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Ascent of Bubbles in Magma Conduits Using Boundary Elements and Particles

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Abstract

We investigate the use of the Multipole-accelerated Boundary Element Method (BEM) and of the Singularity Method for studying the interaction of many bubbles rising in a volcanic conduit. Observation shows that the expression of volcanic eruption is extremely variable, from slow release of magma to catastrophic explosive manifestation. We investigate the application of the Fast Multipole Method to the solution of (i) the Boundary Element Formulation of the Stokes flow and of (ii) the particle formulation using the Stokeslets, the Green Function of the Stokes flow law, as a particle kernel. We show how these implementations allow for the first time to numerically model in a dynamic setting a very large number of bubbles, i.e few thousands with the BEM models, allowing investigating the feedback between the single bubble deformation and their collective evolution, and few hundred of thousands of bubbles with the particle approach. We illustrate how this method can be used to investigate the intense interaction of a large number of bubbles and suggest a framework for studying the feedback between many bubbles and a complex thermal nonlinear magmatic matrix.

Keywords: Boundary Elements, Fast Multipole Method, Fluid Dynamics, Bubble Dynamics, Volcanology, Strombolian Activity, Explosive Eruption

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1. Introduction

Magmas are multiphase flows transporting solid and gas components that undergo shear deformation [8, 4], making the calculation of their rheology —or in general their non-steady dynamic response— a complex problem. The gas component is always involved in volcanic eruptions. In fact the three main types of eruptions are (i) magmatic (magma+gas, but magma driven), (ii) phreatomagmatic (magma+gas, but gas driven), (iii) phreatic eruption (steam superheating due to contact with magma). While the observation of the large variety of volcanic activity, from slow eruption to explosive [2], illustrates how the system dynamics can diverge, many volcanic conduit release their gas and magma through a frequent and repetitive controlled explosive manifestation called "Strombolian Activity" [3].

Another way is to distinguish between types of eruptions is to order them in term of activity, Hawaiian, Strombolian, Volcanian, Plinian and Palean, where the last is the most intense. This works aims at analysing bubbles dynamics for the Strombolian case, in which the activity is gas (and therefore bubble) dominated, but in which the dynamics is not yet explosive [6]. In such type of activity the gas component inside the ascending magma is separated in bubbles that expand due to the rise of magma and the consequent reduction of pressure [1]. The bubble expansion, increasing the speed of rise and the amount of interaction with the bubbles with each other, is believed to be responsible of the emergence of a drift in complexities of the bubble collective dynamics, but such results has never been modelled. This work focuses on how this interaction acts, how its numerical modeling can be numerically tackled and what early results suggest [7].

We show here how the Boundary Element Method applied to Stokes flow can be used to study the complex dynamics of a large number of Bubbles, revealing the feedback between their morphological evolution and their collective dynamics. In the limit where the deformation of the bubbles is negligible, i.e. when the surface tension is sufficiently strong to compensate the shear forces applied to a single bubble, the dynamics of a large number of bubbles can be investigated with the simpler Singularity Method, using the fundamental Stokes' flow solutions called Stokeslets, as recently employed in geophysical fluid dynamics [10] and bio-fluid dynamics [11]. If the Strombolian Activity is a manifestation of the nonlinear interaction of the rising bubbles, which spontaneously evolves toward a regime of marginal stability [9]analogously to many other complex systems [5], our approach is the most suited for detecting the phase transition responsible of the observed patterns at the surface.

The present work illustrates how the Fast Multipole (FM) can be combined with (i) the Boundary Element Method (BEM), achieving the most efficient strategy to analyse the two phase flow represented by the bubbles-magma interaction; and with (ii) the Particle approach, where particles are defined by Stokeslets, i.e. Green Functions of the Stokes equation, to model the interaction of a large number (up to $100,000$ per processor) of almost rigid bubbles. The two approaches are used for studying these systems at two different scales. The first for analysing the feedback between the morphological evolution of the single bubble and the evolution of the collectivity of bubbles, and the second to analyse the large scale patterns formed by large sets of above millions of bubbles.

