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Syzygy Modules for Noetherian Rings

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INTRODUCTION

Throughout this paper Λ is a twosided noetherian ring and $\operatorname{mod}\Lambda$ the category of finitely generated Λ-modules. For each integer $k > 0$ we denote by $\Omega^k(\operatorname{mod}\Lambda)$ the full subcategory of $\operatorname{mod}\Lambda$ whose objects are the C in $\operatorname{mod}\Lambda$ which are the kth syzygies of modules in $\operatorname{mod}\Lambda$, that is, for which there is an exact sequence $0 \to C \to P_{k-1} \to \cdots \to P_0 \to B \to 0$ in $\operatorname{mod}\Lambda$ with the P_i projective modules. While projective resolutions of modules have been studied extensively, little seems to be known about the subcategories $\Omega^k(\operatorname{mod}\Lambda)$ of $\operatorname{mod}\Lambda$. One of the few general properties known about these categories is that they are covariantly finite in $\operatorname{mod}\Lambda$ [AR2]. This means that each module M in $\operatorname{mod}\Lambda$ has a left $\Omega^k(\operatorname{mod}\Lambda)$-approximation, that is, we have a morphism $f: M \to L$ with L in $\Omega^k(\operatorname{mod}\Lambda)$ with the property that if $h: M \to L'$ is any morphism with L' in $\Omega^k(\operatorname{mod}\Lambda)$, then there is some $g: L \to L'$ such that $gf = h$. While in general covariantly finite subcategories of $\operatorname{mod}\Lambda$ have proven to be of interest in various situations, for instance, in the representation theory of artin algebras and commutative ring theory, they are particularly interesting when they are extension closed, i.e., if $0 \to A \to B \to C \to 0$ is exact with A and C in $\Omega^k(\operatorname{mod}\Lambda)$, then B is in $\Omega^k(\operatorname{mod}\Lambda)$, in connection with Wakamatsu's lemma type results (see [AR1, AR2, AR3]). The main motivation for this paper was to describe when the subcategories $\Omega^k(\operatorname{mod}\Lambda)$ are extension closed. However, before stating our main result, it is convenient to recall some definitions [AB].

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Let C be in $\operatorname{mod} \Lambda$. We say that the grade of C, written grade C, is greater than or equal to an integer $i \geq 0$ if $\operatorname{Ext}^j_\Lambda(C, \Lambda) = 0$ for all $j < i$. We say the strong grade of C, written s.grade C, is greater than or equal to an integer $i \geq 0$ if grade $X \geq i$ for all submodules X of C. Finally, we say that a ring Λ is a noetherian R-algebra if Λ is an R-algebra where R is a commutative noetherian ring and Λ is a finitely generated R-module. Our main result is the following.

THEOREM 0.1. *Let Λ be a noetherian R-algebra. Then the following conditions are equivalent for an integer $k > 0$.*

(a) $\Omega^i(\operatorname{mod} \Lambda)$ *is extension closed for* $1 \leq i \leq k$.

(b) *If* $0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ *is a minimal injective resolution of Λ as a right Λ-module, then* flatdim $I_i \leq i + 1$ *for $i < k$.*

(c) grade $\operatorname{Ext}^i_\Lambda(C, \Lambda) \geq i$ *for all C in* $\operatorname{mod} \Lambda^{\mathrm{op}}$ *and $i \leq k$.*

(d) s.grade $\operatorname{Ext}^{i+1}_\Lambda(B, \Lambda) \geq i$ *for all B in* $\operatorname{mod} \Lambda$ *and $i \leq k$.*

Actually, the only place the hypothesis that Λ is a noetherian R-algebra is used is in showing that (a) implies (c) when $k > 1$. The proofs of all the other implications only depend on the condition that Λ be a twosided noetherian ring.

We now make some other remarks about this theorem. First, in the course of proving (a) implies (c), it is shown that (a) implies that the modules in $\Omega^i(\operatorname{mod} \Lambda)$ are i-torsionfree for $1 \leq i \leq k$ (see Section 1 for the definition). This implies for $1 \leq i \leq k$ that the $\Omega^i(\operatorname{mod} \Lambda)$ are also closed under summands. It is a measure of our ignorance of the categories $\Omega^i(\operatorname{mod} \Lambda)$ that we do not even know precisely when they are closed under summands.

Second, twosided noetherian rings satisfying (c) were studied in [AB], culminating in what was called the approximation theorem. However, except for the case when Λ is commutative, no criterion for when a twosided noetherian ring satisfied (c) was known. The theorem remedies this situation.

Third, the equivalence of (a) and (b) was proven in [AR3] for artin algebras and it was shown that (d) implies (a) for twosided noetherian rings. All the other results are new, in particular for artin algebras.

1. EXTENSION CLOSURE AND STRONG GRADE

The main aim of this section is to show for a twosided noetherian ring Λ that for an integer $k \geq 1$ the category $\Omega^i(\operatorname{mod} \Lambda)$ is extension closed for $1 \leq i \leq k$ if and only if s.grade $\operatorname{Ext}^{i+1}_\Lambda(B, \Lambda) \geq i$ for $1 \leq i \leq k$.

Throughout this section Λ is an arbitrary twosided noetherian ring, all Λ-modules are assumed to be finitely generated, and $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\mathrm{op}}$ denote the categories of finitely generated left and finitely generated right modules, respectively. For each integer $i \geq 0$ and each C in $\operatorname{mod}\Lambda$, we consider the abelian group $\operatorname{Ext}^i_\Lambda(C, \Lambda)$ a right Λ-module by means of the action induced by right multiplication in Λ. We recall that a module C is k-torsionfree for an integer $k > 0$ if $\operatorname{Ext}^i_\Lambda(\operatorname{Tr} C, \Lambda) = 0$ for $1 \leq i \leq k$, where $\operatorname{Tr} C$ is the transpose of C [AB].

Our first aim in this section is to show how the fact that the categories $\Omega^i(\operatorname{mod}\Lambda)$ are extension closed for $1 \leq i \leq k$ is connected with the fact that the modules in $\Omega^i(\operatorname{mod}\Lambda)$ are the i-torsionfree modules for $1 \leq i \leq k$. The results along these lines are based on the following.

THEOREM 1.1. *Suppose C is k-torsionfree with $k \geq 1$. Then the following are equivalent.*

(a) s.grade $\operatorname{Ext}^1_\Lambda(C, \Lambda) \geq k$.

(b) *If $0 \to A \to B \to C \to 0$ is exact with A k-torsionfree, then B is k-torsionfree.*

(c) *If $0 \to \Lambda^n \to E \to C \to 0$ is exact, then E is k-torsionfree.*

The proof of this result proceeds by induction on $k \geq 1$ as follows. The cases $k = 1$ and $k = 2$ are proven separately with the induction starting with $k \geq 3$.

Recall that C being 1-torsionfree is the same as C being torsionless, i.e., C is in $\Omega^1(\operatorname{mod}\Lambda)$.

LEMMA 1.2. *The following are equivalent for C in $\Omega^1(\operatorname{mod}\Lambda)$.*

(a) s.grade $\operatorname{Ext}^1_\Lambda(C, \Lambda) \geq 1$.

(b) *If $0 \to A \to B \to C \to 0$ is exact with A in $\Omega^1(\operatorname{mod}\Lambda)$, then B is in $\Omega^1(\operatorname{mod}\Lambda)$.*

(c) *If $0 \to \Lambda^n \to E \to C \to 0$ is exact, then E is in $\Omega^1(\operatorname{mod}\Lambda)$.*

Proof. (a) $\Rightarrow$ (b) Applying the functor $X \mapsto X^{**}$, where $X^{**} = \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda(X, \Lambda), \Lambda)$, to the exact sequence $0 \to A \to^f B \to^g C \to 0$ we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle \alpha_C} & & \\ & & A^{**} & \xrightarrow{f^{**}} & B^{**} & \xrightarrow{g^{**}} & C^{**} & & \end{array}$$

where $\alpha_X\colon X \to X^{**}$ is the usual morphism. The fact that A and C are in $\Omega^1(\operatorname{mod}\Lambda)$ means that α_A and α_C are monomorphisms. It is then easily checked that α_B is also a monomorphism if $f^{**}\colon A^{**} \to B^{**}$ is a monomorphism. But f^{**} can be seen to be a monomorphism as follows. The exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ gives rise to the exact sequence $0 \to C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \to \operatorname{Ext}^1_\Lambda(C, \Lambda)$. Since s.grade $\operatorname{Ext}^1_\Lambda(C, \Lambda) \geq 1$, we have that $L^* = 0$, where $L = \operatorname{Coker} f^*$. This implies that $f^{**}\colon A^{**} \to B^{**}$ is a monomorphism, which finishes the proof.

