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A note on some Ostrowski-like type inequalities

Seak Weng Vong *

Department of Mathematics, University of Macau, Av. Padre Tomás Pereira, Taipa, Macau

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ABSTRACT

Error estimate plays an important role for numerical integration. Some Ostrowski-like type inequalities concerning a new type of quadrature formula are established recently by Huy and Ngô (2009, 2010) [6,7]. In this note, improvement and extension to these inequalities are given.

1. Introduction

Error inequalities on quadrature formulas are important for the implementation of numerical integration. The midpoint and the trapezoid rules are typical quadrature rules and their error estimates are studied in [1–5]. Recently, Huy and Ngô introduced a new type of quadrature formula in [6,7] and they also establish some new Ostrowski-like type inequalities. The main concern of this note is to give some improvements on these inequalities.

We now give a brief review on these inequalities. Suppose that we want to evaluate the integral of a function f on an interval $[a, b]$ numerically. By considering the values of f at some points $\xi_1, \ldots, \xi_n$ on $[a, b]$, the integral can be approximated by $\frac{b-a}{n}\sum_{i=1}^{n} f(\xi_i)$. In [6,7], the relative coordinates of these points are referred to as knots. That is, for a point $\xi_i = a + x_i(b-a) \in [a, b]$, it is then called as the knot x_i. By choosing the knots to satisfy

$$x_1^j + x_2^j + \cdots + x_n^j = \frac{n}{j+1} \tag{1}$$

for some integer j, the following Ostrowski-like type inequality is established [7].

Theorem 1.1. *Suppose that $f : [a, b] \to \mathbb{R}$ is an m-th differentiable function. If $x_1, \ldots, x_n$ satisfy (1) for $j = 1, \ldots, m$, then we have*

$$|I(f) - Q(f, n, m, x_1, \ldots, x_n)| \le \frac{2m+5}{4}\frac{(b-a)^{m+1}}{(m+1)!}(S-s), \tag{2}$$

where $S = \sup_{a\le x\le b} f^{(m)}(x)$, $s = \inf_{a\le x\le b} f^{(m)}(x)$, $I(f) = \int_a^b f(x)dx$, and

$$Q(f, n, m, x_1, \ldots, x_n) = \frac{b-a}{n}\sum_{i=1}^{n} f(a + x_i(b-a)).$$

* Tel.: +853 83974359; fax: +853 28838314.
E-mail address: swvong@umac.mo.

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As mentioned in the remarks of [7], for some particular cases (see Example 2.1 in the next section for more details), the best constants for the inequalities in the form of (2) have been given in [8]. However, the result in [7] gives a new way to think about these Ostrowski-like type inequalities by enlarging the number of knots involved.

We point out here that their analysis depends heavily on the Grüss Inequality. In Section 2, we find that Theorem 1.1 can be improved by estimating the difference directly.

Another Ostrowski-like type inequality with which this note is concerned is an inequality involving the L^p-norm. It can be stated as [6]:

Theorem 1.2. *Suppose that* $f : [a, b] \to \mathbb{R}$ *is an m-th differentiable function such that* $f^{(m)} \in L^p(a, b)$. *If* $x_1, \ldots, x_n$ *satisfy* (1) *for* $j = 1, \ldots, m-1$, *then we have*

$$|I(f) - Q(f, n, m, x_1, \ldots, x_n)| \le \frac{1}{m!}\left(\left(\frac{1}{mq+1}\right)^{\frac{1}{q}} + \left(\frac{1}{(m-1)q+1}\right)^{\frac{1}{q}}\right)\|f^{(m)}\|_p (b-a)^{m+\frac{1}{q}}, \tag{3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\|f\|_r = \begin{cases} \left(\int_a^b |f(x)|^r dx\right)^{\frac{1}{r}}, & \text{when } 1 \le r < \infty, \\ \operatorname{ess\,sup}_{[a,b]} |f|, & \text{when } r = \infty. \end{cases}$

In the next section, we find that (3) can be improved by further imposing that (1) holds for $j = m$.

2. The main results

We first give an improvement of (2).

Theorem 2.1. *Suppose that* $f^{(m-1)}$ *is absolutely continuous and* $s \le f^{(m)} \le S$ *a.e. on* $[a, b]$. *If* $x_1, \ldots, x_n$ *satisfy* (1) *for* $j = 1, \ldots, m$, *then we have*

$$|I(f) - Q(f, n, m, x_1, \ldots, x_n)| \le \frac{(b-a)^{m+1}}{(m+1)!}(S - s). \tag{4}$$

Proof. For the completeness of our presentation, we first give some expressions in [7].