This paper is structured as follows. In Sect. 2, we summarise our approach and present several resolution and scaling tests. In Sect. 3 and Sect. 4 we show the Boundary Element Method-based and the Stokeslet-based models respectively. Insights on the use of this approach for understanding the real cases such as the Strombolian Activity are finally suggested in Sect. 5.

2. Simulation of Stokes Flow

2.1. Governing Equations

We consider the Stokes equation for a steady, highly viscous flow

$$\nabla \cdot \sigma + \rho \mathbf{b} = 0 \tag{1}$$

where $\mathbf{b}$ is the body force, and the stress tensor term $\nabla \cdot \sigma = -\nabla P + \mu \nabla^2 \mathbf{u}$ is related to P the pressure; $\mathbf{u}$, the velocity field and μ, the dynamic viscosity.

The Stokes equation can be recast into a variety of integral equations. We follow here the formulation of [13], to which we refer for further details. We denote as D is the domain where (1) holds and we write

$$u_i(\mathbf{x}_o) = \frac{1}{8\pi\mu} \int_D \sigma_{ik}(\mathbf{x}) n_k G_{ij}(\mathbf{x}, \mathbf{x}_o) dS(\mathbf{x}) + \frac{1}{8\pi} \int_D u_i(\mathbf{x}) n_k T_{ijk}(\mathbf{x}, \mathbf{x}_o) dS(\mathbf{x}) \tag{2}$$

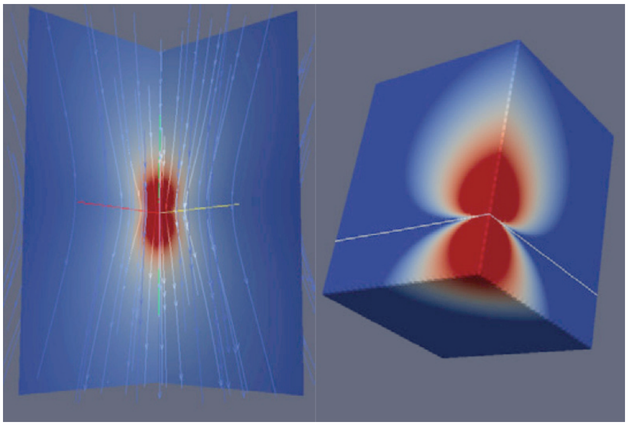

Figure 1: Stokeslet fundamental solutions. On the left the representation of the Velocity Green Function, on the right the induced pressure.

where G_{ij} and T_{ijk} are the steady, Green's functions for velocity and stress respectively, also known as the *Stokeslet* and the *Stresslet*

$$G_{ij}(\mathbf{x}-\mathbf{x}_o) = \frac{\delta_{ij}}{r} + \frac{\hat{x}_i\hat{x}_j}{r^3};\ \hat{\mathbf{x}} = \mathbf{x}-\mathbf{x}_o \text{ and } r = |\hat{\mathbf{x}}| \tag{3}$$

$$T_{ijk}(\mathbf{x}-\mathbf{x}_o) = -6\frac{\hat{x}_i\hat{x}_j\hat{x}_k}{r^5}. \tag{4}$$

In turn, (2) is cast into a form more appropriate for quasi-steady multiphase flows in the presence of a gravity field. Hence for $\mathbf{x} \in S_i$ we obtain

$$\frac{1+\lambda_i}{2}\mathbf{u}(\mathbf{x}) - \sum_j^N \frac{1-\lambda_j}{8\pi}\int_{S_j}^{PV} \mathbf{n}\cdot\mathsf{T}\cdot\mathbf{u}\,dS = -\frac{1}{8\pi\mu_0}\sum_j^N\int_{S_j}\mathsf{G}\cdot\Delta\mathbf{f}\,dS\ , \tag{5}$$

where PV denotes the principal value of the integral, μ_0 is the viscosity of the mantle taken as a reference for $\lambda_i = \mu_i/\mu_0$, and the normal stress jump $\Delta\mathbf{f}$ that accounts for gravity is given by

$$\Delta\mathbf{f} = \Delta\rho\,(\mathbf{b}\cdot\mathbf{x})\,\mathbf{n}\,. \tag{6}$$