(b) ⇒ (c) This is trivial.

(c) ⇒ (a) Let L be a submodule of $\operatorname{Ext}^1_\Lambda(C, \Lambda)$. We want to show that $L^* = 0$. Since $\operatorname{Ext}^1_\Lambda(C, \Lambda)$ is a finitely generated Λ^{op}-module, L is also a finitely generated submodule of $\operatorname{Ext}^1_\Lambda(C, \Lambda)$. From this it follows that there is an exact sequence $0 \to \Lambda^n \to E \xrightarrow{g} C \to 0$ such that the induced exact sequence $E^* \to (\Lambda^n)^* \to \operatorname{Ext}^1_\Lambda(C, \Lambda)$ has the property that $E^* \to (\Lambda^n)^* \to L \to 0$ is exact. We therefore have the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Lambda^n & \longrightarrow & E & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^* & \longrightarrow & (\Lambda^n)^{**} & \longrightarrow & E^{**} & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

from which it follows that $L^* = 0$. This shows that s.grade $\operatorname{Ext}^1_\Lambda(C, \Lambda) \geq 1$, completing the proof of the lemma. ∎

Having proven Theorem 1.1 when $k = 1$, we now turn our attention to proving it when $k = 2$. Remember, a module C is 2-torsionfree if and only if $\alpha_C\colon C \to C^{**}$ is an isomorphism, i.e., C is reflexive.

LEMMA 1.3. *The following are equivalent for a reflexive Λ-module C.*

(a) s.grade $\operatorname{Ext}^1_\Lambda(C, \Lambda) \geq 2$.

(b) *If $0 \to A \to B \to C \to 0$ is exact with A reflexive, then B is reflexive.*

(c) *If $0 \to \Lambda^n \to E \to C \to 0$ is exact, then E is reflexive.*

Proof. (a) ⇒ (b) We have the exact sequence $0 \to C^* \to B^* \to A^* \to L \to 0$ with grade $L \geq 2$ since $L \subset \operatorname{Ext}^1_\Lambda(C, \Lambda)$ and s.grade $\operatorname{Ext}^1_\Lambda(C, \Lambda) \geq 2$. Then the exact sequence $0 \to K \to A^* \to L \to 0$ gives rise to the exact sequence $0 \to L^* \to A^{**} \to K^* \to \operatorname{Ext}^1_\Lambda(L, \Lambda)$, which shows that the map $A^{**} \to K^*$ is an isomorphism. This shows that $0 \to A^{**} \to B^{**} \to C^{**}$ is exact and so we have the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha_A}\downarrow\wr & & {\scriptstyle\alpha_B}\downarrow & & {\scriptstyle\alpha_C}\downarrow\wr & & \\ 0 & \longrightarrow & A^{**} & \longrightarrow & B^{**} & \longrightarrow & C^{**} & & \end{array}$$

which shows that B is reflexive.

(b) $\Rightarrow$ (c) This is trivial.

(c) $\Rightarrow$ (a) Let L be a submodule of $\operatorname{Ext}^1_\Lambda(C, \Lambda)$. Then $L^* = 0$ since C is in $\Omega^1(\operatorname{mod} \Lambda)$ and so $\operatorname{s.grade} \operatorname{Ext}^1_\Lambda(C, \Lambda) \geq 1$. We know there is an exact sequence $0 \to \Lambda^n \to E \to C \to 0$ such that $\operatorname{Im}((\Lambda^n)^* \to \operatorname{Ext}^1_\Lambda(C, \Lambda)) = L$. Thus we have the exact sequences $0 \to C^* \to E^* \to K \to 0$ and $0 \to K \to (\Lambda^n)^* \to L \to 0$. Then we get the exact sequence $0 \to L^* \to (\Lambda^n)^{**} \to K^* \to \operatorname{Ext}^1_\Lambda(L, \Lambda) \to 0$. So we have to show that $0 \to (\Lambda^n)^{**} \to K^* \to 0$ is exact. We have the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda^n & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \longrightarrow & K^* & \longrightarrow & E^{**} & \longrightarrow & C^{**}, & & \end{array}$$

from which it follows that $\Lambda^n \to K^*$ is an isomorphism, giving our desired result. ∎

Having established Theorem 1.1 when $k = 1$ and 2, we now prove it for $k \geq 3$. This proof is based on the following.

LEMMA 1.4. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be exact with* C *k-torsionfree, with* $k \geq 3$ *and* B *reflexive. Define the module* K *by the exact sequence* $0 \to C^* \xrightarrow{g^*} B^* \to K \to 0$. *Then* B *is k-torsionfree if and only if* $\operatorname{Ext}^i_\Lambda(K, \Lambda) = 0$ *for* $1 \leq i \leq k - 2$.

Proof. Since C is k-torsionfree with $k \geq 3$, we have that $\operatorname{Ext}^i_\Lambda(C^*, \Lambda) = 0$ for $i = 1, \ldots, k - 2$. We have the long exact sequence $0 \to K^* \to B^{**} \to C^{**} \to \operatorname{Ext}^1_\Lambda(K, \Lambda) \to \operatorname{Ext}^1_\Lambda(B^*, \Lambda) \to \operatorname{Ext}^1_\Lambda(C^*, \Lambda) \to \operatorname{Ext}^2_\Lambda(K, \Lambda) \to \operatorname{Ext}^2_\Lambda(B^*, \Lambda) \to \operatorname{Ext}^2_\Lambda(C^*, \Lambda) \to \cdots$. Now $B^{**} \to C^{**} \to 0$ is exact since B and C are reflexive. Hence we have isomorphisms $\operatorname{Ext}^i_\Lambda(K, \Lambda) \xrightarrow{\sim} \operatorname{Ext}^i_\Lambda(B^*, \Lambda)$ for $i = 1, \ldots, k - 2$. Therefore B is k-torsionfree if and only if $\operatorname{Ext}^i_\Lambda(K, \Lambda) = 0$ for $i = 1, \ldots, k - 2$, since a reflexive module B is k-torsionfree for $k \geq 3$ if and only if $\operatorname{Ext}^i_\Lambda(B^*, \Lambda) = 0$ for $i = 1, \ldots, k - 2$. ∎

We now prove Theorem 1.1 by induction on k. Clearly we can assume that $k \geq 3$.

(a) $\Rightarrow$ (b) Let $0 \to A \to B \to C \to 0$ be exact with A k-torsionfree, where $k \geq 3$. Since $\operatorname{s.grade} \operatorname{Ext}^1_\Lambda(C, \Lambda) \geq k \geq 3$ we have by Lemma 1.3 that B is reflexive. Therefore to show that B is k-torsionfree, it suffices to show that $\operatorname{Ext}^i_\Lambda(K, \Lambda) = 0$ for $i = 1, \dots, k-2$, where $K = \operatorname{Coker}(C^* \to B^*)$. Since $0 \to K \to A^* \to L \to 0$ is exact with $L \subset \operatorname{Ext}^1_\Lambda(C, \Lambda)$, we know that $\operatorname{Ext}^i_\Lambda(L, \Lambda) = 0$ for $i = 0, \dots, k-1$, where $k - 1 \geq 2$. Hence we have isomorphisms $\operatorname{Ext}^i_\Lambda(K, \Lambda) \xrightarrow{\sim} \operatorname{Ext}^i_\Lambda(A^*, \Lambda)$ for $1 \leq i \leq k-2$. Therefore $\operatorname{Ext}^i_\Lambda(K, \Lambda) = 0$ for $1 \leq i \leq k-2$ since the fact that A is k-torsionfree implies $\operatorname{Ext}^i_\Lambda(A^*, \Lambda) = 0$ for $i = 1, \dots, k-2$ when $k \geq 3$.

(b) $\Rightarrow$ (c) This is trivial.

