Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Stochastic Processes and their Applications 121 (2011) 1464–1491
stochastic processes and their applications
www.elsevier.com/locate/spa

An approximation scheme for reflected stochastic differential equations

Lawrence Christopher Evans*, Daniel W. Stroock

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, United States

Received 26 August 2010; received in revised form 8 March 2011; accepted 8 March 2011
Available online 16 March 2011

Abstract

In this paper, we consider the Stratonovich reflected SDE $\mathrm{d}X_t = \sigma(X_t) \circ \mathrm{d}W_t + b(X_t)\mathrm{d}t + \mathrm{d}L_t$ in a bounded domain $\mathcal{O}$. Letting W_t^N be the N-dyadic piecewise linear interpolation of W_t, we show that the distribution of the solution (X_t^N, L_t^N) to the reflected ODE $\dot{X}_t^N = \sigma(X_t^N)\dot{W}_t^N + b(X_t^N) + \dot{L}_t^N$ converges weakly to that of (X_t, L_t). Hence, we prove a distributional version for reflected diffusions of the famous result of Wong and Zakai.

In particular, we apply our result to derive some geometric properties of coupled reflected Brownian motion, especially those properties which have been used in the recent work on the "hot spots" conjecture for special domains.

MSC: primary 60J50; 60F17; secondary 60J55; 60J60

Keywords: Wong–Zakai approximation; Reflected stochastic differential equation

1. Introduction

1.1. Motivation

As is well known, Itô stochastic differential equations can be very misleading from a geometric standpoint. The classic example of this observation is the Itô stochastic differential

* Corresponding address: Department of Mathematics, University of Missouri, Columbia, MO 65211, United States.
E-mail addresses: evanslc@missouri.edu (L.C. Evans), dws@math.mit.edu (D.W. Stroock).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2011.03.005

equation (SDE)

$$\mathrm{d}X(t) = \sigma(X(t))\mathrm{d}W_t \quad \text{with } X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

where W_t is a 1-dimensional Brownian motion. If one makes the mistake of thinking that Itô differentials of Brownian motion behave like classical differentials, then one would predict that $X(t)$ should live on the unit circle. On the other hand, Itô's formula, which is a quantitative statement of the extent to which they do not behave like classical differentials, says that $\mathrm{d}|X(t)|^2 = |X(t)|^2\mathrm{d}t$, and so $|X(t)|^2 = \mathrm{e}^t$.

To avoid the sort of misinterpretation to which Itô SDE's lead, it is convenient to replace Itô SDE's by their Stratonovich counterparts. When one does so, then the Wong–Zakai theorem [16] shows that the solution to the SDE can be approximated by solutions to the ordinary differential equation (ODE) which one obtains by piecewise linearizing the Brownian paths. In this way, one can transfer to solutions of the SDE geometric properties which one knows for the solutions to the ODE's. The purpose of this paper is to carry out the analogous program for SDE's for diffusions which are reflected at the boundary of some region. This is not the first time that such a program has been attempted. For example, R. Petterson proved in [7] a result of this sort under the assumption that the domain is convex. Unfortunately, convexity is too rigid a requirement for applications of the sort which appear in papers like [2] by Banuelos and Burdzy, and so it is important to replace convexity by a more general condition, like the one given in [6] by Sznitman and Lions. Finally, it should be mentioned that the article [5] by Kohatsu-Higa contains a very general, highly abstract approximation procedure which may be applicable to the situation here.

1.2. Background for reflected SDE's

We begin by recalling the (deterministic) Skorohod problem.

Let $\mathcal{O} \subset \mathbb{R}^d$ be a domain and to each $x \in \partial\mathcal{O}$ assign a nonempty collection $\nu(x) \subseteq \mathbb{S}^{d-1}$, to be thought of as the set of directions in which a path can be "pushed" when it hits x. Given a continuous path $w : [0, \infty) \to \mathbb{R}^d$ with $w_0 \in \bar{\mathcal{O}}$, known as the "input", we say that a *solution to the Skorohod problem for* $(\mathcal{O}, \nu(x))$ is a pair $(x., \ell.)$ consisting of a continuous path $t \in [0, \infty) \longmapsto x_t \in \bar{\mathcal{O}}$ and a continuous function of locally bounded variation $t \in [0, \infty) \longmapsto \ell_t \in \mathbb{R}^d$ such that

$$x_t = w_t + \ell_t, \qquad |\ell|_t = \int_0^t 1_{\partial\mathcal{O}}(x_s)\mathrm{d}|\ell|_s, \quad \text{and} \quad \ell_t = \int_0^t \nu(x_s)\mathrm{d}|\ell|_s, \tag{1}$$

where $|\ell|_t$ denotes the total variation of ℓ_t on the interval $[0, t]$, and the third line is a shorthand way of saying that

$$\frac{\mathrm{d}\ell_t}{\mathrm{d}|\ell|_t} \in \nu(x_t), \quad \mathrm{d}|\ell|_t - \text{a.e.}$$

When a unique solution exists for each input, we will call the map $w. \rightsquigarrow (x., \ell.)$ the *Skorohod map* and will denote it by Γ. Also, the path $x.$ will be referred to as the "output".

Throughout this paper, we will take $\nu(x)$ to be the collection of inward pointing proximal normal vectors

$$\nu(x) \equiv \{\nu \in \mathbb{S}^{d-1} : \exists C > 0\ \forall x' \in \bar{\mathcal{O}}\ (x' - x) \cdot \nu + C|x - x'|^2 \geq 0\}. \tag{2}$$

Elementary algebra shows that

$$(x' - x) \cdot \nu + C|x - x'|^2 < 0 \Longleftrightarrow \left|x' - \left(x - \frac{\nu}{2C}\right)\right|^2 < \left(\frac{1}{2C}\right)^2 \tag{3}$$

which shows that, geometrically, $\nu(x)$ is the collection of unit vectors based at $x \in \partial\mathcal{O}$ such that there exists an open ball touching the base of ν but not intersecting $\mathcal{O}$.

The class of domains which we will consider was described by Lions and Sznitman in [6]. Namely, we will say that $\mathcal{O}$ is *admissible* if the definition given below holds.

Definition 1.1. 1. $\forall x \in \partial\mathcal{O}$, $\nu(x) \neq \phi$, and there exists a $C_0 \geq 0$ such that

$$(x' - x) \cdot \nu + C_0|x - x'|^2 \geq 0 \quad \text{for all } x' \in \mathcal{O},\ x \in \partial\mathcal{O}, \text{ and } \nu \in \nu(x).$$

2. There exists a function $\phi \in C^2(\mathbb{R}^d; \mathbb{R})$ and $\alpha > 0$ such that

$$\nabla\phi(x) \cdot \nu \geq \alpha > 0 \quad \text{for all } x \in \partial\mathcal{O} \text{ and } \nu \in \nu(x).$$

3. There exist $n \geq 1$, $\lambda > 0$, $R > 0$, $a_1, \ldots, a_n \in \mathbb{S}^{d-1}$, and $x_1, \ldots, x_n \in \partial O$ such that

$$\partial O \subseteq \bigcup_{i=1}^{n} B(x_i, R) \quad \text{and}$$

$$x \in \partial O \cap B(x_i, 2R) \Longrightarrow \nu \cdot a_i \geq \lambda > 0 \quad \text{for all } \nu \in \nu(x).$$

In view of (3), Part 1 of Definition 1.1 can be seen as a sort of uniform exterior ball condition. More precisely, it says that not only can every point $x \in \partial\mathcal{O}$ be touched by an exterior ball but also that the exterior ball touching x can be scaled to have a uniformly large radius. In the convex analysis literature, the closure of a set $\mathcal{O}$ satisfying Part 1 of Definition 1.1 is said to be *uniformly prox-regular* (see [8], especially Theorem 4.1, for more on the properties of uniformly prox-regular sets).