2.2. Numerical Method

The surfaces S_i and the supported quantities $\mathbf{u}, \Delta\mathbf{f}, \ldots$ are discretized with panels. The boundary integral equation (5) thus becomes a linear system

$$((1+\lambda)/2 + \mathsf{T})\,\mathbf{U} = \mathbf{F}\,. \tag{7}$$

Where U and F are the vectors defining the velocity u and the right hand side of equation (5), respectively. Many approaches rely to the construction of the matrix T, a computation that scales as N^2_{panels} both memory- and computation time-wise, making them impractical for large systems.

We use a fast multipole method (FMM) [14, 15, 16] for the evaluation of the integrals in (5). The FMM scales as $N\log(N)$, which is far more tractable and still allows the use of a Generalised Minimal Residual method (GMRES) or any Krylov space based method that does not rely on the storage of the full matrix.

A multipole method exploits the decay of the kernel to convolve and makes a controlled approximation. More explicitly, let us compute

$$u(\mathbf{x}_o) = \int_D G(\mathbf{x}_o-\mathbf{x})\rho(\mathbf{x})dV(\mathbf{x})\,. \tag{8}$$

We consider the contribution from D_i, a part of D that is far *enough* from our evaluation point $\mathbf{x}_o$ and proceed with a Taylor expansion of the kernel G about $\mathbf{x}_c \in D_i$

$$\begin{aligned} u(\mathbf{x}_o) &= \int_{D_i} G(\mathbf{x}_o - \mathbf{x})\rho(\mathbf{x})dV(\mathbf{x}) \\ &\simeq \int_{D_i} (G(\mathbf{x}_o - \mathbf{x}_c) - \nabla G(\mathbf{x}_o - \mathbf{x}_c)\cdot(\mathbf{x}_o - \mathbf{x}_c) + \ldots)\rho(\mathbf{x})dV(\mathbf{x}) \\ &\simeq G(\mathbf{x}_o - \mathbf{x}_c)\int_{D_i} \rho(\mathbf{x})dV(\mathbf{x}) \\ &\quad -\nabla G(\mathbf{x}_o - \mathbf{x}_c)\cdot \int_{D_i} (\mathbf{x}_o - \mathbf{x}_c)\rho(\mathbf{x})dV(\mathbf{x}) + \ldots \end{aligned} \tag{9}$$

We note that the equation involves successive moments of the ρ distribution in D_i. The FMM algorithm sorts the sources in a tree structure whose cells contain the moment integrals —or multipoles— and carries out a field evaluation through a tree traversal. The refinement of the interactions is determined by a tree traversal stopping criterion based on a prescribed tolerance. The reader is referred to [14, 15, 16] for further details.

The present FMM code can handle convolutions with the Green's functions for the Poisson equation, the Stokeslet or the Stresslet. It employs up to the second order moments of the source distributions (quadrupoles).

2.3. Performances

The FMM-BEM drastically improves the computational cost of the method. For the coarse resolutions, the method displays the nominal N^2 scaling of a direct interaction code. The scaling then quickly approaches a nearly linear one ($N\log(N)$) for the finer resolutions. In its current implementation the FMM-BEM uses a shared tree, thus reducing the communication load at the expense of memory requirements.

The FMM-BEM has been parallelised using MPI and tested on a Opteron cluster with Quadrics connections, displaying a very good scaling up to 64 CPUS, still keeping 90% of efficiency. Another test has been performed on the Silica Science and Engineering High Performance Computing system at The University of Sydney. Silica is a SGI Altix XE1200 Cluster with Infiniband connections in which each computing node has 8 processors. On this system, taking the maximum load of a node (8 CPUs) as a reference, was always above 80%, up to 64 processors.

Because parallel efficiency is mainly affected by geometrical mesh query and manipulation routines, more tests with efficiency vs. number of particles and mesh density per particle are however still necessary.