(c) $\Rightarrow$ (a) Let $L \subset \operatorname{Ext}^1_\Lambda(C, \Lambda)$. Then there is an exact sequence $0 \to \Lambda^n \to E \to C \to 0$ such that $\operatorname{Im}((\Lambda^n)^* \to \operatorname{Ext}^1_\Lambda(C, \Lambda)) = L$. Thus we have exact sequences $0 \to C^* \to E^* \to K \to 0$ and $0 \to K \to (\Lambda^n)^* \to L \to 0$. Since E is k-torsionfree, we know that $\operatorname{Ext}^i_\Lambda(E^*, \Lambda) = 0$ for $i = 1, \dots, k-2$. So by Lemma 1.4 we have that $\operatorname{Ext}^i_\Lambda(K, \Lambda) = 0$ for $i = 1, \dots, k-2$. Hence we have $\operatorname{Ext}^{i+1}_\Lambda(L, \Lambda) \simeq \operatorname{Ext}^i_\Lambda(K, \Lambda) = 0$ for $i = 1, \dots, k-2$ so $\operatorname{Ext}^j_\Lambda(L, \Lambda) = 0$ for $j = 2, \dots, k-1$. But since E is reflexive, we know that $\operatorname{Ext}^i_\Lambda(L, \Lambda) = 0$ for $i = 0, 1$, and so grade $L \geq k$. This finishes the proof of Theorem 1.1.

We will be using Theorem 1.1 in the following form, which is an immediate consequence of the theorem.

COROLLARY 1.5. *The following are equivalent for an integer* $k \geq 1$.

(a) *For each* i *with* $1 \leq i \leq k$, *we have that the subcategory of* i-*torsionfree modules is extension closed*.

(b) *If* $1 \leq i \leq k$ *and* C *is* i-*torsionfree, then* $\operatorname{s.grade} \operatorname{Ext}^1_\Lambda(C, \Lambda) \geq i$.

We now turn our attention to applying our results about the i-torsionfree modules being extension closed to the main question of this paper, when the categories $\Omega^i(\operatorname{mod} \Lambda)$ are extension closed for $1 \leq i \leq k$ for some integer $k \geq 1$. The following is the first result along these lines. We denote by $\mathscr{X}_i = \operatorname{add} \Omega^i(\operatorname{mod} \Lambda)$ the subcategory of mod Λ whose objects are those modules which are summands of ith syzygies.

PROPOSITION 1.6. *Let* Λ *be a twosided noetherian ring. Then we have the following for an integer* $k \geq 1$.

(a) $\operatorname{grade} \operatorname{Ext}^{i+1}_\Lambda(B, \Lambda) \geq i$ *for* $1 \leq i < k$ *and all modules* B *if and only if* $\Omega^i(\operatorname{mod} \Lambda)$ *coincides with the i-torsionfree modules for all* $1 \leq i \leq k$.

(b) *If* $\Omega^i(\operatorname{mod}\Lambda)$ *is extension closed for* $1 \le i < k$, *then* $\Omega^i(\operatorname{mod}\Lambda)$ *consists of the i-torsionfree modules for* $1 \le i \le k$, *and hence* $\mathscr{X}_i = \Omega^i(\operatorname{mod}\Lambda)$ *for* $1 \le i \le k$.

(c) *If* $\mathscr{X}_i$ *is extension closed for each* $1 \le i \le k$, *then* $\mathscr{X}_i = \Omega^i(\operatorname{mod}\Lambda)$ *for each* $1 \le i \le k$.

Proof. (a) This is Proposition 2.26 in [AB].

(b) Proceed by induction on k. Since $\Omega^1(\operatorname{mod}\Lambda)$ consists of the 1-torsionfree modules, or equivalently, the torsionless modules, we have our desired result for $k = 1$.

Suppose $k > 1$. Then by assumption each $\Omega^i(\operatorname{mod}\Lambda)$ consists of the i-torsionfree modules for $i < k$ and is also extension closed. Therefore the i-torsionfree modules are extension closed for $1 \le i < k$. Hence by Corollary 1.5 we have that s.grade $\operatorname{Ext}^1_\Lambda(C, \Lambda) \ge i$ for $1 \le i < k$ and all i-torsionfree modules C. Since $\Omega^i(\operatorname{mod}\Lambda)$ consists of the i-torsionfree modules for $1 \le i < k$, we have that grade $\operatorname{Ext}^{i+1}_\Lambda(B, \Lambda) \ge i$ for $1 \le i < k$ and all Λ-modules B. Then by part (a) we have that $\Omega^k(\operatorname{mod}\Lambda)$ consists of the k-torsionfree modules. It is then obvious that $\mathscr{X}_k = \Omega^k(\operatorname{mod}\Lambda)$.

(c) Proceed by induction on k. Since $\Omega^1(\operatorname{mod}\Lambda) = \mathscr{X}_1$, we have our result for $k = 1$. Let $k > 1$. Then $\mathscr{X}_i = \Omega^i(\operatorname{mod}\Lambda)$ is extension closed for $1 \le i < k$ by the induction hypothesis. By part (b) we have that $\Omega^k(\operatorname{mod}\Lambda)$ consists of the k-torsionfree modules, and is hence closed under summands. It follows that $\Omega^k(\operatorname{mod}\Lambda) = \mathscr{X}_k$. ∎

We now summarize our results so far as follows.

THEOREM 1.7. *Suppose* Λ *is a twosided noetherian ring. Then the following are equivalent for an integer* $k \ge 1$.

(a) s.grade $\operatorname{Ext}^{i+1}_\Lambda(B, \Lambda) \ge i$ *for all* $1 \le i \le k$ *and all* Λ-*modules* B.

(b) $\Omega^i(\operatorname{mod}\Lambda)$ *is extension closed for each* $1 \le i \le k$.

(c) $\Omega^i(\operatorname{mod}\Lambda)$ *is extension closed and consists of the i-torsionfree modules for* $1 \le i \le k$.

(d) $\mathscr{X}_i$ *is extension closed for each* $1 \le i \le k$.

Proof. (b) ⇔ (c) Clearly (c) implies (b). That (b) implies (c) is Proposition 1.6(b).

(c) ⇔ (d) That (d) implies (c) follows from Proposition 1.6(c). That (c) implies (d) is trivial.

(c) ⇒ (a) Since $\Omega^i(\operatorname{mod}\Lambda) = \mathscr{X}_i$ is extension closed, it follows from Corollary 1.5 that s.grade $\operatorname{Ext}^1(C, \Lambda) \ge i$ for all C in $\Omega^i(\operatorname{mod}\Lambda)$ with

$1 \leq i \leq k$. Hence s.grade $\operatorname{Ext}^{i+1}(B, \Lambda) \geq i$ for $1 \leq i \leq k$ and for all B in mod Λ.

(a) ⇒ (b) By Corollary 1.5 we have that the subcategories of i-torsionfree modules are each extension closed for $1 \leq i \leq k$. But also by Proposition 1.6(a) we have that $\Omega^i(\operatorname{mod} \Lambda)$ is the subcategory consisting of the i-torsionfree modules for $1 \leq i \leq k$. Hence $\Omega^i(\operatorname{mod} \Lambda)$ is extension closed for each $1 \leq i \leq k$. ∎

2. STRONG GRADE AND FLAT DIMENSION

Let $0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ be a minimal injective resolution of Λ as a right Λ-module. In this section we describe the property that s.grade $\operatorname{Ext}_\Lambda^{i+1}(C, \Lambda) \geq i$ for all C in mod Λ and $i \leq k$ in terms of the flat dimension of the first k injective modules in this resolution. The main idea is to express flat dimension in terms of maps from modules of the form $\operatorname{Ext}_\Lambda^i(C, \Lambda)$ to injective modules.

LEMMA 2.1. *Let I be an injective Λ^{op}-module and n a positive integer. Then* flatdim $I < n$ *if and only if* $\operatorname{Hom}_\Lambda(\operatorname{Ext}_\Lambda^n(C, \Lambda), I) = 0$ *for all C in* mod Λ.

Proof. From [AB, Th. 2.8] we have the exact sequence $0 \to \operatorname{Ext}_\Lambda^1(\operatorname{Tr} \Omega^n C, X) \to \operatorname{Tor}_n(X, C) \to \operatorname{Hom}_\Lambda(\operatorname{Ext}_\Lambda^n(C, \Lambda), X) \to \operatorname{Ext}_\Lambda^2(\operatorname{Tr} \Omega^n C, X)$ for C in mod Λ and X in mod Λ^{op}.

