{i+1} : L_{i+1} \to L_i$, $\alpha_i : L_i \to K_i$ such that

(a) $L_i = K_i^{[n]}$ and $\alpha_i : L_i \to K_i$ is the embedding. The simplicial structure of L_1 is induced from $K_1^{[n]}$ and the simplicial structure of L_i, $i > 1$ is defined as a sufficiently small barycentric subdivision of $K_i^{[n]}$. We will refer to this simplicial structure while constructing standard resolutions of L_i. It is clear that α_i is always a combinatorial map;

(b) the maps h_{i+1}, g_{i+1}, α_{i+1} and α_i combinatorially commute. Recall that this means that for every simplex Δ of K_i, $(\alpha_i \circ g_{i+1})((h_{i+1} \circ \alpha_{i+1})^{-1}(\Delta)) \subset \Delta$.

We will construct L_i in such a way that $Z = \varprojlim (L_i, g_i)$ will admit a map $r : Z \to X$ such that Z and r satisfy the conclusions of the theorem. Assume that the construction is completed for i. We proceed to $i + 1$ as follows.

Let $\dim_{\mathbb{Z}_p} X \leqslant k$, $2 \leqslant k \leqslant n$ and let $f : N \to K(\mathbb{Z}_p, k)$ be a cellular map from a subcomplex N of L_i, $L_i^{[k]} \subset N$. Let $\omega_L : EW(L_i, k) \to L_i$ be the standard resolution of L_i for f. We are going to construct from $\omega_L : EW(L_i, k) \to L_i$ a resolution $\omega : EW(K_i, k) \to K_i$ of K_i suitable for X. If $\dim K_i \leqslant k$ set $\omega = \alpha_i \circ \omega_L : EW(K_i, k) = EW(L_i, k) \to K_i$.

If $\dim K_i > k$ set $\omega_k = \alpha_i \circ \omega_L : EW_k(K_i, k) = EW(L_i, k) \to K_i$ and we will construct by induction resolutions $\omega_j : EW_j(K_i, k) \to K_i$, $k+1 \leqslant j \leqslant \dim K_i$ such that $EW_j(K_i, k)$ is a subcomplex of $EW_{j+1}(K_i, k)$ and ω_{j+1} extends ω_j for every $k \leqslant j < \dim K_i$. Note that saying that an $(n+1)$-cell is attached to a CW-complex by a map of degree p we mean that the cell is attached by a map ϕ from the boundary $\mathbb{S}^n$ of the cell to the CW-complex such that ϕ factors through a map $\mathbb{S}^n \to \mathbb{S}^n$ of degree p.

Assume that $\omega_j : EW_j(K_i, k) \to K_i$, $k \leqslant j < \dim K_i$ is constructed. For every simplex Δ of K_i of $\dim = j + 1$ consider the subcomplex ${\omega_j}^{-1}(\Delta)$ of $EW_j(K_i, k)$. Enlarge ${\omega_j}^{-1}(\Delta)$ by attaching cells of $\dim = n + 1$ by maps of degree p in order to kill the elements of $p\pi_n({\omega_j}^{-1}(\Delta)) = \{pa\colon a \in \pi_n({\omega_j}^{-1}(\Delta))\}$ and attaching cells of $\dim > n + 1$ in order to get a subcomplex with trivial homotopy groups in $\dim > n$. Let $EW_{j+1}(K_i, k)$ be $EW_j(K_i, k)$ with all the cells attached for all $(j + 1)$-dimensional simplexes Δ of K_i and let $\omega_{j+1} : EW_{j+1}(K_i, k) \to K_i$ be an extension of ω_j sending the interior points of the attached cells to the interior of the corresponding Δ.

Finally denote $EW(K_i, k) = EW_j(K_i, k)$ and $\omega = \omega_j : EW_j(K_i, k) \to K_i$ for $j = \dim K_i$. Note that since we attach cells only of $\dim > n$, the n-skeleton of $EW(K_i, k)$ coincides with the n-skeleton of $EW(L_i, k)$.