Parts 2 and 3 of Definition 1.1 are regularity requirements on $\partial\mathcal{O}$ which ensure that the "normal vectors" do not fluctuate too wildly. In this connection, notice that Part 3 is implied by Part 2 when $\mathcal{O}$ is bounded.

In their paper [6], Lions and Sznitman show that for each $w_\cdot \in C([0, \infty); \mathbb{R}^d)$ with $w_0 \in \bar{\mathcal{O}}$, there exists a unique solution $(x_\cdot, \ell_\cdot)$ to the deterministic Skorohod problem when the domain $\mathcal{O}$ is admissible. The map Γ which takes $w_\cdot$ to $x_\cdot$ is called the deterministic Skorohod map.

We turn next to the formulation of reflected diffusions in terms of a Skorohod problem for an SDE. Until further notice, we will be looking at Itô SDE's and will only reformulate them as Stratonovich SDE's when it is important to do so.

Let $\mathcal{O} \subset \mathbb{R}^d$ be an admissible domain, and let $\sigma : \bar{\mathcal{O}} \longrightarrow \mathrm{Hom}(\mathbb{R}^r; \mathbb{R}^d)$ and $b : \bar{\mathcal{O}} \longrightarrow \mathbb{R}^d$ be uniformly Lipschitz continuous maps. Given an r-dimensional Brownian motion $W_\cdot$ and $x_0 \in \bar{\mathcal{O}}$, a solution to $(X_\cdot, L_\cdot)$ to the reflected SDE (4) is a continuous process $\{(X_t, L_t) : t \geq 0\}$ which is progressively measurable with respect to $W_\cdot$ and satisfies the conditions that $(X_t, L_t) \in \bar{\mathcal{O}} \times \mathbb{R}^d$ and $|L|_t < \infty$ for all $t \geq 0$, and, almost surely,

$$\begin{aligned} X_t &= x_0 + \int_0^t \sigma(X_s)\mathrm{d}W_s + \int_0^t b(X_s)\mathrm{d}s + L_t, \\ |L|_t &= \int_0^t 1_{\partial\mathcal{O}}(X_s)\mathrm{d}|L|_s, \quad \text{and} \quad L_t = \int_0^t \nu(X_s)\mathrm{d}|L|_s, \end{aligned} \tag{4}$$

where $|L|_t$ denotes the total variation of L_t by time t, and the third line is shorthand for $\frac{\mathrm{d}L_t}{\mathrm{d}|L|_t} \in \nu(X_t)$, $\mathrm{d}|L|_t -$ a.e.

Existence and uniqueness of solution to reflected SDE's was proved by Tanaka in [13] when $\mathcal{O}$ is convex. The extension of his result to admissible domains was made by Lions and Sznitman in [6] and Saisho in [9]. We refer the reader to those papers for an overview of the subject.

1.3. Our result

With these preliminaries in place, our main result can be summarized as follows. Given a bounded admissible domain $\mathcal{O}$, suppose that $\sigma \in C^2(\bar{\mathcal{O}}; \mathrm{Hom}(\mathbb{R}^r; \mathbb{R}^d))$ and $b : \bar{\mathcal{O}} \longrightarrow \mathbb{R}^d$ is uniformly Lipschitz continuous. Then, given an r-dimensional Brownian motion $W_\cdot$, denote by $W_\cdot^N$ the polygonal path obtained by linearly interpolating $W_\cdot$ over the N-dyadic intervals (defined precisely in Section 3). Then, for each $x_0 \in \bar{\mathcal{O}}$, there exists a unique solution to the reflected ODE

$$X_t^N = x_0 + \int_0^t \sigma(X_s^N)\mathrm{d}W_s^N + \int_0^t b(X_s^N)\mathrm{d}s + L_t^N,$$
$$|L^N|_t = \int_0^t 1_{\partial\mathcal{O}}(X_s^N)\mathrm{d}|L^N|_s, \quad \text{and} \quad L_t^N = \int_0^t \nu(X_s^N)\mathrm{d}|L^N|_s.$$

Moreover, if (X, L) is the solution to (4) when the stochastic integrals are taken in the sense of Stratonovich, then the distribution of (X^N, L^N, W^N) on $C([0,\infty); \bar{\mathcal{O}}) \times C([0,\infty); \mathbb{R}^d) \times C([0,\infty); \mathbb{R}^r)$ converges to that of (X, L, W). This is the content of Theorem 5.4. Finally, in Section 6 we show how our result can be applied to better understand the geometry of certain reflected SDE.

In [16,17] Wong and Zakai show, in the unreflected case, that X^N almost surely converges to X in $C([0,\infty); \mathbb{R}^d)$ when $d = 1$ and in the L^1 sense when $d \geq 2$. In [7] Petterson shows that in the reflected case with a convex domain, X^N converges to X almost surely when either $d = 1$ or $d \geq 2$ and σ is constant. In view of his results, it would be interesting to know whether almost sure convergence, or at least convergence in measure, holds in the more general setting in which we are working here. In light of the cited results, a counter-example would require that $d \geq 2$ and that either the domain is non-convex or σ is non-constant. To date, the authors are unaware of either a proof of stronger convergence or a counter-example to it. Be that as it may, at least for the applications we consider in Section 6, convergence in distribution is sufficient.

2. Equations with reflection

2.1. Properties of solutions to reflected ODE's

Suppose that $\mathcal{O}$ is a bounded, admissible domain and that $\sigma : \bar{\mathcal{O}} \longrightarrow \mathrm{Hom}(\mathbb{R}^r; \mathbb{R}^d)$ is uniformly Lipschitz continuous. In this section, we will show that, for each $x_0 \in \bar{\mathcal{O}}$ and piecewise smooth $w_\cdot : [0,\infty) \longrightarrow \mathbb{R}^r$, there is precisely one solution $(x_\cdot, \ell_\cdot)$ to the reflected ODE

$$x_t = x_0 + \int_0^t \sigma(x_s)\mathrm{d}w_s + \ell_t,$$
$$|\ell|_t = \int_0^t 1_{\partial\mathcal{O}}(x_s)\mathrm{d}|\ell|_s, \quad \text{and} \quad \ell_t = \int_0^t \nu(x_s)\mathrm{d}|\ell|_s, \tag{5}$$

where $x_\cdot \in C([0,\infty); \bar{\mathcal{O}})$ and $\ell_t : [0,\infty) \longrightarrow \mathbb{R}^d$ is a continuous function having finite variation $|\ell|_t$ on $[0, t]$ for all $t > 0$. In addition, we will give a geometrically appealing alternate description of this solution. Previously, existence and uniqueness results for variants of (5) are well known in the convex analysis literature. For example, see [3] for such a recent result as well as a good overview of other known results.

Although the proofs of existence and uniqueness are implicit in the contents of other articles, we, mimicking the proof of Theorem 3.1 in [6], will prove them here. For this purpose, consider the map $F_w : C([0,\infty); \bar{O}) \longrightarrow C([0,\infty); \bar{O})$ given by $F_w(y_\cdot) = \Gamma(x_0 + \int_0^\cdot \sigma(y_s)\mathrm{d}w_s)$, where Γ is the Skorohod map (we will henceforth suppress the dependence of F on w). We will show that F has a unique fixed point, and the key to doing so is contained in the next lemma.