3. Modeling Bubble Dynamics with the Boundary Element Approach

As in existing numerical models in the literature [7], the bubbles are modelled as non-viscous nearly dense-less bodies immersed in a viscous matrix. Many implementations of such setting exist but they all need extremely high numerical effort to model a relatively low number of bubbles. For example the largest system modelled today with FD or FE is composed by $1000 \times 1000 \times 1000$ cells, which allows less than a hundred of highly resolved bubbles. Other methods, such as spectral, do not allow the high and sharp viscosity ratio required by the exact setting. BEM is instead the ideal setup for solving such kind of problems, allowing modelling already $1,000$ bubbles at high resolution (128 elements per bubble) with a relatively little computational effort, consisting in 64 processors and less than $24h$ for calculating $1,000$ steps.

A first example of such simulations is displayed in Figure 2, where 4 steps of the evolution of 625 regularly spaced bubbles with different radiiand negligible surface tension illustrate their collective behavior. Although each bubble has a radically different radius the resulting cooperative pattern resembles the one of a single coherent plume. This highlights a striking characteristic of this model: the overall collective dynamics is unaffected by bubble heterogeneity. Bubbles, instead, undergo a very strong deformation that depends on the relative position of the single bubble from the axis of the collectively controlled diapirism, but is largely independent from the single bubble size. It is particularly striking to observe that the final stage of the model approximates a completely symmetric solution, which could be obtained with a quarter sized setup, however the bubbles sizes are not symmetric respect to vertical planes, so the

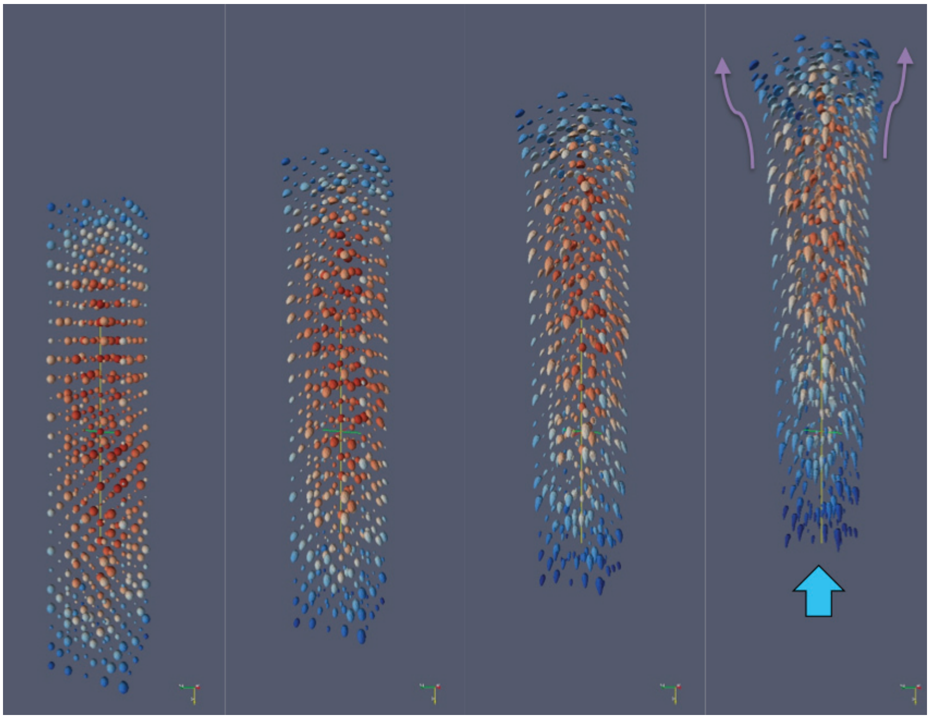

Figure 2: Four snapshots of 625 bubbles rising. The bubbles, initially regularly spaced, deform and evolve in a complex fashion responding to the surrounding flow created by their collective forces

symmetric outcome is purely the results of the natural smoothing due the superposition of the far field forcing of each bubble.

While this simulation allows a detailed look at the single bubble morphology, the study of the long term behaviour of a much greater set of bubbles can be more efficiently performed with the alternative approach described in the following section.