Let us show that $EW(K_i, k)$ is suitable for X. Fix a simplex Δ of K_i and denote $T = \alpha_i^{-1}(\Delta)$. First note that T is $(n - 1)$-connected, $\omega^{-1}(\Delta)$ is $(k - 1)$-connected, $\pi_n(\omega^{-1}(\Delta))$ is p-torsion, $\pi_j(\omega^{-1}(\Delta)) = 0$ for $j \geqslant n + 1$ and $\pi_j(\omega^{-1}(\Delta)) = \pi_j(\omega_L^{-1}(T))$ for $j \leqslant n - 1$.

By Proposition 2.1, $\pi_j(\omega_L^{-1}(T))$ is p-torsion for $j \leqslant n - 1$. Then $\pi_j(\omega^{-1}(\Delta))$ is p-torsion for $k \leqslant j \leqslant n$. Therefore by Bockstein Theory $\dim_{\pi_j(\omega^{-1}(\Delta))} X \leqslant \dim_{\mathbb{Z}_p} X \leqslant k$, $k \leqslant j \leqslant n$ and hence by Proposition 2.2, e-dim $X \leqslant \omega^{-1}(\Delta)$.

Thus we have shown that $EW(K_i, k)$ is suitable for X. Now replacing K_{i+1} by K_j with a sufficiently large j we may assume by (i) of Proposition 2.3 that there is a combinatorial lifting of h_{i+1} to $h'_{i+1} : K_{i+1} \to EW(K_i, k)$. By (ii) of Proposition 2.3 we replace h'_{i+1} by its cellular approximation preserving the property of h'_{i+1} of being a combinatorial lifting of h_{i+1}.

Then h'_{i+1} sends the n-skeleton of K_{i+1} to the n-skeleton of $EW(K_i, k)$. Recall that the n-skeleton of $EW(K_i, k)$ is contained in $EW(L_i, k)$ and hence one can define $g_{i+1} = \omega_L \circ h'_{i+1}|_{K_{i+1}^{[n]}} : L_{i+1} = K_{i+1}^{[n]} \to L_i$. Finally define a simplicial structure on L_{i+1} to be a sufficiently small barycentric subdivision of $K_{i+1}^{[n]}$ such that

(c) $\operatorname{diam} g_{i+1}^j(\Delta) \leqslant 1/i$ for every simplex Δ in L_{i+1} and $j \leqslant i$ where $g_i^j = g_{j+1} \circ g_{j+2} \circ \cdots \circ g_i : L_i \to L_j$.

It is easy to check that the properties (a) and (b) are satisfied.

Denote $Z = \varprojlim (L_i, g_i)$ and let $r_i : Z \to L_i$ be the projections.

Clearly $\dim Z \leqslant n$. For constructing L_{i+1} we used an arbitrary map $f : N \to K(\mathbb{Z}_p, k)$ such that $\dim_{\mathbb{Z}_p} X \leqslant k$, $2 \leqslant k \leqslant n$ and N is a subcomplex of L_i containing $L_i^{[k]}$. By a standard reasoning described in detail in the proof of Theorem 1.6, [9] one can show that choosing $\mathbb{Z}_p$ and f in an appropriate way for each i we can achieve that $\dim_{\mathbb{Z}_p} Z \leqslant k$ for every integer $2 \leqslant k \leqslant n$ and every $\mathbb{Z}_p$ such that $\dim_{\mathbb{Z}_p} X \leqslant k$. Then by the Bockstein theorem $\dim_G Z \leqslant k$ for every finitely generated torsion Abelian group G such that $\dim_G X \leqslant k$, $2 \leqslant k \leqslant n$.

By (iii) of Proposition 2.3, the property (b) implies that for every $x \in X$ and $z \in Z$ the following holds:

(d1) $g_{i+1}(\alpha_{i+1}^{-1}(\mathrm{st}(p_{i+1}(x)))) \subset \alpha_i^{-1}(\mathrm{st}(p_i(x)))$, and
(d2) $h_{i+1}(\mathrm{st}((\alpha_{i+1} \circ r_{i+1})(z))) \subset \mathrm{st}((\alpha_i \circ r_i)(z))$.