By using Taylor's theorem with the integral remainder (see [9]), the following expression can readily be obtained

$$I(f) = \sum_{k=0}^{m-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) + \int_a^b \frac{(b-x)^m}{m!} f^{(m)}(x)dx.$$

With the condition (1) for $j = 1, \ldots, m-1$, one can get that

$$\sum_{i=1}^{n} f(a + x_i(b-a)) = \sum_{k=0}^{m-1} \frac{n(b-a)^k}{(k+1)!} f^{(k)}(a) + \sum_{i=1}^{n} \int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!} f^{(m)}((1-x_i)a + x_i x)dx.$$

Thus

$$I(f) - Q(f, n, m, x_1, \ldots, x_n) = \int_a^b \frac{(b-x)^m}{m!} f^{(m)}(x)dx - \frac{b-a}{n}\sum_{i=1}^{n} \int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!} f^{(m)}((1-x_i)a + x_i x)dx. \tag{5}$$

Huy and Ngô then estimate (5) with the Grüss Inequality. However, we find that it can be estimated directly instead.

To this end, we first notice that $(b-x)^m$ and $(b-x)^{m-1}$ are nonnegative on $[a, b]$. Thus, we can make a change of the integration variable to get (see [10])

$$\int_a^b \frac{(b-x)^m}{m!} f^{(m)}(x)dx = \frac{1}{(m+1)!}\int_0^{(b-a)^{m+1}} f^{(m)}(b - y^{\frac{1}{m+1}})dy.$$

Since the transformation $y = (b-x)^{m+1}$ maps the $[a, b]$ one-to-one on $[0, (b-a)^{m+1}]$, we get that $f^{(m)}(b - y^{\frac{1}{m+1}}) \le S$ for almost every y on $[0, (b-a)^{m+1}]$. We can thus conclude that

$$\int_a^b \frac{(b-x)^m}{m!} f^{(m)}(x)dx \le \frac{1}{(m+1)!}(b-a)^{m+1} S.$$

On the other hand, by (1) for $j = m$, we have

$$\frac{b-a}{n}\sum_{i=1}^{n}\int_a^b \frac{x_i^m(b-x)^{m-1}}{(m-1)!}f^{(m)}((1-x_i)a+x_ix)dx = \frac{b-a}{nm!}\sum_{i=1}^{n}x_i^m\int_0^{(b-a)^m} f^{(m)}((1-x_i)a+x_i(b-y^{\frac{1}{m}}))dy$$
$$\geq \frac{b-a}{nm!}\sum_{i=1}^{n}x_i^m(b-a)^m s = \frac{1}{(m+1)!}(b-a)^{m+1}s.$$

We thus get that

$$I(f) - Q(f,n,m,x_1,\ldots,x_n) \leq \frac{1}{(m+1)!}(b-a)^{m+1}(S-s).$$

Similarly, one can show that

$$I(f) - Q(f,n,m,x_1,\ldots,x_n) \geq \frac{1}{(m+1)!}(b-a)^{m+1}(s-S)$$

and (4) follows. □

Remark. If $f^{(m)}$ is continuous, the argument above can be replaced directly by the mean value theorem for integrals.

Example 2.1. In [8], it was shown that if $f^{(m-1)}$ is absolutely continuous and $s_m \leq f^{(m)} \leq S_m$ a.e. on $[a,b]$ for $m = 1, 2, 3$, then

$$\left|I(f) - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leq C_m(S_m - s_m)(b-a)^{m+1},$$

where $C_1 = \frac{5}{72}$, $C_2 = \frac{1}{162}$, and $C_3 = \frac{1}{1152}$. These constants are sharp in the sense that one cannot replace them with smaller numbers. The quadrature formula corresponds to taking the knots as $x_1 = 0$, $x_2 = x_3 = x_4 = x_5 = \frac{1}{2}$ and $x_6 = 1$. One can easily see that these knots satisfy (1) for $j = 1, 2, 3$. The coefficients given by the estimate of Theorem 2.1 are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{24}$ for $m = 1, 2, 3$ respectively. We see that Theorem 2.1 gives closer values to the sharp ones than those given in [7].