Lemma 2.1. *For each $T > 0$ there exists a $C = C_w(T) < \infty$ such that for any pair of paths $y_\cdot$ and $y'_\cdot$,*

$$|F(y_\cdot)_t - F(y'_\cdot)_t|^2 \le \int_0^t \mathrm{e}^{C(t-\tau)}|y_\tau - y'_\tau|^2\mathrm{d}\tau \quad \text{for all } t \in [0,T]. \tag{6}$$

Proof. Set $z_\cdot = F(y_\cdot)$ and $z'_\cdot = F(y'_\cdot)$. Given $T > 0$, we will show that there is a $C < \infty$ such that

$$|z_t - z'_t|^2 \le C\left(\int_0^t |z_\tau - z'_\tau|^2\,\mathrm{d}\tau + \int_0^t |y_\tau - y'_\tau|^2\,\mathrm{d}\tau\right), \quad t \in [0,T].$$

Once this is proved, the required estimate follows immediately from Gronwall's inequality.

Let ϕ be the function associated with $\mathcal{O}$ (see part 2 of Definition 1.1). For any constant γ, we have that

$$\begin{aligned}
&\mathrm{e}^{-\gamma[\phi(z_t)+\phi(z'_t)]}d(\mathrm{e}^{\gamma[\phi(z_t)+\phi(z'_t)]}|z_t - z'_t|^2)\\
&\quad = 2(z_t - z'_t)\cdot[(\sigma(y_t)\mathrm{d}w_t + \mathrm{d}\ell_t) - (\sigma(y'_t)\mathrm{d}w_t + \mathrm{d}\ell'_t)]\\
&\qquad + |z_t - z'_t|^2\gamma[\nabla\phi(z_t)\cdot(\sigma(y_t)\mathrm{d}w_t + \mathrm{d}\ell_t) + \nabla\phi(z'_t)(\sigma(y'_t)\mathrm{d}w_t + \mathrm{d}\ell'_t)]\\
&\quad = [(2(z_t - z'_t) + \gamma|z_t - z'_t|^2\nabla\phi(z_t))\cdot\nu(z_t)]\mathrm{d}|\ell|_t\\
&\qquad + [(2(z'_t - z_t) + \gamma|z_t - z'_t|^2\nabla\phi(z'_t))\cdot\nu(z'_t)]\mathrm{d}|\ell'|_t + [2(z_t - z'_t)\cdot(\sigma(y_t) - \sigma(y'_t))\\
&\qquad + \gamma|z_t - z'_t|^2(\nabla\phi(z_t)\sigma(y_t) + \nabla\phi(z'_t)\sigma(y'_t))]\mathrm{d}w_t.
\end{aligned}$$

Taking $\gamma = \frac{-2C_0}{\alpha}$, we have that (cf. Part 1 of Definition 1.1) the first two terms are less than or equal to 0. Since σ and $\nabla\phi$ are Lipschitz continuous and $\frac{\mathrm{d}w}{\mathrm{d}t}$ is bounded on finite intervals, we know that there exists a $C = C_w(T) < \infty$ such that

$$|z_t - z'_t|^2 \le C\left(\int_0^t |z_\tau - z'_\tau|^2\mathrm{d}\tau + \int_0^t |z_\tau - z'_\tau||y_\tau - y'_\tau|\mathrm{d}\tau\right)$$

for $t \in [0,R]$. Thus, because $|z_\tau - z'_\tau||y_\tau - y'_\tau| \le \frac{1}{2}|z_\tau - z'_\tau|^2 + \frac{1}{2}|y_\tau - y'_\tau|^2$, we get our estimate after replacing C by $2C$. $\square$

Once we have Lemma 2.1, one can apply a standard Picard iteration argument to show that F_w has a unique fixed point and that this fixed point is the first component of the one and only pair (x_t, ℓ_t) which solves (5).

We now want to describe a couple of important properties of the solution $(x_\cdot, \ell_\cdot)$.

Lemma 2.2. *Let $(x_\cdot, \ell_\cdot)$ be the solution to (5) for a given input $w_\cdot$ and starting point $x_0 \in \bar{\mathcal{O}}$. Then there exists a constant C, depending only on σ, b, and $\mathcal{O}$, such that*

$$\mathrm{d}|x|_t \le C\mathrm{d}|w|_t.$$

Proof. Set $y_t = x_0 + \int_0^t \sigma(x_s)\mathrm{d}w_s$. Then, $x_. = \Gamma(y_.)$, and so it follows from Theorem 2.2 in [6] that $\mathrm{d}|\ell|_t \le \mathrm{d}|y|_t$. Since σ is bounded on $\bar{\mathcal{O}}$, there exists a $C < \infty$ such that $\mathrm{d}|y|_t \le C\mathrm{d}|w|_t$, and therefore, because $x_t = y_t + \ell_t$, we have that $\mathrm{d}|x|_t \le \mathrm{d}|y|_t + \mathrm{d}|\ell|_t \le C(\mathrm{d}|w|_t + \mathrm{d}|w|_t)$, from which the lemma follows immediately. □

We now introduce a more geometric representation of Eq. (5). For a closed set $D \subseteq \mathbb{R}^d$ and $z \in \mathbb{R}^d$, let $d_D(z) \equiv \inf_{y \in D} |y - z|$ denote the distance from z to D and denote by

$$T_D(z) \equiv \left\{ v \in \mathbb{R}^d : \liminf_{h \searrow 0} \frac{d_D(z + hv)}{h} = 0 \right\}$$

the *tangent cone* (a.k.a. the contingent cone) to D at z. Finally, let $\mathrm{proj}_D(z)$ denote the (possibly multi-valued) projection of z onto D.

The following is a version of a representation result which was introduced originally in [4].

Theorem 2.3. *Let $\mathcal{O}$ be a bounded, admissible set and $w.$ a fixed, piecewise smooth input. If $(x., \ell.)$ is the unique solution to (5), then*

$$\dot{x}_t = proj_{T_{\bar{\mathcal{O}}}(x_t)}(\sigma(x_t)\dot{w}_t), \quad t\text{-a.e.} \tag{7}$$

Conversely, given a solution $x.$ to (7), there exists an $\ell.$ such that $(x., \ell.)$ is a solution to (5).

Remark 2.4. In general, the tangent cone $T_D(z)$ is only closed and not necessarily convex. However, Part 1 of Definition 1.1 guarantees that $T_{\bar{\mathcal{O}}}(z)$ is convex for all $z \in \bar{\mathcal{O}}$ (cf. Lemma 2.5 below) and so $\mathrm{proj}_K(\cdot)$ is single valued.

In order to prove Theorem 2.3, we will need to introduce some concepts from convex analysis. For more information about these concepts and their properties, we refer the reader to the texts [14,15].

A nonempty set $K \subseteq \mathbb{R}^d$ is called a *cone* if $v \in K \implies \lambda v \in K$ for all $\lambda \ge 0$. Given a cone K, we denote by K^* its *polar cone* K^* to be the set $\{w : v \cdot w \le 0,\ \forall v \in K\}$. Next, for a given closed set $D \subseteq \mathbb{R}^d$ and a $z \in D$, we define the *proximal normal cone* to D at z to be the set

$$N_D^p(z) \equiv \{v \in \mathbb{R}^d : \exists C > 0 \text{ s.t. } (y - z) \cdot v \le C|z - y|^2,\ \forall y \in D\}$$

and the *Clarke tangent cone* to D at z to be the set

$$\hat{T}_D(z) \equiv \{v \in \mathbb{R}^d : \forall z_n \in D \text{ s.t. } z_n \to z,\ \exists v_n \in T_D(z_n) \text{ s.t. } v_n \to v\}.$$

Note that $\hat{T}_D(z)$ is always convex.

We now present a lemma which records the properties of an admissible set $\mathcal{O}$ in terms of these concepts.