Letting $X = I$ we get $\operatorname{Tor}_n(I, C) \simeq \operatorname{Hom}_\Lambda(\operatorname{Ext}_\Lambda^n(C, \Lambda), I)$ for all C in mod Λ. This also follows from [CE, p. 120]. Then we use that flatdim $I < n$ if and only if $\operatorname{Tor}_n(I, C) = 0$ for all C in mod Λ. ∎

We now give the main result of this section.

PROPOSITION 2.2. *Let $0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ be a minimal injective resolution of Λ as a right Λ-module and let k be a positive integer. Then* flatdim $I_i \leq i + 1$ *for $i < k$ if and only if* s.grade $\operatorname{Ext}_\Lambda^{i+1}(C, \Lambda) \geq i$ *for all C in* mod Λ *and $i \leq k$.*

Proof. We first treat the case $k = 1$. So if $I = I_0$ we want to show that flatdim $I \leq 1$ if and only if s.grade $\operatorname{Ext}_\Lambda^2(C, \Lambda) \geq 1$ for all C in mod Λ. We have by Lemma 2.1 that flatdim $I \leq 1$ if and only if $\operatorname{Hom}_\Lambda(\operatorname{Ext}_\Lambda^2(C, \Lambda), I) = 0$ for all C in mod Λ. Now $f\colon \operatorname{Ext}_\Lambda^2(C, \Lambda) \to I$ is not zero if and only if $\operatorname{Im} f \cap \Lambda \neq 0$, which is the case if and only if there is a submodule A of $\operatorname{Ext}_\Lambda^2(C, \Lambda)$ such that $\operatorname{Hom}_\Lambda(A, \Lambda) \neq 0$. Hence we are done.

Assume now that $k > 1$, and that s.grade $\operatorname{Ext}_\Lambda^{i+1}(C, \Lambda) \geq i$ for $i \leq k$ and all C in mod Λ. By the induction assumption we have flatdim $I_i \leq i + 1$ for $i < k - 1$. To show flatdim $I_{k-1} \leq k$, it suffices to show that

$\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Lambda^{k+1}(C,\Lambda), I_{k-1}) = 0$. Consider the short exact sequences $0 \to K_{k-2} \to I_{k-2} \to K_{k-1} \to 0$ obtained from the injective resolution of Λ. Then $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Lambda^{k+1}(C,\Lambda), I_{k-1}) \neq 0$ if and only if there is some submodule A of $\mathrm{Ext}_\Lambda^{k+1}(C,\Lambda)$ with $\mathrm{Hom}_\Lambda(A, K_{k-1}) \neq 0$.

Consider for $A \subset \mathrm{Ext}_\Lambda^{k+1}(C,\Lambda)$ the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(A, K_{k-2}) \to \mathrm{Hom}_\Lambda(A, I_{k-2}) \to \mathrm{Hom}_\Lambda(A, K_{k-1}) \to \mathrm{Ext}_\Lambda^1(A, K_{k-2}) \to 0. \qquad (*)$$

We have $\mathrm{Hom}_\Lambda(A, I_{k-2}) = 0$, since flatdim $I_{k-2} \leq k-1$, and $\mathrm{Ext}_\Lambda^1(A, K_{k-2}) \simeq \mathrm{Ext}_\Lambda^{k-1}(A,\Lambda) = 0$ since grade $A \geq k$ by assumption. Hence we get $\mathrm{Hom}_\Lambda(A, K_{k-1}) = 0$, which shows $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Lambda^{k+1}(C,\Lambda), I_{k-1}) = 0$.

Assume now that flatdim $I_i \leq i+1$ for $i < k$. We have s.grade $\mathrm{Ext}_\Lambda^{k+1}(C,\Lambda) \geq k-1$ by the induction assumption. Consider the sequence $(*)$, where $\mathrm{Hom}_\Lambda(A, K_{k-1}) = 0$ since $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Lambda^{k+1}(C,\Lambda), I_{k-1}) = 0$. Hence $\mathrm{Ext}_\Lambda^1(A, K_{k-2}) \simeq \mathrm{Ext}_\Lambda^{k-1}(A,\Lambda) = 0$, so that grade $A \geq k$, and we are done. ∎

3. ADJOINTNESS

Let Λ be a twosided noetherian ring. For each C in mod Λ we choose a fixed epimorphism $f: P_C \to C$, where P_C is projective. We define $\Omega^1(C)$ to be Ker f, and we define $\Omega^n(C)$ for each $n \geq 1$ by induction. Let $\underline{\mathrm{mod}}\,\Lambda$ be the category mod Λ modulo projectives. There are induced functors $G = \Omega^n$: $\underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda$ and $F = \mathrm{Tr}\,\Omega^n\,\mathrm{Tr}$: $\underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda$ for each $n \geq 1$, and we know from [AR2] that they are adjoint functors. In this section we shall need a more explicit description of why Ω^k and $\mathrm{Tr}\,\Omega^k\,\mathrm{Tr}$ are adjoint functors. This is obtained through studying maps between complexes. We give some applications, including establishing an exact sequence which we shall need later.

Let J be the integers in an interval $\langle -\infty, \infty\rangle, [i,\infty\rangle, \langle -\infty, i]$ or $[i,j]$, where $i < j$ are integers. Let

$$P^{\cdot} \qquad \cdots \to P_i \xrightarrow{p_i} P_{i-1} \xrightarrow{p_{i-1}} P_{i-2} \to \cdots$$

and

$$Q^{\cdot} \qquad \cdots \to Q_i \xrightarrow{q_i} Q_{i-1} \xrightarrow{q_{i-1}} Q_{i-2} \to \cdots$$

be complexes of projective modules, where $i \in J$. If $I \subset J$ and I is the integers in an interval, we denote by $P_I^{\cdot}$ and $Q_I^{\cdot}$ the corresponding restrictions of the complexes, and for a map $f\colon P^{\cdot} \to Q^{\cdot}$ of complexes we have the restriction map $f_I\colon P_I^{\cdot} \to Q_I^{\cdot}$. For $i \in J$ let $I = J_{\le i}$ be the integers $j \in J$ with $j \le i$ and let $I = J_{\ge i}$ be the integers $j \in J$ with $j \ge i$. Consider the subgroup $U(P^{\cdot}, Q^{\cdot})$ of $\mathrm{Hom}(P^{\cdot}, Q^{\cdot})$ consisting of the maps $f\colon P^{\cdot} \to Q^{\cdot}$ for which there are maps $h_i\colon P_{i-1} \to Q_i$ such that $q_i h_i p_i = f_{i-1} p_i$ for $\{i-1, i\} \subset J$. We then have the following easily verified result.

LEMMA 3.1. *Let $P^{\cdot}$ and $Q^{\cdot}$ be complexes of projective modules on some J as above.*

(a) *Assume that $Q^{\cdot}$ is exact, and let $I = J_{\ge i}$ for some $i \in J$.*
  (i) *Given $g\colon P_I^{\cdot} \to Q_I^{\cdot}$ there is some $\tilde{g}\colon P^{\cdot} \to Q^{\cdot}$ with $\tilde{g}_I = g$.*
  (ii) *If $g \in U(P_I^{\cdot}, Q_I^{\cdot})$, then $\tilde{g} \in U(P^{\cdot}, Q^{\cdot})$.*

(b) *Assume that $P^{\cdot *}$ is exact and let $I = J_{\le i}$ for some integer i.*
  (i) *Given $g\colon P_I^{\cdot} \to Q_I^{\cdot}$ there is some $\tilde{g}\colon P^{\cdot} \to Q^{\cdot}$ with $\tilde{g}_I = g$.*
  (ii) *If $g \in U(P_I^{\cdot}, Q_I^{\cdot})$, then $\tilde{g} \in U(P^{\cdot}, Q^{\cdot})$.*

Writing $\underset{\sim}{\mathrm{Hom}}(P^{\cdot}, Q^{\cdot}) = \mathrm{Hom}(P^{\cdot}, Q^{\cdot})/U(P^{\cdot}, Q^{\cdot})$, we have the following direct consequence.