4. Using the Green Functions of the Stokes flow for Modelling the Collective behaviour of a Very Large Number of Bubbles

Figure 3 represents the long term evolution of a very large number of particles, each representing one bubble with different size characterised by different buoyancy compared to the external magma and undergoing a differentiated drag. We assume here that the surface tension of the bubble restrains the geometrical deformation of the single bubble, so the system is a different end-member case compared to the last section where the surface tension was instead negligible. Physically the predominance of the role of the surface tension arises in sets of particularly small bubbles.

The calculations of figure 3 have been performed on a laptop on one processor in less than one hour. This means that using an appropriate Parallel Multipole solver, such as the recently developed PetFMM, or alternative implementations, it will be possible to scale this approach to many millions or tens of millions of particles, with the possibility to study the interactions of very large sets of bubbles and investigate the possible emergence of waves and other solitonic solutions, which might explain the observation of the regular low energy Strombolian and other volcanic activities.

This simple approach is numerically comparable to a standard particle technique as the Vortex method [17, 18, 19]. The sum of the linear contribution of each particle's kernel, defined by the Stokeslet Green Functions, is performed employing the Fast Multipole Method. As the particles tend to cluster, it is predictable that in the long term the bubbles will coalesce. Although this has not been implemented yet, no technical complications are foreseen for adding this

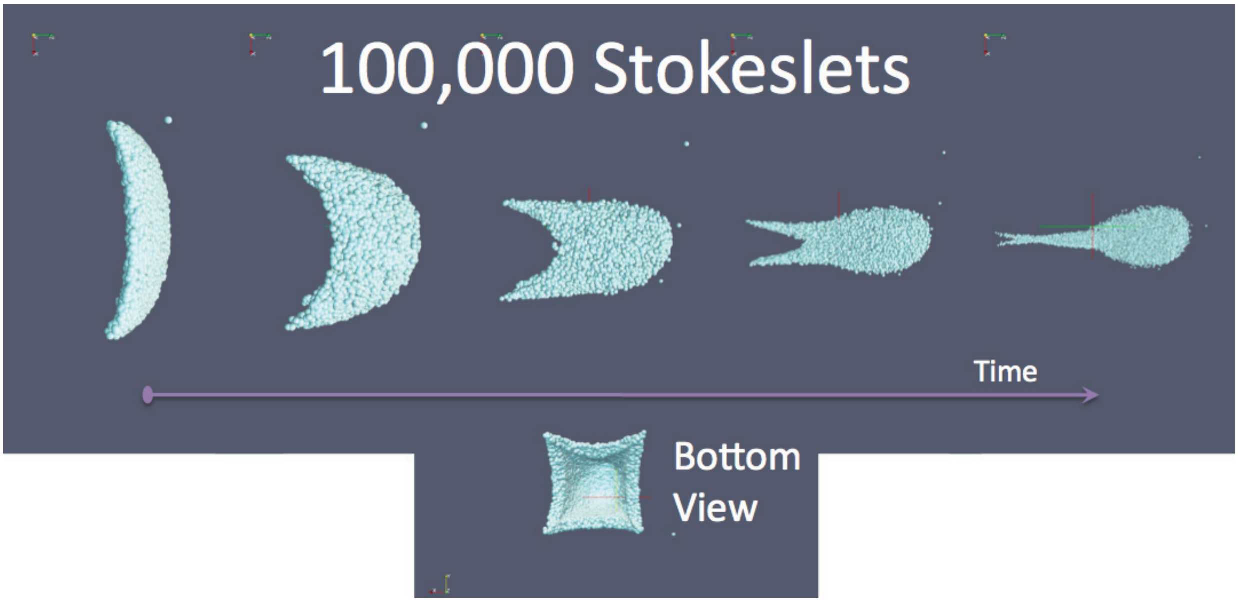

Figure 3: Simulation of 100,000 bubbles each modelled as a rigid sphere. The conduit size is much bigger than the dimensions of the modelled configuration, therefore the model is laterally unbounded. The bubbles collectively join attracted by the presence of their neighbours. It is striking how almost all bubbles merge into the collective dynamics, even when each one has a different size and therefore different buoyancy. The flow induced in the surrounding magma is the main driver of their dynamics. This model illustrates how a big sets of separate bubbles is able to collectively rise and coalesce into a larger bubble.

extra feature. Defining a critical distance threshold inferior to the sum of the radius of two neighbouring particles, it will be possible to implement a merging "event" in which a particle whose volume is the sum of each bubble will be produced.