Lemma 2.5. *Let $\mathcal{O}$ be admissible. Then we have the following.*

(1) *For each $z \in \partial\mathcal{O}$,*

$$v \in N_{\bar{\mathcal{O}}}^p(z) \iff \frac{-v}{|v|} \in \nu(z) \quad \text{for } v \ne 0.$$

(2) *The graph of $z \longrightarrow N_{\bar{\mathcal{O}}}^p(z)$ is closed. That is, if $z_i \in \bar{\mathcal{O}}$, $v_i \in N_{\bar{\mathcal{O}}}^p(z_i)$, $z_i \to z$, and $v_i \to v$, then $v \in N_{\bar{\mathcal{O}}}^p(z)$.*

(3) *$T_{\bar{\mathcal{O}}}(z) = \hat{T}_{\bar{\mathcal{O}}}(z)$, and so it is convex for all $z \in \bar{\mathcal{O}}$.*

(4) *$N_{\bar{\mathcal{O}}}^p(z) = T_{\bar{\mathcal{O}}}(z)^*$ for all $z \in \bar{\mathcal{O}}$.*

Proof. (1) is immediate from our definitions.

(2) follows from (1) and Part 1 of Definition 1.1. Indeed, there exists a $C_0 > 0$ such that for each i,

$$(z_i - y) \cdot v_i + C_0 |v_i| |z_i - y|^2 \geq 0, \quad \forall y \in \bar{\mathcal{O}} \tag{8}$$

(note that when $z_i \in \mathcal{O}$, $v_i = 0$ and (8) holds trivially). Taking $i \to \infty$ we see that $(z - y) \cdot v + C_0 |v| |z - y|^2 \geq 0$ for all $y \in \bar{\mathcal{O}}$, from which it follows that $v \in N^p_{\bar{\mathcal{O}}}(z)$.

(3) and (4) follow in a standard way from (2). See Chapter 4. of [15] and Chapter 6 of [14] (in particular Corollary 6.29) for details. □

Using ideas from [4], we now prove Theorem 2.3.

Proof of Theorem 2.3. First suppose $(x_., \ell_.)$ is a solution to (5). From Lemma 2.2 and its proof, we see that $x_.$ and $\ell_.$ are locally Lipschitz and therefore $\dot{x}_t = \sigma(x_t)\dot{w}_t + \dot{\ell}_t$, t-a.e. Since x_{t+h} and x_{t-h} are in $\bar{\mathcal{O}}$, we have that

$$\dot{x}_t \in -T_{\bar{\mathcal{O}}}(x_t) \cap T_{\bar{\mathcal{O}}}(x_t), \quad t\text{-a.e.}, \tag{9}$$

and, because $T_{\bar{\mathcal{O}}}(x_t)$ is convex, $\dot{x}_t$ is the projection of $\sigma(x_t)\dot{w}_t$ onto $T_{\bar{\mathcal{O}}}(x_t)$ if and only if $(\sigma(x_t)\dot{w}_t - \dot{x}_t) \cdot (v - \dot{x}_t) \leq 0$ for all, $v \in T_{\bar{\mathcal{O}}}(x_t)$. Note that by property (1) of Lemma 2.5, $-\dot{\ell}_t \in N^p_{\bar{\mathcal{O}}}(x_t)$ (when $x_t \in \mathcal{O}$ this holds trivially), and so, by property (4) of Lemma 2.5 and (9), we have that

$$\dot{x}_t \cdot \dot{\ell}_t \leq 0, \qquad \dot{x}_t \cdot \dot{\ell}_t \geq 0 \Longrightarrow \dot{x}_t \cdot \dot{\ell}_t = 0.$$

Therefore, using property (4) again, we have that

$$(\sigma(x_t)\dot{w}_t - \dot{x}_t) \cdot (v - \dot{x}_t) = -\dot{\ell}_t \cdot (v - \dot{x}_t) = -\dot{\ell}_t \cdot v \leq 0$$

as desired.

Conversely, suppose $x_.$ is a solution to (7), and set $\ell_t \equiv \int_0^t \dot{x}_s - \sigma(x_s)\dot{w}_s \mathrm{d}s$. Then $\ell_0 = 0$ and, since σ is bounded, $\ell_.$ is a continuous function of locally bounded variation. Finally, because $\dot{x}_t$ is the projection of $\sigma(x_t)\dot{w}_t$ onto the convex set $T_{\bar{\mathcal{O}}}(x_t)$, we have that

$$-\dot{\ell}_t \cdot (v - \dot{x}_t) = (\sigma(x_t)\dot{w}_t - \dot{x}_t) \cdot (v - \dot{x}_t) \leq 0, \quad \forall v \in T_{\bar{\mathcal{O}}}(x_t).$$

Since $\dot{x}_t \in T_{\bar{\mathcal{O}}}(x_t)$ and $T_{\bar{\mathcal{O}}}(x_t)$ is a convex cone, for each $v \in T_{\bar{\mathcal{O}}}(x_t)$, $x_t + v \in T_{\bar{\mathcal{O}}}(x_t)$. Thus, by replacing v with $v + x_t$ in the inequality above, we find that $-\dot{\ell}_t \cdot v \leq 0$ for all $v \in T_{\bar{\mathcal{O}}}(x_t)$, and so $-\dot{\ell}_t \in T_{\bar{\mathcal{O}}}(x_t)^* = N^p_{\bar{\mathcal{O}}}(x_t)$. Finally, by property (1) of Lemma 2.5, this implies that (x_t, ℓ_t) is a solution to (5). □

3. Tightness of the approximating measures

Let $(C([0,\infty); \mathbb{R}^r), \mathcal{F}, \mathbb{W})$ be the standard r-dimensional Wiener space. That is, $\mathcal{F}$ is the Borel field for $C([0,\infty); \mathbb{R}^r)$ and $\mathbb{W}$ is the standard Wiener measure. We will use $W_.$ to denote a generic Wiener path and $\mathcal{F}_t$ to denote the σ-algebra generated by $W_. \upharpoonright [0, t]$. Finally, for each positive integer N, let $W^N_.$ denote the N-dyadic linear polygonalization of $W_.$. That is, $W^N_{m2^{-N}} = W_{m2^{-N}}$ and $W^N_.$ is linear on $[m2^{-N}, (m+1)2^{-N}]$ for each $m \in \mathbb{N}$.

Next, $\mathcal{O} \subset \mathbb{R}^d$ will be a bounded, admissible domain, and $b : \bar{\mathcal{O}} \longrightarrow \mathbb{R}^d$ and $\sigma : \bar{\mathcal{O}} \longrightarrow \mathrm{Hom}(\mathbb{R}^r; \mathbb{R}^d)$ will be uniformly Lipschitz continuous functions. Given a starting point $x_0 \in \bar{\mathcal{O}}$, for each $W^N_\cdot$, $(X^N_\cdot, L^N_\cdot)$ will denote the solution to the reflected ODE (5) with w_t and $\sigma(x)$ replaced by, respectively,

$$\begin{pmatrix} W^N_t \\ t \end{pmatrix} \in \mathbb{R}^r \times \mathbb{R} \quad \text{and} \quad \begin{pmatrix} \sigma(x) \\ b(x) \end{pmatrix} \in \text{Hom}(\mathbb{R}^r \times \mathbb{R}; \mathbb{R}^d \times \mathbb{R}).$$

$\{X^N_t : t \geq 0\}$ and $\{L^N_t : t \geq 0\}$ are then progressively measurable with respect to $\{\mathcal{F}_t : t \geq 0\}$, and we will use $\mathbb{P}^N$ on the (X, L, W)-*pathspace* $C([0, \infty); \bar{\mathcal{O}}) \times C([0, \infty); \mathbb{R}^d) \times C([0, \infty); \mathbb{R}^r)$ to denote the distribution of the triple (X^N_t, L^N_t, W^N_t) under $\mathbb{W}$.