PROPOSITION 3.2. *Let $P^{\cdot}$ and $Q^{\cdot}$ be complexes of projective modules on an interval J, where $P^{\cdot *}$ and $Q^{\cdot}$ are exact, and let $I = J_{\ge i} \cap J_{\le j}$, where i and j are in J and $i < j$. Then the natural map $\underset{\sim}{\mathrm{Hom}}(P^{\cdot}, Q^{\cdot}) \to \underset{\sim}{\mathrm{Hom}}(P_I^{\cdot}, Q_I^{\cdot})$ is an isomorphism.*

Note that if $P^{\cdot}$ is the complex $P_i \xrightarrow{p_i} P_{i-1}$ and $Q^{\cdot}$ the complex $Q_i \xrightarrow{q_i} Q_{i-1}$, then it is not hard to see that we have a natural isomorphism $\underset{\sim}{\mathrm{Hom}}(P^{\cdot}, Q^{\cdot}) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\mathrm{Coker}\, p_i, \mathrm{Coker}\, q_i)$. Hence we get the following consequence.

COROLLARY 3.3. *Let $P^{\cdot}$ and $Q^{\cdot}$ be complexes of projective modules indexed by some J, where $P^{\cdot *}$ and $Q^{\cdot}$ are exact. For each integer i with $\{i-1, i\} \subset J$ we have a natural isomorphism $\underset{\sim}{\mathrm{Hom}}(P^{\cdot}, Q^{\cdot}) \to \underline{\mathrm{Hom}}(\mathrm{Coker}\, p_i, \mathrm{Coker}\, q_i)$, and hence a natural isomorphism $\underline{\mathrm{Hom}}(\mathrm{Coker}\, p_i, \mathrm{Coker}\, q_i) \to \underline{\mathrm{Hom}}(\mathrm{Coker}\, p_j, \mathrm{Coker}\, q_j)$ when $\{j-1, j\} \subset J$.*

Now let M and N be in $\mathrm{mod}\,\Lambda$, and consider the projective resolutions

$$\cdots \to Q_2 \xrightarrow{q_2} Q_1 \xrightarrow{q_1} Q_0 \to N \to 0$$

and

$$\cdots \to L_2 \xrightarrow{l_2} L_1 \xrightarrow{l_1} L_0 \to \mathrm{Tr}\, M \to 0.$$

Denote by $Q^{\cdot}$ and $L^{\cdot}$ the corresponding complexes of projective modules, indexed by the set J of nonnegative integers. For the dual complex $L^{\cdot *}$

and a given positive integer k, let $P^{\cdot}$ be the shifted complex where $P_i = (L_{k+1-i})^*$. Let $I = J_{\leq k+1}$. Then $P_I^{\cdot}$ and $Q_I^{\cdot}$ are projective complexes on I such that $P_I^{\cdot *}$ and $Q_I^{\cdot}$ are exact. Hence we have the complexes

$$\begin{array}{ccccccccc} L_0^* & \xrightarrow{q_1^* = p_{k+1}} & L_1^* & \to & \cdots & \to & L_k^* & \to & L_{k+1}^* \\ \| & & \| & & & & \| & & \| \\ P_{k+1} & & P_k & & & & P_1 & & P_0 \end{array}$$

and

$$Q_{k+1} \to Q_k \to \cdots \to Q_1 \to Q_0 .$$

Then we have the following consequence of Corollary 3.3.

COROLLARY 3.4. *Let M and N be in* $\operatorname{mod} \Lambda$. *Then we have natural isomorphisms*

$$\underline{\operatorname{Hom}}(M, \Omega^k N) \xrightarrow{\sim} \underline{\operatorname{Hom}}(\operatorname{Tr} \Omega^1 \operatorname{Tr} M, \Omega^{k-1} N) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \underline{\operatorname{Hom}}(\operatorname{Tr} \Omega^k \operatorname{Tr} M, N).$$

In particular the induced isomorphism $\underline{\operatorname{Hom}}(M, \Omega^k N) \xrightarrow{\sim} \underline{\operatorname{Hom}}(\operatorname{Tr} \Omega^k \operatorname{Tr} M, N)$ *gives an adjointness isomorphism between the functors* $\underline{\Omega^k}$ *and* $\operatorname{Tr} \Omega^k \operatorname{Tr}$ *on* $\underline{\operatorname{mod}}\, \Lambda$.

Write, as in [AB], $J_k(C) = \operatorname{Tr} \Omega^k C$ and $D_k(C) = \Omega^k \operatorname{Tr} C$. Letting $M = \Omega^k N$ in Corollary 3.4 we obtain associated with the identity on $\Omega^k N$ a map $t\colon J_k^2(N) \to N$ in $\underline{\operatorname{mod}\, \Lambda}$, which we know is (and is easily seen to be) a right $J_k(\operatorname{mod} \Lambda^{\mathrm{op}})$-approximation [AR3]. Similarly we get $s\colon M \to D_k^2(M)$ in $\underline{\operatorname{mod}\, \Lambda}$ by setting $N = \operatorname{Tr} \Omega^k \operatorname{Tr} M$. Note that it follows from [AB, Appendix] that the maps t and s coincide with those defined in [AB].

We give the following application of these considerations, which we shall need in the next section. We denote by $\alpha_{M,N}$ the adjointness isomorphism $\underline{\operatorname{Hom}}(M, \Omega^k N) \xrightarrow{\sim} \underline{\operatorname{Hom}}(\operatorname{Tr} \Omega^k \operatorname{Tr} M, N)$.

PROPOSITION 3.5. *Let M and N be in* $\operatorname{mod} \Lambda$. *Then we have the commutative diagram*

$$\begin{array}{ccc} \underline{\operatorname{Hom}}(M, \Omega^k N) & \xrightarrow[\sim]{\alpha_{M,N}} & \underline{\operatorname{Hom}}(\operatorname{Tr} \Omega^k \operatorname{Tr} M, N) \\ \downarrow \wr\, \operatorname{Tr} & & \downarrow \wr\, \operatorname{Tr} \\ \underline{\operatorname{Hom}}(\operatorname{Tr} \Omega^k N, \operatorname{Tr} M) & \xrightarrow[\sim]{(\alpha_{\operatorname{Tr} N, \operatorname{Tr} M})^{-1}} & \underline{\operatorname{Hom}}(\operatorname{Tr} N, \Omega^k \operatorname{Tr} M) \end{array}$$

Proof. Consider the complexes $P_I^{\cdot}$ and $Q_I^{\cdot}$ for I the integers in $[0, k+1]$ as above. Then we have a natural isomorphism $\operatorname{Hom}(P_I^{\cdot},$

$Q_I^{\cdot}) \xrightarrow{\sim} \operatorname{Hom}(Q_I^{\cdot *}, P_I^{\cdot *})$, which induces a natural isomorphism $\underset{\sim}{\operatorname{Hom}}(P_I^{\cdot}, Q_I^{\cdot}) \xrightarrow{\sim} \underset{\sim}{\operatorname{Hom}}(Q_I^{\cdot *}, P_I^{\cdot *})$. Using Corollary 3.3 we obtain our desired result. ∎

Consider again the adjointness morphism $s\colon M \to D_k^2(M) = \Omega^k \operatorname{Tr} \Omega^k \operatorname{Tr}(M)$ in $\overline{\operatorname{mod}} \Lambda$. When $n = 2$ we have $\Omega^2 \operatorname{Tr} M \simeq M^*$ in $\overline{\operatorname{mod}} \Lambda$, and it follows from [AB, p. 143] that if $f\colon M \to M^{**}$ is the natural map, then $\underline{f} = s$. In this case we have an exact sequence $0 \to \operatorname{Ext}_\Lambda^1(\operatorname{Tr} M, \Lambda) \to M \xrightarrow{f} M^{**} \to \operatorname{Ext}_\Lambda^2(\operatorname{Tr} M, \Lambda) \to 0$, and in particular, if M is torsionless, an exact sequence $0 \to M \xrightarrow{f} M^{**} \to \operatorname{Ext}_\Lambda^2(\operatorname{Tr} M, \Lambda) \to 0$. The following generalization of this will be useful.

PROPOSITION 3.6. *If $k \geq 2$ and M in* mod Λ *is $(k-1)$-torsionfree, we have an exact sequence* $0 \to M \xrightarrow{f} \Omega^k \operatorname{Tr} \Omega^k \operatorname{Tr} M \amalg Q \to \Omega^{k-2} \operatorname{Ext}_\Lambda^k(\operatorname{Tr} M, \Lambda) \to 0$, *where Q is projective and $\underline{f}\colon M \to \Omega^k \operatorname{Tr} \Omega^k \operatorname{Tr} M$ is the adjointness morphism.*

Proof. We already know the result for $k = 2$, so we can assume $k \geq 3$.

