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On a hamiltonian cycle in which specified vertices are not isolated

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Abstract

Let G be a graph with n vertices and minimum degree at least $n/2$, and B a set of vertices with at least $3n/4$ vertices. In this paper, we show that there exists a hamiltonian cycle in which every vertex in B is adjacent to some vertex in B.

Keywords: Hamiltonian cycles; A minimum degree; Dirac-type condition; Specifed vertices

1. Introduction

Dirac [2] showed that a graph with n vertices and minimum degree at least $n/2$ is hamiltonian. From Theorem 1 of Egawa et al. [3], we have that, for any edge, there is a hamiltonian cycle which contains the specified edge, but with several exceptions. On the other hand, for any two vertices, there exists a hamiltonian cycle in which the vertices are not adjacent. Furthermore, it holds that, for any vertex subset A containing at most $n/4$ vertices, there is a hamiltonian cycle such that any two vertices in A are not adjacent. Kaneko and Yoshimoto generalized the fact as follows.

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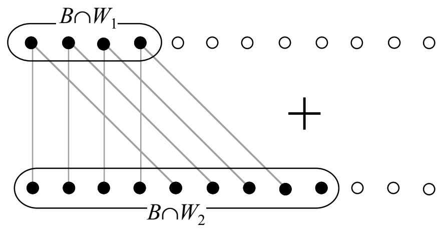

Fig. 1.

Theorem 1 (Kaneko and Yoshimoto [5]). *Let G be a graph with n vertices and minimum degree at least $n/2$, and let d be a positive integer such that $d \leqslant n/4$. Then, for any vertex subset A with at most $n/2d$ vertices, there exists a hamiltonian cycle C such that $d_C(u,v) \geqslant d$ for any vertices u and v in A.*

In this theorem, the distance $d_C(u,v)$ is defined as the number of edges in a shortest subpath joining u and v. Suppose that $d=3$, and let A be a vertex subset with at most $n/6$ vertices. Then, there is a hamiltonian cycle which satisfies the condition in the above theorem. In the cycle, any vertex in $B=V(G)\setminus A$ is adjacent to some vertex in B. In other words, for any vertex subset B with at least $5n/6$ vertices, there is a hamiltonian cycle C such that $d_C(u,B)=1$ for all vertex $u \in B$. In this paper, we improve the lower bound of the number of vertices in B as follows.

Theorem 2. *Let G be a graph with n vertices and the minimum degree at least $n/2$, and let B be a vertex subset with at least $3n/4$ vertices. Then, there exists a hamiltonian cycle in which every vertex in B is adjacent to some vertex in B.*

We show that the lower bound is the best possible one. A vertex subset is called *isolated* if the subset contains a vertex which has no neighbours in the subset. Assume that a graph is the balanced complete bipartite graph $K_{n/2,n/2}$ with partite sets W_1 and W_2, and let B be a vertex subset of $K_{n/2,n/2}$ of cardinality at most $3n/4-1$ such that $|B\cap W_2| \geqslant 2|B\cap W_1|+1$. Then, in any hamiltonian cycle, B contains at least $|B\cap W_2|-2|B\cap W_1|$ vertices which have no neighbours in B, i.e., B is isolated. See Fig. 1. Thus, the desired cycle does not exist.

Finally, we prepare notations used in the subsequent argument. We denote by $N_G(x)$ the set of vertices which is adjacent to x in a graph G and its cardinality by $\deg_G(x)$. The cardinality of S is denoted by $|S|$ and the subgraph induced by a vertex subset S is denoted by $\langle S\rangle$. A spanning subgraph F is called *a path factor* if F consists of paths of length of at least one. In the proof, we shall use the following theorem.