In first subsection, we show that the family $\{\mathbb{P}^N : N \geq 0\}$ is tight on the (X, L, W)-pathspace. In second subsection, we also develop some estimates which will needed for the next section.

3.1. Tightness of the $\mathbb{P}^N$

By Kolmogorov's Continuity Criterion, we will know that $\{\mathbb{P}^N : N \geq 0\}$ is tight as soon as we prove that for each $m \in \mathbb{N}$ and $T > 0$ there exists a $C_m(T) < \infty$, which is independent of N, such that for all $0 \leq s < t \leq T$,

$$\mathbb{E}[|W^N_t - W^N_s|^{2^{m+1}}] \leq C_m(T)(t-s)^{2^m} \tag{10}$$

$$\mathbb{E}[|X^N_t - X^N_s|^{2^{m+1}}] \leq C_m(T)(t-s)^{2^m} \tag{11}$$

$$\mathbb{E}[|L^N_t - L^N_s|^{2^{m+1}}] \leq C_m(T)(t-s)^{2^m}. \tag{12}$$

First note that (10) is an easy consequence of the equality $\mathbb{E}[|W_t - W_s|^{2^{m+1}}] = C_m(t-s)^{2^m}$ where $C_m = \mathbb{E}[|W_1|^{2^{m+1}}]$.

The proofs of (11) and (12) are a little more involved.

Lemma 3.1. *There is a* $C < \infty$ *such that for all* $s < t \leq s + 2^{-N}$,

$$|X^N_t - X^N_s| \leq C|W^N_t - W^N_s| + C(t-s). \tag{13}$$

Proof. When s and t lie in the same N-dyadic interval, this follows more or less immediately from Lemma 2.2. Namely,

$$\begin{aligned} |X^N_t - X^N_s| &\leq |X^N|_t - |X^N|_s \leq C((|W^N|_t - |W^N|_s) + (t-s)) \\ &= C(|W^N_t - W^N_s| + (t-s)), \end{aligned}$$

where the last equality comes from the fact that s and t lie in the same N-dyadic interval. When they are in adjacent N-dyadic intervals, one can reduce to the case when they are in the same N-dyadic interval by an application of Minkowski's inequality. □

It remains to handle s and t with $t - s > 2^{-N}$, and for this we will need the next three lemmas. Here, and elsewhere, $\lfloor u \rfloor_N$ is shorthand for the largest N-dyadic number $m2^{-N}$ dominated by u. That is, $\lfloor u \rfloor_N$ equals 2^{-N} times the integer part of $2^N u$. When the choice of N is clear from the context, we simply write $\lfloor u \rfloor$. We will also use the notation $\Delta W^N_{m2^{-N}} := W^N_{(m+1)2^{-N}} - W^N_{m2^{-N}}$ which gives the convenient algebraic relation $W^N_u - W^N_{\lfloor u \rfloor} = (u - \lfloor u \rfloor)2^N \Delta W^N_{\lfloor u \rfloor}$.

Lemma 3.2. *For $m \geq 0$ there exists a $C_m < \infty$ such that for all $s < t$*

$$\mathbb{E}\left[\left(\int_s^t |W_u^N - W_{\lfloor u\rfloor}^N| \, \mathrm{d}|W^N|_u\right)^{2^m}\right] \leq C_m(t-s)^{2^m} \tag{14}$$

and

$$\mathbb{E}\left[\left(\int_s^t (u - \lfloor u\rfloor) \, \mathrm{d}|W^N|_u\right)^{2^m}\right] \leq C_m(t-s)^{2^m}. \tag{15}$$

Proof. If $s < t$ lie in the same N-dyadic interval we have that

$$\int_s^t |W_u^N - W_{\lfloor u\rfloor}^N|\mathrm{d}|W^N|_u = 4^N|\Delta W_{\lfloor s\rfloor}^N|^2 \int_s^t (u - \lfloor u\rfloor)\mathrm{d}u \leq 2^N|\Delta W_{\lfloor s\rfloor}^N|^2(t-s)$$
$$\int_s^t (u - \lfloor u\rfloor)\mathrm{d}|W^N|_u = 2^N|\Delta W_{\lfloor s\rfloor}^N| \int_s^t (u - \lfloor u\rfloor)\mathrm{d}u \leq |\Delta W_{\lfloor s\rfloor}^N|(t-s)$$

and so

$$\begin{aligned}\mathbb{E}\left[\left(\int_s^t |W_u^N - W_{\lfloor u\rfloor}^N|\mathrm{d}|W^N|_u\right)^{2^m}\right]^{2^{-m}} &\leq C_m(t-s) \\ \mathbb{E}\left[\left(\int_s^t (u - \lfloor u\rfloor)\mathrm{d}|W^N|_u\right)^{2^m}\right]^{2^{-m}} &\leq C_m 2^{-N2^{m-1}}(t-s) \leq C_m(t-s).\end{aligned} \tag{16}$$

Applying the Minkowski inequality, we see that the inequalities (16) continue to hold for general $s < t$. □

Lemma 3.3. *Let ϕ and α be as in Part 2 of Definition 1.1, and set $\gamma = -\frac{2C_0}{\alpha}$, where C_0 is the constant in Part 1 of that definition. Given $s \geq 0$, there exist $\{\mathcal{F}_t : t \geq 0\}$ progressively measurable functions $\{Z_{\tau,s} : \tau \geq s\}$ and $\{V_{\tau,s} : \tau \geq s\}$ satisfying*

$$|Z_{u,s}^N| \leq C|X_u^N - X_s^N|, \qquad |Z_{u_2,s}^N - Z_{u_1,s}^N| \leq C|X_{u_2}^N - X_{u_1}^N|, \quad \text{and} \quad |V_{u,s}^N| \leq C, \tag{17}$$

with a constant $C < \infty$, which is independent of s and N, such that

$$\mathrm{e}^{\gamma\phi(X_t^N)}|X_t^N - X_s^N|^2 \leq \int_s^t Z_{u,s}^N \mathrm{d}W_u^N + \int_s^t V_{u,s}^N \mathrm{d}u \quad \text{for all } t > s. \tag{18}$$

Proof. Just as in the proof of Lemma 2.1,

$$\begin{aligned}&d(\mathrm{e}^{\gamma\phi(X_t^N)}|X_t^N - X_s^N|^2) \\ &\quad\leq \mathrm{e}^{\gamma\phi(X_t^N)}(2(X_t^N - X_s^N) + \gamma|X_t^N - X_s^N|^2\nabla\phi(X_t^N))\sigma(X_t^N)\mathrm{d}W_t^N \\ &\qquad+ \mathrm{e}^{\gamma\phi(X_t^N)}(2(X_t^N - X_s^N) + \gamma|X_t^N - X_s^N|^2\nabla\phi(X_t^N))b(X_t^N)\mathrm{d}t.\end{aligned}$$

from which (18) follows with

$$Z_{u,s}^N = \mathrm{e}^{\gamma\phi(X_u^N)}(2(X_u^N - X_s^N) + |X_u^N - X_s^N|^2\gamma\nabla\phi(X_u^N))\sigma(X_u^N)$$

and

$$V_{u,s}^N = \mathrm{e}^{\gamma\phi(X_u^N)}(2(X_u^N - X_s^N) + |X_u^N - X_s^N|^2\gamma\nabla\phi(X_u^N)) \cdot b(X_u^N).$$

Since $\nabla\phi$, b, and σ are Lipschitz continuous functions on the bounded domain $\mathcal{O}$, it is clear how to choose the C in (17). □

The next lemma is a variant of Burkholder's inequality.