Let M in mod Λ be k-torsionfree and let $N = \operatorname{Tr} \Omega^k \operatorname{Tr} M$. Consider the projective resolutions

$$\cdots \to Q_1 \to Q_0 \to \operatorname{Tr} \Omega^k \operatorname{Tr} M \to 0$$

and

$$\cdots \to L_1 \to L_0 \to \operatorname{Tr} M \to 0.$$

Then we have the commutative diagram

$$\begin{array}{ccccccccccccc}
L_0^* & \longrightarrow & L_1^* & \longrightarrow & L_2^* & \longrightarrow & \cdots & \longrightarrow & L_k^* & \to & L_{k+1}^* & \longrightarrow & \operatorname{Tr} \Omega^k \operatorname{Tr} M \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \wr & & \downarrow \wr & & \downarrow \| \\
Q_{k+1} & \longrightarrow & Q_k & \longrightarrow & Q_{k-1} & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & \operatorname{Tr} \Omega^k \operatorname{Tr} M \longrightarrow 0,
\end{array}$$

where we can assume that $L_k^* \to Q_1$ and $L_{k+1}^* \to Q_0$ are isomorphisms.

Since $\operatorname{Ext}_\Lambda^i(\operatorname{Tr} M, \Lambda) = 0$ for $i = 1, \ldots, k-1$, the top sequence is exact except possibly at L_k^*. Hence we get an exact commutative diagram

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & M & \longrightarrow & L_2^* & \longrightarrow & L_3^* & \longrightarrow & \cdots & \longrightarrow & L_{k-1}^* & \longrightarrow & X & \longrightarrow 0 \\
& \downarrow f & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow s & \\
0 \longrightarrow & \Omega^k \operatorname{Tr} \Omega^k \operatorname{Tr} M & \longrightarrow & Q_{k-1} & \longrightarrow & Q_{k-2} & \longrightarrow & \cdots & \longrightarrow & Q_2 & \longrightarrow & Y & \longrightarrow 0
\end{array}$$

and an exact sequence $0 \to X \xrightarrow{s} Y \to \operatorname{Ext}_\Lambda^k(\operatorname{Tr} M, \Lambda) \to 0$. Using the mapping cone we get an exact sequence $0 \to M \xrightarrow{(f, f')} \Omega^k \operatorname{Tr} \Omega^k \operatorname{Tr} M \amalg L_2^* \to Q_{k-1} \amalg L_3^* \to \cdots \to Q_3 \amalg P_{n-1}^* \to Q_2 \to \operatorname{Ext}_\Lambda^k(\operatorname{Tr} M,$

$\Lambda) \to 0$ and consequently an exact sequence $0 \to M \xrightarrow{(f, f')} \Omega^k \operatorname{Tr} \Omega^k \operatorname{Tr} M \amalg Q \to \Omega^{k-2} \operatorname{Ext}^k_\Lambda(\operatorname{Tr} M, \Lambda) \to 0$ with Q projective. ∎

We shall show in the next section that $\underline{f}\colon M \to D^2_k(M)$ in $\underline{\mathrm{mod}}\,\Lambda$ is sometimes left minimal, that is, if

$$\begin{array}{ccc} M & \xrightarrow{\underline{f}} & D^2_k(M) \\ & \searrow^{\underline{f}} & \downarrow{\scriptstyle \underline{g}} \\ & & D^2_k(M) \end{array}$$

is commutative, then $\underline{g}\colon D^2_k(M) \to D^2_k(M)$ is an isomorphism.

4. GRADE AND EXTENSION CLOSURE

The aim of this section is to show that if Λ is a noetherian R-algebra, that is, R is commutative noetherian and Λ is a finitely generated R-module, then $\Omega^i(\mathrm{mod}\,\Lambda)$ is closed under extensions for $i \leq k$ if and only if $\operatorname{grade} \operatorname{Ext}^i_\Lambda(B, \Lambda) \geq i$ for all B in $\mathrm{mod}\,\Lambda^{\mathrm{op}}$ and $i \leq k$.

We first give a part of the result which is valid for arbitrary twosided noetherian rings. We start out with the following preliminary result.

LEMMA 4.1. *Assume that grade $Z \geq k$ for some Z in* $\mathrm{mod}\,\Lambda$ *and some integer $k \geq 1$. Then we have* $\operatorname{Ext}^{k-1}_\Lambda(Z, \mathscr{X}_k) = 0$, *that is,* $\operatorname{Ext}^{k-1}_\Lambda(Z, X) = 0$ *for all X in* $\mathscr{X}_k = \operatorname{add} \Omega^k(\mathrm{mod}\,\Lambda)$.

Proof. If $k = 1$, then $Z^* = 0$ clearly implies $\operatorname{Hom}_\Lambda(Z, \mathscr{X}_1) = 0$. For $k > 1$ consider for C in $\mathrm{mod}\,\Lambda$ the exact sequence $0 \to \Omega^k C \to P \to \Omega^{k-1} C \to 0$ with P projective and the induced exact sequence $\operatorname{Ext}^{k-2}_\Lambda(Z, \Omega^{k-1} C) \to \operatorname{Ext}^{k-1}_\Lambda(Z, \Omega^k C) \to 0$. Since $\operatorname{grade} Z \geq k - 1$, we have $\operatorname{Ext}^{k-2}_\Lambda(Z, \Omega^{k-1} C) = 0$ by the induction assumption, and consequently $\operatorname{Ext}^{k-1}_\Lambda(Z, \Omega^k C) = 0$. ∎

PROPOSITION 4.2. *Let Λ be a twosided noetherian ring.*

(a) $\Omega^1(\mathrm{mod}\,\Lambda)$ *is closed under extensions if and only if* $\operatorname{Ext}^1_\Lambda(B, \Lambda)^* = 0$ *for all B in* $\mathrm{mod}\,\Lambda^{\mathrm{op}}$.

(b) *Let $k \geq 1$ be an integer. If* $\operatorname{grade} \operatorname{Ext}^i_\Lambda(C, \Lambda) \geq i$ *for all C in* $\mathrm{mod}\,\Lambda^{\mathrm{op}}$ *and $i \leq k$, then* $\Omega^i(\mathrm{mod}\,\Lambda)$ *is closed under extensions for $i \leq k$.*

Proof. (a) Assume first that $\mathscr{X}_1 = \Omega^1(\mathrm{mod}\,\Lambda)$ is closed under extensions, and consider for C in $\mathrm{mod}\,\Lambda$ the exact sequence $0 \to t(C) \to C \to C/t(C) \to 0$, where $t(C)$ is given by the exact sequence $0 \to t(C) \to C \to$

C^{**}. We know that $t(C) \simeq \operatorname{Ext}^1_\Lambda(\operatorname{Tr} C, \Lambda)$ (see [AB, Prop. 2.6]). Assume that $t(C)^* \neq 0$. Then we have $t(t(C)) \neq t(C)$, and hence $V = t(C)/t(t(C)) \neq 0$. Consider the pushout diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t(C) & \xrightarrow{f} & C & \xrightarrow{g} & C/t(C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & \overset{h}{\swarrow} & \Vert & & \\
0 & \longrightarrow & V & \xrightarrow{s} & W & \longrightarrow & C/t(C) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since V and $C/t(C)$ are in $\mathscr{X}_1$ and $\mathscr{X}_1$ is closed under extensions, we have that W is in $\mathscr{X}_1$. Then there is some $h\colon C/t(C) \to W$ such that $hg = q$. But then $sp(t(C)) = qf(t(C)) = hgf(t(C)) = 0$, so that $s(V) \simeq V$ is 0, which is a contradiction. Hence $t(C)^* = 0$, and consequently $\operatorname{grade} \operatorname{Ext}^1_\Lambda(\operatorname{Tr} C, \Lambda) \geq 1$ for all C in $\operatorname{mod} \Lambda$. It follows that $\operatorname{grade} \operatorname{Ext}^1_\Lambda(B, \Lambda) \geq 1$ for all B in $\operatorname{mod} \Lambda^{\mathrm{op}}$.