Theorem 3 (Johansson [4]). *Let G be a connected graph with n vertices and suppose that $n=\sum_{i=1}^{k} n_i$, where $n_i \geqslant 2$ is an integer. If the minimum degree of G is at least $\sum_{i=1}^{k}\lfloor n_i/2\rfloor$, then H has a path factor consisting of k components of orders $n_1,n_2,\ldots,n_k$.*

All the notations and terminologies not explained here are given in [1].

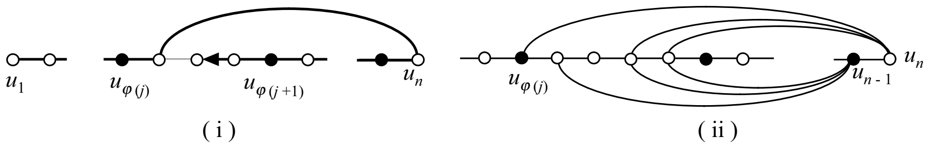

Fig. 2.

2. The Proof of Theorem 2

Suppose that there is a counterexample G for contradiction. Since the complete graph has the desired hamiltonian cycle, without loss of generality, we may assume that $G\cup xy$ is not a counterexample for any edge $xy\in E(\bar{G})$. If C is one of the desired hamiltonian cycles in $G\cup xy$, then $P=C-xy$ is a hamiltonian path in G. In this paper, we call such a hamiltonian path of G *a base path* joining x and y. By the minimum degree condition, there is an edge $y'x'$ in P such that xx' and $y'y\in E(G)$, and then $P\cup\{xx', y'y\}\backslash y'x'$ is a hamiltonian cycle of G. Thus, the following fact can be obtained easily.

Fact 1. *There exists a desired hamiltonian cycle if there is a base path joining vertices in B in which each end vertex is adjacent to a vertex in B.*

Let $A=V(G)\backslash B$, and we divide the argument into two cases.

Case 1: *There are two vertices $u\in A$ and $v\in B$ such that $uv\notin E(G)$.* Assume that a vertex $v_1\in B$ is not adjacent to a vertex v_n in A, then there is a base path $(v_1, v_2, \ldots, v_n)$ joining v_1 and v_n. We can show that there exists a base path $(u_1, u_2, \ldots, u_n)$ such that $\{u_1, u_2, u_n\}\subset B$ and $u_{n-1}\in A$ as follows. If there is no base path $(u_1, u_2, \ldots, u_n)$ with $\{u_1, u_2, u_n\}\subset B$, then the vertex v_n cannot have a neighbour in $\{v_i \mid v_{i-1}\in B, v_{i+1}\in B\}$, and hence all the neighbours are in the complement of this set, this complement is as follows:

$$\{v_{i-1}, v_{i+1} \mid v_i\in A\backslash v_n\}\cup v_{n-1}.$$

Since $|A\backslash v_n|\leqslant n/4-1$, we have $|N_G(v_n)|\leqslant 2(n/4-1)+1=n/2-1$. This is a contradiction. Also, by Fact 1, we may assume that $u_{n-1}\in A$.

Let $P=(u_1, u_2, \ldots, u_n)$ be a base path such that $\{u_1, u_2, u_n\}\subset B$ and $u_{n-1}\in A$. Let $u_{\varphi(1)}, u_{\varphi(2)}, \ldots, u_{\varphi(|A|)}$ be the vertices of A taken in the order as they occur on P, and let

$$Q_j=(u_{\varphi(j)}, u_{\varphi(j)+1}, \ldots, u_{\varphi(j+1)-1})$$

for all $j\leqslant|A|-1$. We note that the vertex u_n is adjacent to none of $\{u_i \mid \varphi(j)+1\leqslant i\leqslant \varphi(j+1)-3\}$ with $j\leqslant|A|-1$, otherwise G has a base path which satisfies the condition of Fact 1. See Fig. 2(i). Therefore, we have

$$N_G(u_n)\cap V(Q_j)\subset\{u_{\varphi(j)}, u_{\varphi(j+1)-2}, u_{\varphi(j+1)-1}\} \tag{1}$$

for all $j\leqslant|A|-1$. See Fig. 2(ii). Furthermore, we prove the following claim.