Lemma 3.4. *Let* $\alpha : [0,\infty) \longrightarrow \mathrm{Hom}(\mathbb{R}^r;\mathbb{R}^m)$ *be an* $\{\mathcal{F}_t : t \ge 0\}$*-progressively measurable function and let* $p \ge 1$*. Then there exists a constant* $C_p > 0$ *depending only on* p *such that*

$$E\left[\left|\int_s^t \alpha(\lfloor\tau\rfloor)\mathrm{d}W_\tau^N\right|^p\right] \le C_p E\left[\left(\int_s^t |\alpha(\lfloor\tau\rfloor)|^2 \mathrm{d}\tau\right)^{\frac{p}{2}}\right].$$

Proof. For simplicity of exposition, we will assume that s is an N-dyadic rational. The "N-dyadic remainder" on the left side of the interval $[s,t]$ is handled the same as that on the right side.

$$\begin{aligned}
E\left[\left|\int_s^t \alpha(\lfloor\tau\rfloor)\mathrm{d}W_\tau^N\right|^p\right] &\le C_p E\left[\left|\int_s^{\lfloor t\rfloor} \alpha(\lfloor\tau\rfloor)\mathrm{d}W_\tau^N\right|^p\right] + C_p E\left[\left|\int_{\lfloor t\rfloor}^t \alpha(\lfloor\tau\rfloor)\mathrm{d}W_\tau^N\right|^p\right] \\
&\le C_p E\left[\left|\int_s^{\lfloor t\rfloor} \alpha(\lfloor\tau\rfloor)\mathrm{d}W_\tau\right|^p\right] \\
&\quad + C_p 2^{Np}(t-\lfloor t\rfloor)^p E[|\alpha(\lfloor t\rfloor)|^p] E[|\Delta W_{\lfloor t\rfloor}^N|^p] \\
&\le C_p E\left[\left(\int_s^{\lfloor t\rfloor} |\alpha(\lfloor\tau\rfloor)|^2 \mathrm{d}\tau\right)^{\frac{p}{2}}\right] + C_p (t-\lfloor t\rfloor)^{\frac{p}{2}} E[|\alpha(\lfloor t\rfloor)|^p] \\
&\le C_p E\left[\left(\int_s^t |\alpha(\lfloor\tau\rfloor)|^2 \mathrm{d}\tau\right)^{\frac{p}{2}}\right]. \quad \square
\end{aligned}$$

We now prove (11) in the case that $t - s > 2^{-N}$ by induction on m. Taking into account the fact that ϕ is bounded, we can use (18) to derive the estimate

$$\begin{aligned}
\mathbb{E}[|X_t^N - X_s^N|^{2^{m+1}}] &\le C_m \mathbb{E}\left[\left(\int_s^t (Z_{u,s}^N - Z_{\lfloor u\rfloor\vee s,s}^N)\mathrm{d}W_u^N\right)^{2^m}\right] \\
&\quad + C_m \mathbb{E}\left[\left(\int_s^t Z_{\lfloor u\rfloor\vee s,s}^N \mathrm{d}W_u^N\right)^{2^m}\right] + C_m \mathbb{E}\left[\left(\int_s^t V_{u,s}^N \mathrm{d}u\right)^{2^m}\right] \qquad (19)
\end{aligned}$$

for some $C_m < \infty$. Because $V_{u,s}^N$ is bounded (see (17)), the third term is bounded by a constant times $(t-s)^{2^m}$. For the first term we have that, for some constants $C < \infty$,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_s^t (Z_{u,s}^N - Z_{\lfloor u\rfloor\vee s,s}^N)\mathrm{d}W_u^N\right)^{2^m}\right] &\le C\mathbb{E}\left[\left(\int_s^t |X_u^N - X_{\lfloor u\rfloor\vee s}^N| \mathrm{d}|W^N|_u\right)^{2^m}\right] \\
&\le C\mathbb{E}\left[\left(\int_s^t |W_u^N - W_{\lfloor u\rfloor\vee s}^N| \mathrm{d}|W^N|_u\right)^{2^m}\right]
\end{aligned}$$

$$+C\mathbb{E}\left[\left(\int_s^t (u-(\lfloor u\rfloor \vee s))\mathrm{d}|W^N|_u\right)^{2^m}\right]$$
$$\le C(t-s)^{2^m},$$

where the first inequality follows from (17), the second inequality from (13), and the third inequality from (14) and (15). Finally, for the second term we have that

$$\begin{aligned}\mathbb{E}\left[\left(\int_s^t Z^N_{\lfloor u\rfloor\vee s,s}\mathrm{d}W^N_u\right)^{2^m}\right] &\le C\mathbb{E}\left[\left(\int_s^t |Z^N_{\lfloor u\rfloor\vee s,s}|^2\mathrm{d}u\right)^{2^{m-1}}\right]\\ &\le C(t-s)^{(2^{m-1}-1)}\mathbb{E}\left[\int_s^t |Z^N_{\lfloor u\rfloor\vee s,s}|^{2^m}\mathrm{d}u\right]\\ &\le C(t-s)^{(2^{m-1}-1)}\mathbb{E}\left[\int_s^t |X^N_{\lfloor u\rfloor\vee s}-X^N_s|^{2^m}\mathrm{d}u\right]\\ &\le C(t-s)^{(2^{m-1}-1)}\mathbb{E}\left[\int_s^t ((\lfloor u\rfloor\vee s)-s)^{2^{m-1}}\mathrm{d}u\right]\\ &\le C(t-s)^{(2^{m-1}-1)}\mathbb{E}\left[\int_s^t (t-s)^{2^{m-1}}\mathrm{d}u\right]\le C(t-s)^{2^m},\end{aligned}$$

where the first inequality is an application of Lemma 3.4, the third inequality follows from (17), the fourth inequality is our induction hypothesis, and the fifth inequality follows from our assumption that $t-s>2^{-N}$. Hence we will be done once we show that (11) holds when $m=0$. But we can handle the base case by the same estimates as above, only now noting that the second term of (19) is 0 in this case.

Finally, we must prove (12). Since

$$\mathrm{d}X^N_t=\sigma(X^N_t)\mathrm{d}W^N_t+b(X^N_t)\mathrm{d}t+\mathrm{d}L^N_t$$

we have that

$$\mathbb{E}[|L^N_t-L^N_s|^{2^{m+1}}]\le C\mathbb{E}[|X^N_t-X^N_s|^{2^{m+1}}]+CE\left[\left(\int_s^t\sigma(X^N_u)\mathrm{d}W^N_u\right)^{2^{m+1}}\right]$$
$$+C\mathbb{E}\left[\left(\int_s^t b(X^N_u)\mathrm{d}u\right)^{2^{m+1}}\right].$$

We already know that the first term is bounded from above by $C(t-s)^{2^m}$. Moreover, because b is bounded and $0\le s<t\le T$, the third term is bounded above by a constant depending on T times $(t-s)^{2^m}$. For the second term we have that

$$\mathbb{E}\left[\left(\int_s^t\sigma(X^N_u)\mathrm{d}W^N_u\right)^{2^{m+1}}\right]\le C\mathbb{E}\left[\left(\int_s^t\sigma(X^N_u)-\sigma(X^N_{\lfloor u\rfloor})\mathrm{d}W^N_u\right)^{2^{m+1}}\right]$$
$$+C\mathbb{E}\left[\left(\int_s^t\sigma(X^N_{\lfloor u\rfloor})\mathrm{d}W^N_u\right)^{2^{m+1}}\right]$$

$$
\begin{aligned}
&\le C\mathbb{E}\left[\left(\int_s^t |X_u^N - X_{\lfloor u\rfloor}^N|\mathrm{d}|W^N|_u\right)^{2^{m+1}}\right] + C(t-s)^{2^m} \\
&\le C\left(\mathbb{E}\left[\left(\int_s^t |W_u^N - W_{\lfloor u\rfloor}^N|\mathrm{d}|W^N|_u\right)^{2^m}\right]\right. \\
&\quad \left. + \ \mathbb{E}\left[\left(\int_s^t (u-\lfloor u\rfloor)\mathrm{d}|W^N|_u\right)^{2^m}\right]\right) + C(t-s)^{2^m} \\
&\le C(t-s)^{2^{m-1}}
\end{aligned}
$$

where the second inequality follows from Lemma 3.4 and the fact that σ is bounded, the third inequality follows from (13), and the last inequality follows from (14) and (15). Putting these inequalities together we get (12).