Now we turn to consider the L^p-norm bound. By further imposing that (1) holds for $j = m$, we extend the result in [6] as:

Theorem 2.2. *Suppose that $f : [a,b] \to \mathbb{R}$ is an m-th differentiable function such that $f^{(m)} \in L^p(a,b)$. If $x_1,\ldots,x_n$ satisfies (1) for $j = 1,\ldots,m$, then we have*

$$|I(f) - Q(f,n,m,x_1,\ldots,x_n)| \leq \frac{1}{m!}\left(\left(\frac{1}{mq+1}\right)^{\frac{1}{q}} + \left(\frac{m}{(m+1)((m-1)q+1)}\right)^{\frac{1}{q}}\right)\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}$$

for $p \in (1,\infty]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We give the proof for $p \in (1,\infty)$ only. The case for $p = \infty$ can be proved with a slight modification. First note that, from (5), we have

$$|I(f) - Q(f,n,m,x_1,\ldots,x_n)| \leq \left\|\frac{(b-\cdot)^m}{m!}f^{(m)}\right\|_1 + \frac{b-a}{n}\sum_{i=1}^{n}\left\|\frac{x_i^m(b-\cdot)^{m-1}}{(m-1)!}f^{(m)}((1-x_i)a+x_i\cdot)\right\|_1.$$

As in [6], we can estimate the first term as

$$\left\|\frac{(b-\cdot)^m}{m!}f^{(m)}\right\|_1 \leq \frac{1}{m!}\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}\|f^{(m)}\|_p(b-a)^{m+\frac{1}{q}}.$$

To estimate the second term, notice that

$$\|f^{(m)}((1-x_i)a+x_i\cdot)\|_p = x_i^{-\frac{1}{p}}\left(\int_a^{(1-x_i)a+x_ib}|f^{(m)}(\xi)|^p d\xi\right)^{\frac{1}{p}}.$$

By the additional condition (1) for $j = m$ and the Hölder inequality, we have

$$\sum_{i=1}^{n}\frac{x_i^{m-\frac{1}{p}}}{n} = \sum_{i=1}^{n}\frac{x_i^{m-1+\frac{1}{q}}}{n} = \sum_{i=1}^{n}\frac{x_i^{(m-1)/p}}{n^{1/p}}\cdot\frac{x_i^{((m-1)+1)/q}}{n^{1/q}}$$

$$
\begin{aligned}
&\leq \left(\sum_{i=1}^{n} \frac{x_i^{m-1}}{n}\right)^{1/p} \left(\sum_{i=1}^{n} \frac{x_i^{m}}{n}\right)^{1/q} \\
&= \left(\frac{1}{m}\right)^{1/p} \left(\frac{1}{m+1}\right)^{1/q} \\
&= \frac{1}{m} \left(\frac{m}{m+1}\right)^{1/q}.
\end{aligned}
$$

With this, following the argument of Huy and Ngô, we have

$$
\begin{aligned}
\frac{b-a}{n} \sum_{i=1}^{n} \left\| \frac{x_i^m (b-\cdot)^{m-1}}{(m-1)!} f^{(m)}((1-x_i)a + x_i \cdot) \right\|_1 &\leq \frac{b-a}{n} \sum_{i=1}^{n} \frac{x_i^m}{(m-1)!} \left\| (b-\cdot)^{m-1} \right\|_q \left\| f^{(m)}((1-x_i)a + x_i \cdot) \right\|_p \\
&\leq \frac{b-a}{(m-1)!} \sum_{i=1}^{n} \frac{x_i^{m-\frac{1}{p}}}{n} \left\| (b-\cdot)^{m-1} \right\|_q \|f^{(m)}\|_p \\
&\leq \frac{1}{m!} \left(\frac{m}{(m+1)((m-1)q+1)} \right)^{\frac{1}{q}} \|f^{(m)}\|_p (b-a)^{m+\frac{1}{q}}
\end{aligned}
$$

and the theorem follows. □

Remark. For $p = 1$, the argument in Theorem 2.2 can only yield the same estimate as that in [6].

We conclude this note with an example illustrating Theorem 2.2.

Example 2.2. Suppose that f is 3-times-differentiable. Consider once again the knots $x_1 = 0, x_2 = x_3 = x_4 = x_5 = \frac{1}{2}$ and $x_6 = 1$ as in Example 2.1. Theorem 2.2 then gives

$$
\left| \int_a^b f(x)dx - \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right) \right| \leq \frac{1}{12} \|f'''\|_\infty (b-a)^4.
$$

Notice that, if we apply Theorem 1.2 instead, we can get a similar estimate but the constant is replaced by $\frac{7}{72}$.

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