Claim 1. *There is an edge* $xy \in E(P)$ *such that* $P \backslash \{xy, u_{n-1}u_n\} \cup \{xu_n, yu_n\}$ *is also a base path.*

Proof. Let $R_j = (u_{\varphi(j-1)+1}, u_{\varphi(j-1)+2}, \dots, u_{\varphi(j)})$ for all $j \leqslant |A|$, where we set $\varphi(0) = 0$. As in the previous argument, we have

$$N_G(u_n) \cap V(R_j) \subset \{u_{\varphi(j)-2}, u_{\varphi(j)-1}, u_{\varphi(j)}\}.$$

If equality holds for some j, then the statement is true. Therefore, we can assume that $|N_G(u_n) \cap V(R_j)| \leqslant 2$ for all j. Since

$$\frac{n}{2} \leqslant |N_G(u_n)| = \sum_{j=1}^{|A|} |N_G(u_n) \cap V(R_j)| \leqslant 2|A| \leqslant \frac{n}{2},$$

we have $|A| = n/4$ and $N_G(u_n) \cap V(R_j) = \{u_{\varphi(j)-2}, u_{\varphi(j)}\}$ for all j. Since the vertex u_1 is not adjacent to u_n, it holds that $|V(R_1)| \geqslant 4$. Therefore, there is an integer $l \geqslant 2$ such that $|V(R_l)| = 3$. Then both $u_{\varphi(l)-3} = u_{\varphi(l-1)}$ and $u_{\varphi(l)-2}$ are adjacent to u_n, and $P \backslash \{u_{\varphi(l-1)}u_{\varphi(l)-2}, u_{n-1}u_n\} \cup \{u_{\varphi(l-1)}u_n, u_{\varphi(l)-2}u_n\}$ is a base path, i.e., $u_{\varphi(l-1)}u_{\varphi(l)-2}$ is the desired edge. □

If the vertex u_{n-1} is adjacent to u_1, then we can obtain the desired cycle from the above claim. Next we consider vertices in Q_j which can be adjacent to u_{n-1}. Assume that $|V(Q_j)| \neq 1$, i.e., $|V(Q_j)| \geqslant 3$. Suppose that the vertex u_{n-1} is adjacent to $u_{\varphi(j)}$. By Claim 2, there is an edge $xy \in E(P)$ such that $P \backslash \{xy, u_{n-1}u_n\} \cup \{xu_n, yu_n\}$ is a base path. If $xy \neq u_{\varphi(j)}u_{\varphi(j)+1}$, then the base path $P \backslash \{u_{\varphi(j)}u_{\varphi(j)+1}, xy, u_{n-1}u_n\} \cup \{u_{\varphi(j)}u_{n-1}, xu_n, yu_n\}$ also yields the desired hamiltonian cycle from Fact 1. In the case of $xy = u_{\varphi(j)}u_{\varphi(j)+1}$, the base path $P \backslash \{u_{\varphi(j)}u_{\varphi(j)+1}, u_{n-1}u_n\} \cup \{u_{\varphi(j)}u_{n-1}, u_{\varphi(j)+1}u_n\}$ also yields the desired cycle. Thus, we may assume that u_{n-1} is not adjacent to $u_{\varphi(j)}$. Similarly, it holds that the vertex u_{n-1} is adjacent to none of $\{u_i \mid \varphi(j) + 2 \leqslant i \leqslant \varphi(j+1) - 3\}$. Therefore, for all $j \leqslant |A| - 1$, we have

$$N_G(u_{n-1}) \cap V(Q_j) \subset \{u_{\varphi(j)+1}, u_{\varphi(j+1)-2}, u_{\varphi(j+1)-1}\} \tag{2}$$

if $|V(Q_j)| \neq 1$. See Fig. 2(ii). The following claim is important.