Given a $\psi : [0,\infty) \longrightarrow \mathbb{R}^d$, $\beta \in (0,1]$, and $t > s > 0$, set

$$
\|\psi\|_{\beta,[s,t]} = \sup_{s\le u_1<u_2\le t} \frac{|\psi(u_2)-\psi(u_1)|}{(u_2-u_1)^\beta}.
$$

As an immediate consequence of the estimates in (10), (11), and (12) combined with Kolmogorov's Continuity Criterion (cf. Theorem 3.1.4 in [10]), we have the following theorem.

Theorem 3.5. *For each $\beta < \frac{1}{2}$, $p \in (1,\infty)$, and $T > 0$, there exists a $K_{\beta,p}(T) < \infty$ such that*

$$
\mathbb{P}(\|W^N\|_{\beta,[0,T]} \vee \|X^N\|_{\beta,[0,T]} \vee \|L^N\|_{\beta,[0,T]} \ge R) \le K_{\beta,p}(T)R^{-p} \quad \text{for } R > 0.
$$

3.2. Controlling the variation of $L_\cdot^N$

In general, the variation of a function cannot be controlled by its uniform norm. Thus, before we can apply the tightness result in the previous subsection to get the sort of result which we are seeking, we must give a separate argument which shows that the variation of $L_\cdot^N$ can be estimated in terms of its uniform norm. To be precise, Theorem 3.6 says that the variation of $L_\cdot^N \restriction [0,t]$ can be estimated in terms of the uniform norm of $L_\cdot^N \restriction [0,t]$ and the Hölder norm of $X_\cdot^N \restriction [0,t]$. Hence, since Theorem 3.5 provides control on the Hölder, and therefore the uniform, norms of the three processes $W_\cdot^N$, $X_\cdot^N$, and $L_\cdot^N$, our tightness result will be sufficient for our purposes (cf. Theorem 4.1 below).

In the following, and elsewhere, $\|\psi\|_{[t_1,t_2]} = \sup_{\tau\in[t_1,t_2]} |\psi(\tau)|$.

Theorem 3.6. *For all $0 \le s < t$,*

$$
|L^N|_t - |L^N|_s \le C((t-s)R^{-4}\|X^N\|^4_{\frac{1}{4},[s,t]} + 1)\|L^N\|_{[s,t]}, \tag{20}
$$

where R is the constant given in Part 3 of Definition 1.1.

Our proof follows the proof of Lemma 1.2 in [6].

Proof. Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ denote the open balls $B(x_1, 2R), \ldots, B(x_n, 2R)$ appearing in Part 3 of Definition 1.1, and choose an open set $\mathcal{O}_0$ such that $\bar{\mathcal{O}}_0 \subseteq \mathcal{O}$ and $\bar{\mathcal{O}} \subseteq \mathcal{O}_0 \cup \bigcup_{i=1}^n B(x_i, R)$.

Given $x \in \bar{\mathcal{O}}$, let $k(x)$ be the smallest $1 \le k \le n$ such that $x \in B(x_k, R)$, or otherwise let $k(x)$ be 0. Next, set $\zeta_0 = s$ and define ζ_m for $m \ge 1$ inductively so that

$$\zeta_{m+1} = t \wedge \inf\{\tau \ge \zeta_m : X_\tau^N \notin \mathcal{O}_{k(X_{\zeta_m}^N)}\}.$$

Consider the time interval $[\zeta_m, \zeta_{m+1}]$. If $\zeta_m < t$ and $k(X_{\zeta_m}^N) = 0$, then $L_\cdot^N \restriction [\zeta_m, \zeta_{m+1}]$ is constant and so $|L^N|_{\zeta_{m+1}} - |L^N|_{\zeta_m} = 0$. If $\zeta_m < t$ and $k_m \equiv k(X_{\zeta_m}^N) \ge 1$, then (cf. Part 3 of Definition 1.1)

$$(L_{\zeta_{m+1}}^N - L_{\zeta_m}^N) \cdot a_{k_m} = \int_{\zeta_m}^{\zeta_{m+1}} \nu(X_\tau^N) \cdot a_{k_m} \mathrm{d}|L^N|_\tau \ge \lambda(|L^N|_{\zeta_{m+1}} - |L^N|_{\zeta_m}).$$

Hence, in either case,

$$|L^N|_{\zeta_{m+1}} - |L^N|_{\zeta_m} \le C|L_{\zeta_{m+1}}^N - L_{\zeta_m}^N| \le C\|L^N\|_{[s,t]}.$$

At the same time, if $\zeta_{m+1} < t$ and $k(X_{\zeta_m}^N) \ge 1$, then $|X_{\zeta_{m+1}}^N - X_{\zeta_m}^N| \ge R$ and so

$$\frac{R}{(\zeta_{m+1} - \zeta_m)^{\frac{1}{4}}} \le \frac{|X_{\zeta_{m+1}}^N - X_{\zeta_m}^N|}{(\zeta_{m+1} - \zeta_m)^{\frac{1}{4}}} \le \|X^N\|_{\frac{1}{4},[0,t]}.$$

Thus if $\mathcal{M} = \sup\{m : \zeta_{m+1} < t\}$, then

$$\frac{\mathcal{M}}{2} \le 1 + |\{m : \zeta_{m+1} < t \text{ and } k(X_{\zeta_m}^N) \ge 1\}| \le 1 + \frac{(t-s)\|X^N\|_{\frac{1}{4},[s,t]}^4}{R^4},$$

which, in conjunction with the preceding, means that

$$\begin{aligned} |L^N|_t - |L^N|_s &\le \sum_{m=0}^{\mathcal{M}-1} (|L^N|_{\zeta_{m+1}} - |L^N|_{\zeta_m}) + (|L^N|_t - |L^N|_{\zeta_{\mathcal{M}}}) \\ &\le (C\mathcal{M} + 2)\|L^N\|_{[s,t]} \le C[(t-s)R^{-4}\|X^n\|_{\frac{1}{4},[s,t]}^4 + 1]\|L^N\|_{[s,t]}. \quad \square \end{aligned}$$

4. Associated martingale and submartingale problems

We know that the sequence of measures $\{\mathbb{P}^N : N \ge 0\}$ is on (X, L, W)-pathspace. Our eventual goal is to show that this sequence converges. Equivalently, we want to show that all limit points are the same. In this section, we will show that every limit solves martingale and submartingale problems, and in the next section we will show that this fact is sufficient to check that convergence takes place.