Assume now that $t(C)^* = 0$ for all C in $\operatorname{mod} \Lambda$, and let C be in the extension closure of $\mathscr{X}_1$. Then we have a chain of submodules $0 = C_n \subset C_{n-1} \subset \cdots \subset C_1 \subset C_0 = C$ such that C_i/C_{i+1} is in $\mathscr{X}_1$ for $i = 0, \ldots, n-1$. Let X be a nonzero submodule of C and let i be largest possible with $X \subset C_i$. Then there is a nonzero map $X \to C_i/C_{i+1}$, so that X^* is not zero. Hence we conclude that $t(C)$ is zero, so that C is in $\mathscr{X}_1$.

(b) For $k = 1$ the claim follows from (a). Assume that $k > 1$ and that the claim has been proved for $k - 1$. Assume that $\operatorname{grade} \operatorname{Ext}^i_\Lambda(C, \Lambda) \geq i$ for all C in $\operatorname{mod} \Lambda^{\mathrm{op}}$ and $i \leq k$. By Theorem 1.7 it is sufficient to show that $\mathscr{X}_i$ is closed under extensions for $1 \leq i \leq k$.

Consider now the exact sequence $0 \to L \to M \to N \to 0$, where L and N are in $\mathscr{X}_k$. Since $\mathscr{X}_{k-1}$ is closed under extensions, M is in $\mathscr{X}_{k-1}$ and is $(k-1)$-torsionfree by Proposition 1.6. Then we have an exact sequence

$$0 \to M \overset{(f, f')}{\to} GF(M) \amalg Q \to \Omega^{k-2} \operatorname{Ext}^k_\Lambda(\operatorname{Tr} M, \Lambda) \to 0 \qquad (*)$$

by Proposition 3.6, where $G = \Omega^k$ and $F = \operatorname{Tr} \Omega^k \operatorname{Tr}$. Since $\operatorname{grade} \operatorname{Ext}^k_\Lambda(\operatorname{Tr} M, \Lambda) \geq k$, we have $\operatorname{Ext}^{k-1}_\Lambda(\operatorname{Ext}^k_\Lambda(\operatorname{Tr} M, \Lambda), \mathscr{X}_k) = 0$ by Lemma 4.1, and hence $\operatorname{Ext}^1_\Lambda(\Omega^{k-2} \operatorname{Ext}^k_\Lambda(\operatorname{Tr} M, \Lambda), \mathscr{X}_k) = 0$. It then follows that $\operatorname{Ext}^1_\Lambda(\Omega^{k-2} \operatorname{Ext}^k_\Lambda(\operatorname{Tr} M, \Lambda), M) = 0$, so that $(*)$ splits. Since $GF(M) \amalg Q$ is in $\mathscr{X}_k$, we conclude that M is in $\mathscr{X}_k$. This completes the proof. ∎

In order to prove the converse of Proposition 4.2(b) the following result will be useful for the induction step.

LEMMA 4.3. *Let* Λ *be a twosided noetherian ring and assume that* $\Omega^i(\text{mod}\,\Lambda)$ *is closed under extensions for* $i \le k$ *and that* $\text{grade}\,\text{Ext}^i(C, \Lambda) \ge i$ *for each* C *in* $\text{mod}\,\Lambda^{\text{op}}$ *and* $i < k$. *Then we have the following.*

(a) *The modules in* $\Omega^i(\text{mod}\,\Lambda)$ *and* $\Omega^i(\text{mod}\,\Lambda^{\text{op}})$ *are i-torsionfree for* $i \le k$.

(b) *If* M *in* $\text{mod}\,\Lambda$ *is* $(k-1)$*-torsionfree, then we have the following.*

(i) *The adjointness morphism* $t\colon J_k^2(\text{Tr}\,M) \to \text{Tr}\,M$ *in* $\underline{\text{mod}}\,\Lambda^{\text{op}}$ *is right minimal.*

(ii) *The adjointness morphism* $s\colon M \to D_k^2(M)$ *in* $\underline{\text{mod}}\,\Lambda$ *is left minimal.*

Proof. (a) $\Omega^i(\text{mod}\,\Lambda)$ is extension closed for $i \le k$ by assumption, and $\Omega^i(\text{mod}\,\Lambda^{\text{op}})$ is extension closed for $i \le k$ by Proposition 1.6 since $\text{grade}\,\text{Ext}^i(C, \Lambda) \ge i$ for each C in $\text{mod}\,\Lambda^{\text{op}}$ and $i < k$. It then follows from Proposition 1.6 that the modules in $\Omega^i(\text{mod}\,\Lambda)$ and $\Omega^i(\text{mod}\,\Lambda^{\text{op}})$ are i-torsionfree.

(b)(i) Let M in $\text{mod}\,\Lambda$ be k-torsionfree and let $N = \text{Tr}\,M$. Since by (a), $\Omega^k N$ is k-torsionfree in $\text{mod}\,\Lambda^{\text{op}}$, it follows from [AB, Prop. 2.21] that the map $\underline{g} = t\colon J_k^2(N) \to N$ has the property that the induced map $\Omega^k(\underline{g})\colon \Omega^k J_k^2(N) \to \Omega^k N$ is an isomorphism.

Consider then the commutative diagram

$$\begin{array}{ccc} J_k^2(N) & \xrightarrow{\underline{g}} & N \\ \Big\downarrow{\scriptstyle \underline{h}} & \nearrow_{\underline{g}} & \\ J_k^2(N) & & \end{array}$$

which induces the commutative diagram

$$\begin{array}{ccc} \Omega^k J_k^2(N) & \xrightarrow{\sim} & \Omega^k N \\ \Big\downarrow{\scriptstyle \Omega^k(\underline{h})} & \nearrow_{\sim} & \\ \Omega^k J_k^2(N) & & \end{array}$$

Then $\Omega^k(\underline{h})$ is an isomorphism. Since $\Omega^k\,\text{Tr}\,\Omega^k N$ is in $\Omega^k(\text{mod}\,\Lambda)$ and is hence k-torsionfree by (a), we have that $\text{Ext}_\Lambda^i(J_k^2(N), \Lambda) = 0$ for $i = 1, \dots, k$. Then it follows that the map $\underline{\text{End}}(J_k^2(N)) \to \underline{\text{End}}(\Omega^k J_k^2(N))$ is an isomorphism, and consequently $\underline{h}\colon J_k^2(N) \to J_k^2(N)$ is an isomorphism. This shows that $\underline{g}\colon J_k^2(N) \to N$ is right minimal.

(ii) Let $f\colon M \to D_k^2(M)$ be the natural map. Then it follows that $\operatorname{Tr} \underline{f}\colon \operatorname{Tr} D_k^2(M) \to \operatorname{Tr} M$ is isomorphic to $\underline{g}\colon J_k^2(\operatorname{Tr} M) \to \operatorname{Tr} M$, and hence $\underline{f}$ is left minimal. ∎

For artin algebras Λ there is the following lemma by Wakamatsu (see [AR1]). Let $\mathscr{B}$ be a covariantly finite subcategory of $\operatorname{mod}\Lambda$ closed under extensions and let $0 \to A \to^f B \to^g C \to 0$ be an exact sequence such that $f\colon A \to B$ is a minimal left $\mathscr{B}$-approximation. Then we have $\operatorname{Ext}^1_\Lambda(C, B) = 0$. We shall need an analogue for noetherian R-algebras, and for this the following lemma is useful.

LEMMA 4.4. *Let Λ be a noetherian R-algebra and let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \big\downarrow \| & & \big\downarrow u & & \big\downarrow v & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

be a commutative diagram in $\operatorname{mod}\Lambda$, *and assume that $\underline{u}\colon B \to B$ is an isomorphism in* $\underline{\operatorname{mod}}\,\Lambda$. *Then $\underline{v}\colon C \to C$ is an isomorphism in* $\underline{\operatorname{mod}}\,\Lambda$.