Claim 2. *Suppose that for some* $u \in B$, *the graph* $G - u$ *has a hamiltonian cycle in which every vertex in* $B \backslash u$ *is adjacent to some vertex in* $B \backslash u$. *Then,* G *contains a desired hamiltonian cycle.*

Proof. Let $D = (v_1, v_2, \dots, v_{n-1})$ be the cycle of $G - u$ satisfying the condition. Let $v_{\psi(1)}, v_{\psi(2)}, \dots, v_{\psi(|A|)}$ be the vertices of A taken in the order as they occur on P, and $R_j = (v_{\psi(j)}, v_{\psi(j)+1}, \dots, v_{\psi(j+1)-1})$. If $|R_j| \geqslant 5$, then we have $R_j \cap N_G(u) \subset \{v_{\psi(j)}\}$, otherwise there is a desired cycle from Fact 1. See Fig. 3(i). Of course, $|R_j \cap N_G(u)| \leqslant 1$ if $|R_j| = 1$. Similarly, it holds that if $3 \leqslant |R_j| \leqslant 4$, then $|R_j \cap N_G(u)| \leqslant 2$. Since $|A| \leqslant n/4$, it holds that $n/2 \leqslant |N_G(u)| = \sum_{j=1}^{|A|} |N_G(u) \cap V(R_j)| \leqslant 2|A| \leqslant n/2$. Therefore, we have that $|A| = n/4$, $3 \leqslant |R_j| \leqslant 4$ and $|N_G(u) \cap R_j| = 2$ for all $j \leqslant |A|$. In particular, u is adjacent to

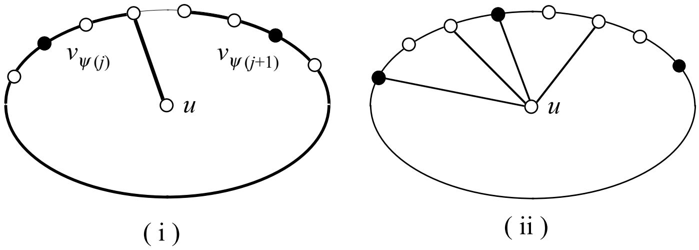

Fig. 3.

all $v_{\psi(j)}$. Since D has $n-1$ vertices, there is j such that $|R_j|=3$, and then we can find a desired hamiltonian cycle of G as in the proof of Claim 1. See Fig. 3(ii). □

Let

$$d_j = |\{u_{\varphi(j)+1}, u_{\varphi(j)+2}, \ldots, u_{\varphi(j+1)}\} \cap N_G(u_1)| + |V(Q_j) \cap N_G(u_{n-1})| + |V(Q_j) \cap N_G(u_n)|,$$

then the following claim holds.

Claim 3. $d_j \leqslant |V(Q_j)| + 2$ *for all* $j \leqslant |A| - 1$.

Proof. If $|V(Q_j)|=1$, then the statement is trivial, and $|V(Q_j)| \neq 2$, because P is a base path. Thus, we show the claim in the case of $|V(Q_j)| \geqslant 3$. Notice that $|V(Q_j) \cap N_G(u_{n-1})|$ and $|V(Q_j) \cap N_G(u_n)|$ are at most three by (1) and (2), and thus, $d_j \leqslant |V(Q_j)| + 6$. Especially, if $|V(Q_j)|=3$, then $d_j \leqslant |V(Q_j)|+5$, because $u_{\varphi(j)+1} = u_{\varphi(j+1)-2}$.