A striking observation of the many bubbles models such as the one of Figure 3 is that the formation of a collective morphology is mainly driven by the global induced flow. Although the bubbles are different, being characterised by different sizes, they are dragged and clustered by the flow itself. This suggests that the global dynamics of bubbles is described by fewer parameters representing the initial spatial distribution, maybe even disregarding particular bubble size. We stress again that the bubble sizes are not symmetric respect to vertical planes, so the global symmetric outcome is the result of the collective effect of the bubbles and not by symmetric initial conditions.

We propose that the regular pulsation characteristic of Strombolian Activity has its origin in the non-linear collective behaviour of the bubbles [1, 2, 3]. However to establish the effects of such behaviour in the dynamics of Strombolian and other more general intricate magma, it is necessary to study in more detail the complex interaction between bubbles, crystals and the background magma. The next section illustrates a method to integrate the non-linear and temperature-dependent complexities of the magma rheology into this numerical setting.

5. Coupling Boundary Elements and finite differences for Modelling Thermal and Non-linear Magmas

Many works have been devoted to the study of the role of nonlinear rheologies in magma. Experiments have shown how a slight variation of few tens of degrees in magma temperature produces several orders of magnitude of variation in the viscosity. How such strong non-linearities in the dynamics of the rising bubbles and suspended crystals might influence the models above is uncertain. We propose here a simple strategy to "perturb" the numerical approach presented before through its coupling with a standard finite difference (FD) setup.

Figure 4 shows a simulation in which temperature was calculated in the FD background and advected using the velocity field from the Fast BEM solution projected to the Nodes of the FD mesh. By superimposing a FD Multigrid advection-diffusion solver we simulate the thermodynamical evolution of the system, which is not accessible with the BEM solver, and use it to change the rheology of the magma. In fact if each node of the grid represents a different