Proof. Consider the exact commutative diagram

$$\begin{array}{ccccccccccc}
(C,\ ) & \longrightarrow & (B,\ ) & \longrightarrow & (A,\ ) & \xrightarrow{\alpha} & \operatorname{Ext}^1_\Lambda(C,\ ) & \longrightarrow & \operatorname{Ext}^1_\Lambda(B,\ ) & \longrightarrow & \operatorname{Ext}^1_\Lambda(A,\ ) \\
\big\downarrow (v,\ ) & & \big\downarrow (u,\ ) & & \big\downarrow \| & & \big\downarrow \operatorname{Ext}^1(v,\ ) & & \big\downarrow \operatorname{Ext}^1(u,\ ) & & \big\downarrow \| \\
(C,\ ) & \longrightarrow & (B,\ ) & \longrightarrow & (A,\ ) & \xrightarrow{\alpha} & \operatorname{Ext}^1_\Lambda(C,\ ) & \longrightarrow & \operatorname{Ext}^1_\Lambda(B,\ ) & \longrightarrow & \operatorname{Ext}^1_\Lambda(A,\ )
\end{array}$$

Since $\underline{u}\colon B \to B$ is assumed to be an isomorphism, it follows that $\operatorname{Ext}^1(u,\ )\colon \operatorname{Ext}^1_\Lambda(B,\ ) \to \operatorname{Ext}^1_\Lambda(B,\ )$ is an isomorphism. Then diagram chasing shows that $\operatorname{Ext}^1(v,\ )\colon \operatorname{Ext}^1_\Lambda(C,\ ) \to \operatorname{Ext}^1_\Lambda(C,\ )$ is an epimorphism. Since $\operatorname{End}_\Lambda(B)$ is noetherian, it follows that $\operatorname{Ext}^1(v,\ )$ is an isomorphism. By [HR, Theorem 2.4] there is a split exact sequence $0 \to K \to P \coprod C \to^{(s,v)} C \to 0$, where P is projective such that $0 \to \operatorname{Ext}^1_\Lambda(C,\ ) \to^{\operatorname{Ext}^1(v,\ )} \operatorname{Ext}^1_\Lambda(C,\ ) \to \operatorname{Ext}^1_\Lambda(K,\ ) \to 0$ is exact. Then $\operatorname{Ext}^1_\Lambda(K,\ ) = 0$ so that K is projective, and hence $\underline{v}\colon C \to C$ is an isomorphism in $\underline{\operatorname{mod}}\,\Lambda$. ∎

Now we can prove the announced analogue of Wakamatsu's lemma.

LEMMA 4.5. *Let $\mathscr{C}$ be a covariantly finite subcategory of* $\operatorname{mod}\Lambda$ *closed under extensions. Assume that we have an exact sequence* $0 \to M \to^f C_M \to Y_M \to 0$, *where $f\colon M \to C_M$ is a left $\mathscr{C}$-approximation such that $\underline{f}\colon M \to C_M$ is left minimal in* $\underline{\operatorname{mod}}\,\Lambda$. *Then we have* $\operatorname{Ext}^1_\Lambda(Y_M, \mathscr{C}) = 0$.

Proof. Assume that we have an exact sequence $0 \to C \to U \to Y_M \to 0$ with C in $\mathscr{C}$. Consider the pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & U & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & C_M & \longrightarrow & Y_M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\mathscr{C}$ is closed under extensions, V is in $\mathscr{C}$. Using that $f\colon M \to C_M$ is a left $\mathscr{C}$-approximation, we get the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{f} & C_M & \longrightarrow & Y_M & \longrightarrow & 0 \\
 & & \| & & \downarrow s & & \downarrow u & & \\
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & U & \longrightarrow & 0 \\
 & & \| & & \downarrow t & & \downarrow v & & \\
0 & \longrightarrow & M & \xrightarrow{f} & C_M & \longrightarrow & Y_M & \longrightarrow & 0
\end{array}$$

Since $f\colon M \to C_M$ is left minimal in $\underline{\operatorname{mod}\Lambda}$, $\underline{ts}\colon C_M \to C_M$ is an isomorphism in $\underline{\operatorname{mod}\Lambda}$. Hence it follows from Lemma 4.4 that $\underline{vu}\colon Y_M \to Y_M$ is an isomorphism.

Consider now the exact sequence $0 \to C \to U \xrightarrow{v} Y_M \to 0$, and the induced exact sequence of functors $0 \to (Y_M,\ ) \to (U,\ ) \to (C,\ ) \to \operatorname{Ext}^1_\Lambda(Y_M,\ ) \xrightarrow{\operatorname{Ext}^1(v,\ )} \operatorname{Ext}^1_\Lambda(U,\ ) \to \operatorname{Ext}^1_\Lambda(C,\ )$. Since $\overline{vu}\colon Y_M \to Y_M$ is an isomorphism, $\operatorname{Ext}^1(v,\ )\colon \operatorname{Ext}^1_\Lambda(Y_M,\ ) \to \operatorname{Ext}^1_\Lambda(U,\ )$ is a split monomorphism. Hence $(U,\ ) \to (C,\ )$ is an epimorphism so that the sequence $0 \to C \to U \to Y_M \to 0$ splits. Hence we have $\operatorname{Ext}^1_\Lambda(Y_M, \mathscr{C}) = 0$. ∎

We are now ready to prove the converse of Proposition 4.2(b).

PROPOSITION 4.6. *Let Λ be a noetherian R-algebra. For a positive integer k we have that if $\Omega^i(\text{mod}\,\Lambda)$ is closed under extensions for $i \leq k$, then* $\text{grade}\,\text{Ext}^i_\Lambda(C, \Lambda) \geq i$ *for all C in* $\text{mod}\,\Lambda^{\text{op}}$ *and $i \leq k$.*

Proof. We have proved the claim for $k = 1$ in Proposition 4.2(a). Assume that $k > 1$ and that the claim has been proved for $k - 1$. So we assume that $\mathscr{X}_i$ is closed under extensions for $1 \leq i \leq k$, and that $\text{grade}\,\text{Ext}^i(C, \Lambda) \geq i$ for all C in $\text{mod}\,\Lambda^{\text{op}}$ and $i < k$. Let M be $(k - 1)$-torsionfree and consider the exact sequence $0 \to M \to^{(f, f')} GF(M) \amalg Q \to \Omega^{k-2}\,\text{Ext}^k(\text{Tr}\,M, \Lambda) \to 0$ from Proposition 3.6, where $f\colon M \to GF(M)$ is left minimal by Lemma 4.3, and $G = \Omega^n$ and $F = \text{Tr}\,\Omega^n\,\text{Tr}$. Then it follows by Lemma 4.5 that $\text{Ext}^1_\Lambda(\Omega^{k-2}\,\text{Ext}^k_\Lambda(\text{Tr}\,M, \Lambda), \Lambda) \simeq \text{Ext}^{k-1}_\Lambda(\text{Ext}^k_\Lambda(\text{Tr}\,M, \Lambda), \Lambda) = 0$. Since $\text{grade}\,\text{Ext}^k(\text{Tr}\,M, \Lambda) \geq k - 1$ by assumption, we now conclude that $\text{grade}\,\text{Ext}^k_\Lambda(\text{Tr}\,M, \Lambda) \geq k$. Now let N be in $\text{mod}\,\Lambda$. Since the modules in $\mathscr{X}_{k-1}$ are $(k - 1)$-torsionfree by Proposition 1.6, it follows from [AB, Prop. 2.21] that $\text{Ext}^k_\Lambda(\text{Tr}\,N, \Lambda) \simeq \text{Ext}^k_\Lambda(\text{Tr}\,M, \Lambda)$, where M is $(k - 1)$-torsionfree, and consequently $\text{grade}\,\text{Ext}^k_\Lambda(\text{Tr}\,N, \Lambda) \geq k$. ∎

Putting the results from all sections together we have the following.

THEOREM 4.7. *For a noetherian R-algebra Λ the following conditions are equivalent for a positive integer k.*

(a) $\Omega^i(\text{mod}\,\Lambda)$ *is closed under extensions for $i \leq k$.*

(b) $\mathscr{X}_i$ *is closed under extensions for $i \leq k$.*

(c) *If $0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ is a minimal injective resolution of Λ as a right Λ-module, then* $\text{flatdim}\,I_i \leq i + 1$ *for $i < k$.*

(d) $\text{grade}\,\text{Ext}^i_\Lambda(C, \Lambda) \geq i$ *for all C in* $\text{mod}\,\Lambda^{\text{op}}$ *and $i \leq k$.*

(e) $\text{s.grade}\,\text{Ext}^{i+1}_\Lambda(B, \Lambda) \geq i$ *for all B in* $\text{mod}\,\Lambda$ *and $i \leq k$.*

As already pointed out in [AR3], contrary to the condition that Λ be k-Gorenstein, these conditions are not left−right symmetric.

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