Suppose that the vertex u_1 is adjacent to $u_{\varphi(j+1)}$. Then the vertex u_n is adjacent to neither $u_{\varphi(j+1)-2}$ nor $u_{\varphi(j+1)-1}$, otherwise we can find out a desired cycle by using Claim 2. Similarly, the vertex u_{n-1} is not adjacent to $u_{\varphi(j+1)-1}$. Thus, if $|V(Q_j)|=3$, then the inequality holds. In the case where $|V(Q_j)| \geqslant 4$, we have that u_{n-1} is not adjacent to $u_{\varphi(j+1)-2}$ by using Claims 1 and 2, and also the inequality holds. Now, the case where $u_1 u_{\varphi(j+1)} \in E(G)$ is shown.

Assume that $u_1 u_{\varphi(j+1)} \notin E(G)$ and u_1 is adjacent to $u_{\varphi(j+1)-1}$. As in the previous argument, each of u_{n-1} and u_n is not adjacent to $u_{\varphi(j+1)-2}$ if $|V(Q_j)| \geqslant 4$. If all other edges exist, then we can find out a desired hamiltonian cycle. See Fig. 4(i). Otherwise, the inequality holds. The case of $|V(Q_j)|=3$ is similar. See Fig. 4(ii).

Suppose that u_1 is adjacent to neither $u_{\varphi(j+1)}$ nor $u_{\varphi(j+1)-1}$. Assume first that $|V(Q_j)| \geqslant 4$. If $u_{\varphi(j+1)-1} u_n \notin E(G)$ and all other edges exist, we can find a desired cycle. See Fig. 4(iii). Thus we may assume that $u_{\varphi(j+1)-1} u_n \in E(G)$. If u_{n-1} is adjacent to $u_{\varphi(j+1)-2}$, then

$$(u_1, u_2, \ldots, u_{\varphi(j+1)-2}, u_{n-1}, u_{n-2}, \ldots, u_{\varphi(j+1)-1}, u_n)$$

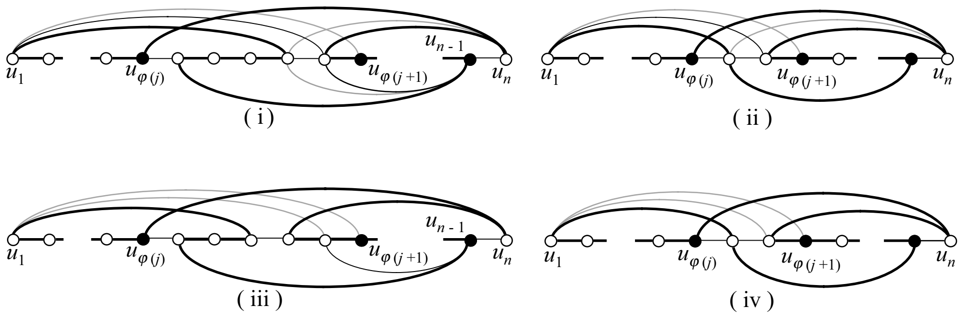

Fig. 4.

is a base path satisfying the condition in Fact 1. Therefore, $u_{\varphi(j+1)-2}u_{n-1} \notin E(G)$, and we may suppose that all other edges exist. Then we can find a desired cycle as in Fig. 4(iii). The case when $|V(Q_j)| = 3$ is similar. See Fig. 4(iv). □

From this claim, we have

$$\sum_{j=1}^{|A|-1} d_j \leqslant \sum_{j=1}^{|A|-1} (|V(Q_j)| + 2) \leqslant \sum_{j=1}^{|A|-1} |V(Q_j)| + \left(\frac{n}{2} - 2\right).$$

Let $Q_0 = (u_1, u_2, \ldots, u_{\varphi(1)-1}) \cup (u_{n-1}, u_n)$, then $\sum_{j=1}^{|A|-1} |V(Q_j)| + (n/2 - 2) = 3n/2 - (|V(Q_0)| + 2)$. Let

$$d_0 = |\{u_2, u_3, \ldots, u_{\varphi(1)}\} \cap N_G(u_1)| + |V(Q_0) \cap N_G(u_{n-1})| + |V(Q_0) \cap N_G(u_n)|.$$

If $d_0 \leqslant |V(Q_0)| + 1$, then

$$|N_G(u_1)| + |N_G(u_{n-1})| + |N_G(u_n)| = \sum_{j=0}^{|A|-1} d_j \leqslant 3n/2 - (|V(Q_0)| + 2) + d_0 < \frac{3n}{2}.$$

This contradicts the minimum degree condition. Thus, to complete the proof in the present case, it suffices to show that $d_0 \leqslant |V(Q_0)| + 1$.