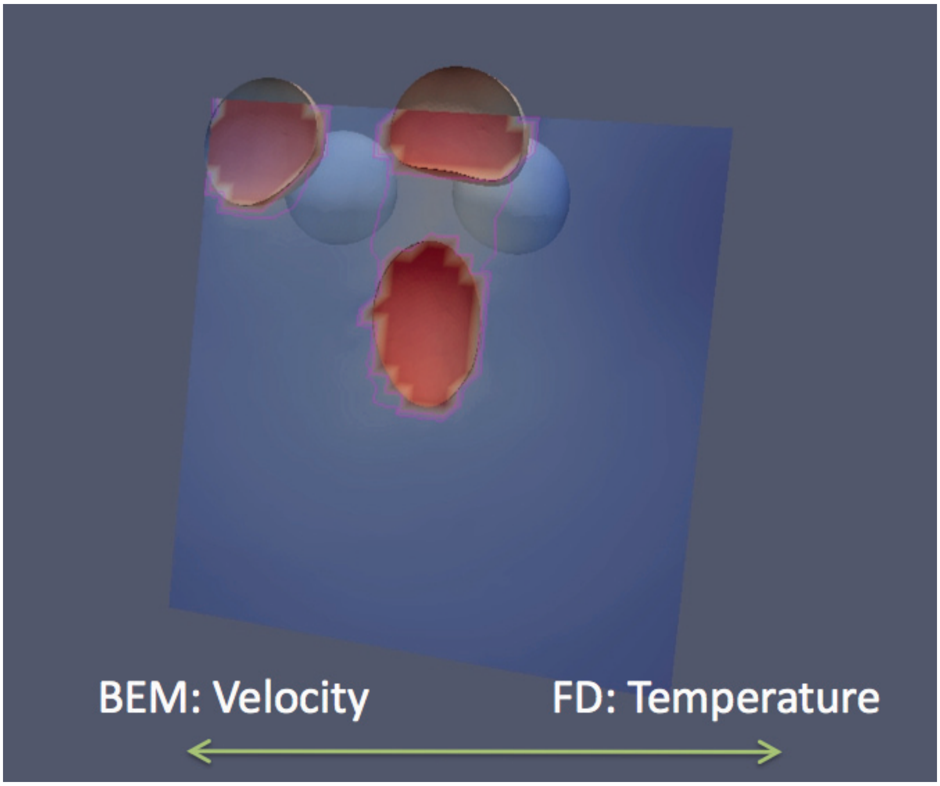

Figure 4: Snapshot of a simulation of the rise of three bubbles modelled with Boundary Elements, surrounded by a FD mesh representing the continuous temperature field. The visualised colours are interpolated from the FD solution, red being the bubble interior temperature (here assumed fixed) and blue the background temperature controlled by a diffusion process. The bottom scheme represents the staggered scheme employed to solve the feedback between Boundary Elements and FD. The velocity field in the 3D space is deduced from the BEM solution and projected into a finite difference grid where it contributes to the advection diffusion of heat in the medium. This approach allows setting non-homogeneous viscosity in the 3D space and projecting than it back to the panels, changing the defining coefficient of the BEM solver.

local viscosity, the response of the magma to the moving bubble can be again recalculated using the particle solver, as shown in [13]. This definition of the local rheology enables the application of the techniques described before to the study of the non-linear response of the mantle to the immersed components.

In simple terms the forcing on the boundary of the bubble is classically expressed by

$$\Delta \mathbf{f}_1 = (\rho_{\text{Bubble}} - \rho_{\text{Magma}})\, g\, z\, \mathbf{n}, \tag{10}$$

where z is the depth of the conduit and g is the gravity acceleration, assumed vertical and constant: $\mathbf{b} = -g\,\mathbf{x}/r$. The complex rheology is instead defined at the bubble boundary through the reference viscosity: μ_{Magma}, and the bubble-magma viscosity ratio $\lambda = \mu_{Bubble}/\mu_{Magma}$. Because this ratio is virtually zero, being the bubble virtually inviscid, the magma complex rheology will only influence the μ_{magma} coefficient. This method can be employed to reformulate the Stokeslet response and the Boundary Element formulation embedding higher moments representing the non-linearity of the magma dynamics. The drawback of this method is that it requires to calculate such response in the entire filled 3D space and not only on the equations conveniently projected on the Boundaries or Particles. Still this allows separating the less time consuming procedure (the calculation of the higher moments) on the FD nodes, from the expensive calculations necessary to invert the system on the bubble boundaries.

6. Conclusions

We have presented a novel computational approach for modeling the interaction of a very large number of bubbles immersed in magma. The method can be extended to any other system in which bubbles are suspended, such as corn syrup, honey, many types of gelatine (solving the elasticity equation as well). We employed the fast multipole acceleration to improve the traditionally slow Boundary Element approach, avoiding the calculation of the dense matrix that characterises it.

Several examples of collective behavior of bubbles have been illustrated, from the highly resolved BEM model of each bubble's free surface evolution, to a simpliified formulation in which each bubble is replaced by sphere whose linear response with the surrounding fluid is entirely described by the Green Functions of the Stokes flow, called Stokeslets. We have shown how both methods show similar results illustrating how the bubbles always tend to cluster. This important result must be tested in other cases for very large number of particles to determine at which length-scale such clustering will break down and determine the interval between two manifestations of the bubble rise as in the Strombolian Activity.

Observation of natural behaviour of the volcanos exhibiting Strombolian Activity indicates that they are really in a state of marginal stability, as shown by the return of the system into the same pulsating mode after each destructive explosive eruption. Whether the approach here proposed is able to explain the complexities and the emergence of waves of bubbles from volcanic conduit can be determined only with much more testing. We have however shown how this is a computationally feasible problem with the present available computational tools, combining the Boundary Element, the Particle (Stokeslets) and the Finite Difference approaches.

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