Define a map $r : Z \to X$ by $r(z) = \bigcap \{p_i^{-1}(\mathrm{st}((\alpha_i \circ r_i)(z)))\colon\ i = 1, 2, \ldots\}$. Then (d2) implies that r is indeed well-defined and continuous.

The properties (d1) and (d2) also imply that for every $x \in X$

$$r^{-1}(x) = \varprojlim \bigl(\alpha_i^{-1}\bigl(\mathrm{st}\bigl(p_i(x)\bigr)\bigr), g_i|_{\alpha_i^{-1}(\mathrm{st}(p_i(x)))}\bigr),$$

where the map $g_i|_{\ldots}$ is considered as a map to $\alpha_{i-1}^{-1}(\mathrm{st}(p_{i-1}(x)))$.

Since $r^{-1}(x)$ is not empty for every $x \in X$, r is a map onto. Fix $x \in X$ and let us show that $r^{-1}(x)$ satisfies the conclusions of the theorem. First note that $M_i = \mathrm{st}(p_i(x))$ is contractible . Since $T_i = \alpha_i^{-1}(M_i)$ is homeomorphic to the n-skeleton of $\mathrm{st}(p_i(x))$, T_i is $(n-1)$-connected and hence $r^{-1}(x)$ is approximately $(n-1)$-connected as the inverse limit of $(n-1)$-connected finite simplicial complexes.

Let $\mathbb{Z}_p$ be such that $\dim_{\mathbb{Z}_p} X \leqslant k$, $2 \leqslant k \leqslant n$ and assume that L_{i+1} is constructed with help of $f : N \to K(\mathbb{Z}_p, k)$. Consider T_{i+1} as the n-skeleton of M_{i+1} and denote

$\beta =$ the inclusion$: T_{i+1} \to M_{i+1}$,
$\tau = h'_{i+1}|_{\ldots} : T_{i+1} \to \omega_L^{-1}(T_i)$,
$\gamma = h'_{i+1}|_{\ldots} : M_{i+1} \to \omega^{-1}(M_i)$ and
$\kappa =$ the inclusion$: \omega_L^{-1}(T_i) \to \omega^{-1}(M_i)$.

Clearly $\gamma \circ \beta = \kappa \circ \tau$. Denote by $\beta^*, \tau^*, \gamma^*$ and κ^* the induced homomorphisms of the n-dimensional cohomology groups modulo $\mathbb{Z}_p$ of the corresponding spaces. Recall that the $(n+1)$-cells of $\omega^{-1}(M_i)$ not contained in $\omega_L^{-1}(T_i)$ are attached to $\omega_L^{-1}(T_i)$ by maps of degree p. Then κ^* is an isomorphism. Since M_{i+1} is contractible, β^* is the zero homomorphism. Thus we obtain that τ^* must also be the zero homomorphism. Then, since $g_{i+1}|_{\ldots} : T_{i+1} \to T_i$ factors through τ, the map $g_{i+1}|_{\ldots}$ induces the zero homomorphism of $H^n(T_i; \mathbb{Z}_p)$ and $H^n(T_{i+1}; \mathbb{Z}_p)$. Now we may assume that $\mathbb{Z}_p$ appears in the construction for infinitely many indices i. Then $\check{H}^n(r^{-1}(x); \mathbb{Z}_p) = 0$ and since $r^{-1}(x)$ is approximately $(n-1)$-connected and of $\dim \leqslant n$ we get that $r^{-1}(x)$ is $\mathbb{Z}_p$-acyclic.

Let G be a finitely generated torsion Abelian group such that $\dim_G X \leqslant n$. By the Bockstein theorem $\dim_{\mathbb{Z}_p} X \leqslant n$ for every $\mathbb{Z}_p \in \sigma(G)$. Then $r^{-1}(x)$ is $\mathbb{Z}_p$-acyclic for every p such that $\mathrm{Tor}_p\, G \neq 0$. Hence $r^{-1}(x)$ is G-acyclic and the theorem follows. □

Remark. It is easy to see that the proof of Theorem 1.4 also works for $\mathbb{Z}$ regarded as $\mathbb{Z}_p$ with $p = 0$. This way one can avoid the use of Theorem 1.3 in the proof of Theorem 1.4 and make the proof self-contained.

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