Let us show that the inequality holds. The vertex u_1 is adjacent to neither u_{n-1} nor u_n, and it holds that

$$N_G(u_{n-1}) \cap V(Q_0) \subset \{u_{\varphi(1)-2}, u_{\varphi(1)-1}, u_n\}$$

and

$$N_G(u_n) \cap V(Q_0) \subset \{u_{\varphi(1)-2}, u_{\varphi(1)-1}, u_{n-1}\}$$

as in the previous argument. Thus, we have $d_0 \leqslant |V(Q_0)| + 4$. Assume that $|V(Q_0)| > 4$. If the vertex u_1 is adjacent to $u_{\varphi(1)}$, then each of u_{n-1} and u_n is adjacent to neither $u_{\varphi(1)-2}$ nor $u_{\varphi(1)-1}$ as in the proof of Claim 3.

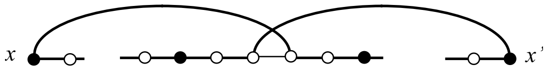

Fig. 5.

Suppose that $u_1u_{\varphi(1)} \notin E(G)$. If u_1 is adjacent to $u_{\varphi(1)-1}$, then each of u_{n-1} and u_n is not adjacent to $u_{\varphi(1)-2}$, and thus, the inequality is true. Also, assume that u_1 is not adjacent to $u_{\varphi(1)-1}$. If there are both the edges $u_{\varphi(1)-2}u_{n-1}$ and $u_{\varphi(1)-1}u_n$, we can find out the desired hamiltonian cycle by Fact 1. Otherwise, the inequality holds. The case when $|V(Q_0)|=4$ can be shown similarly.

Case 2: *All the pairs of vertices* $x\in A$ *and* $y\in B$ *are adjacent*. Suppose that there are two non-adjacent vertices x and x' in A, and let P be a base path joining them. Since $|B|\geqslant 3|A|$, there exist at least four vertices in B such that these appear consecutively in P. Thus, we can easily obtain the desired hamiltonian cycle. See Fig. 5

Assume that $\langle A\rangle$ is complete. If the induced subgraph of B contains a path factor with at most $|A|$ components, then it is a plain fact that the desired cycle can be obtained from the factor because all the pairs of vertices $x\in A$ and $y\in B$ are adjacent. Thus, we show the existence of such a path factor.

If the induced subgraph $\langle B\rangle$ is not connected, then both components in $\langle B\rangle$ contain a hamiltonian path by the minimum degree condition and these constitute a path factor of $\langle B\rangle$. (By the minimum degree condition of G, we have $|A|\geqslant 2$, in this case.) Suppose that $\langle B\rangle$ is connected and let $q=|B|-3(|A|-1)$. Since $|B|\geqslant 3|A|$, we have $q\geqslant 3$ and the minimum degree is at least

$$\left\lceil \frac{|A|+|B|}{2} \right\rceil - |A| = |A| + \left\lceil \frac{q-3}{2} \right\rceil = |A| + \left\lfloor \frac{q-2}{2} \right\rfloor = (|A|-1) + \left\lfloor \frac{q}{2} \right\rfloor$$

$$= \sum_{i=1}^{|A|-1} \left\lfloor \frac{3}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor .$$

Thus, the induced subgraph $\langle B\rangle$ has a path factor with $|A|$ components from Johansson's theorem. Now the proof is complete.

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