Up until now, we have needed only the assumptions that $\mathcal{O}$ is bounded and admissible, and σ and b are Lipschitz continuous. However, starting now, we will be assuming that $\sigma \in C^2(\bar{\mathcal{O}}; \mathrm{Hom}(\mathbb{R}^r; \mathbb{R}^d))$. In addition, it will be convenient to make a change in our notation. Instead of writing the equation which determines (X_t^N, L_t^N) (pathwise) as

$$\mathrm{d}X_t^N = \sigma(X_t^N)\mathrm{d}W_t^N + b(X_t^N)\mathrm{d}t + \mathrm{d}L_t^N, \qquad X_0^N = x_0, \tag{21}$$

we will use the equivalent expression

$$\mathrm{d}X_t^N = \sum_{i=1}^r V_i(X_t^N) d(W_i^N)_t + V_0(X_t^N)\mathrm{d}t + \mathrm{d}L_t^N, \qquad X_0^N = x_0 \tag{22}$$

where V_i is the ith column of the matrix σ and $V_0 = b$. At the same time, we introduce the vector fields $\tilde{V}_i : \bar{\mathcal{O}} \longrightarrow \mathbb{R}^d \times \mathbb{R}^r$ given by $\tilde{V}_i = \binom{V_i}{e_i}$ for $1 \le i \le r$ and $\tilde{V}_0 = \binom{V_0}{0}$, where $\{e_1, \dots, e_r\}$ is the standard, orthonormal basis in $\mathbb{R}^r$. Then, $\mathbb{P}^N$-almost surely,

$$\mathrm{d}Y_t = \sum_{i=1}^r \tilde{V}_i(X_t) d(W_i)_t + \tilde{V}_0(X_t)\mathrm{d}t, \qquad Y_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \tag{23}$$

where $Y_t = \binom{X_t - L_t}{W_t}$. In keeping with this notation, we use D_{V_i} and $D_{\tilde{V}_i}$ to denote the directional derivative operators on $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}^r$ determined, respectively, by V_i and $\tilde{V}_i$. Finally, for $\xi \in \mathbb{R}^d$, T_ξ will denote the translation operator on $C(\mathbb{R}^d \times \mathbb{R}^r; \mathbb{R})$ given by $T_\xi \varphi(x, y) = \varphi(x - \xi, y)$.

Theorem 4.1. *Let $\mathbb{P}$ be any limit point of the sequence $\{\mathbb{P}^N : N \ge 0\}$. Then for all $h \in C_{\mathrm{b}}^2(\mathbb{R}^d \times \mathbb{R}^r; \mathbb{R})$,*

$$h(Y_t) - \int_0^t \left(\frac{1}{2} \sum_{i=1}^r [D_{\tilde{V}_i}^2 T_{L_s} h](X_s, W_s) + [D_{\tilde{V}_0} T_{L_s} h](X_s, W_s) \right) \mathrm{d}s \tag{24}$$

is a $\mathbb{P}$-martingale relative to the filtration $\{\mathcal{B}_t : t \ge 0\}$ generated by the paths in the (X, L, W)-pathspace. Also, for all $f \in C_{\mathrm{b}}^2(\mathbb{R}^d; \mathbb{R})$ satisfying $\frac{\partial f}{\partial \nu}(x) \ge 0$ for every $x \in \partial \mathcal{O}$ and $\nu \in \nu(x)$,

$$f(X_t) - f(x_0) - \int_0^t \left(\frac{1}{2} \sum_{i=1}^r D_{V_i}^2 f(X_s) + D_{V_0} f(X_s) \right) \mathrm{d}s \tag{25}$$

is a $\mathbb{P}$-submartingale relative to the filtration $\{\mathcal{B}_t : t \ge 0\}$.

We will begin with the proof of the martingale property for (24), and, without loss of generality, we will do so under the assumption that h is smooth and compactly supported. What we need to show is that for any limit point $\mathbb{P}$, $0 \le s < t$ and bounded, continuous, $\mathcal{B}_s$-measurable $F : C([0, \infty); \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^r) \longrightarrow [0, \infty)$,

$$\mathbb{E}^{\mathbb{P}} \left[\left(h(Y_t) - h(Y_s) - \int_s^t \tilde{\mathcal{L}} h(u) \mathrm{d}u \right) F \right] = 0 \tag{26}$$

where we have used $\tilde{\mathcal{L}} h(u)$ to denote the integrand in (24), and clearly it suffices to check this when s and t are M-dyadic rationals for some $M \in \mathbb{N}$. Thus, it suffices to show that

$$\mathbb{E}^{\mathbb{P}^N} \left[\left(h(Y_t) - h(Y_s) - \int_s^t \tilde{\mathcal{L}} h(u) \mathrm{d}u \right) F \right] \to 0 \tag{27}$$

for M-dyadic s and t and bounded, $\mathcal{B}_s$-measurable $F \in C([0, \infty); \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^r)$.

For $N \ge M$, write

$$h(Y_t) - h(Y_s) = \sum_{m=2^N s}^{2^N t - 1} h(Y_{(m+1)2^{-N}}) - h(Y_{m 2^{-N}}),$$

and, for each term in the sum, use (23) to see that, $\mathbb{P}^N$-almost surely,

$$h(Y_{(m+1)2^{-N}}) - h(Y_{m2^{-N}}) = \int_{m2^{-N}}^{(m+1)2^{-N}} \sum_{i=1}^{r} [D_{\tilde{V}_i} T_{L_\tau} h](X_\tau, W_\tau)(\dot{W}_{i,m})\mathrm{d}\tau + \int_{m2^{-N}}^{(m+1)2^{-N}} [D_{\tilde{V}_0} T_{L_\tau} h](X_\tau, W_\tau)\mathrm{d}\tau,$$

where $\dot{W}_{i,m} \equiv 2^N (W_i((m+1)2^{-N}) - W_i(m2^{-N}))$.

Since

$$\sum_{m=2^N s}^{2^N t} \int_{m2^{-N}}^{(m+1)2^{-N}} [D_{\tilde{V}_0} T_{L_\tau} h](X_\tau, W_\tau)\mathrm{d}\tau = \int_s^t [D_{\tilde{V}_0} T_{L_\tau} h](X_\tau, W_\tau)\mathrm{d}\tau,$$

the second term on the right causes no problem.

To handle the first term, note that

$$[D_{\tilde{V}_i} T_{L_\tau} h](X_\tau, W_\tau) = [D_{\tilde{V}_i} T_{L_{m2^{-N}}} h](X_\tau, W_\tau) - \sum_{k=1}^{d} \int_{m2^{-N}}^{\tau} [D_{\tilde{V}_i} T_{L_\sigma} \partial_{x_k} h](X_\tau, W_\tau)\, \mathrm{d}L_\sigma.$$

Since the second term on the right is dominated by a constant times $|L|_{(m+1)2^{-N}} - |L|_{m2^{-N}}$, we see that

$$\mathbb{E}^{\mathbb{P}^N}\left[\left|\sum_{m=2^N s}^{2^N t}\left(\int_{m2^{-N}}^{(m+1)2^{-N}} ([D_{\tilde{V}_i} T_{L_\tau} h] - [D_{\tilde{V}_i} T_{L_{m2^{-N}}} h])(X_\tau, W_\tau)\, \mathrm{d}\tau\right)\dot{W}_{i,m}\right|\right] \le C 2^{-\frac{N}{4}} \mathbb{E}^{\mathbb{P}^N}[|L|_t \|W\|_{\frac{1}{4},[0,t]}] \longrightarrow 0$$

as $N \to \infty$.

Next, use (22) to see that

$$[D_{\tilde{V}_i} T_{L_{m2^{-N}}} h](X_\tau, W_\tau) = [D_{\tilde{V}_i} T_{L_{m2^{-N}}} h](X_{m2^{-N}}, W_{m2^{-N}}) + \int_{m2^{-N}}^{\tau} [D_{\tilde{V}_0} D_{\tilde{V}_i} T_{L_{m2^{-N}}} h](X_\sigma, W_\sigma)\, \mathrm{d}\sigma + \sum_{k=1}^{d} [\partial_{x_k} D_{\tilde{V}_i} T_{L_{m2^{-N}}} h](X_\sigma, W_\sigma)\, \mathrm{d}L_\sigma + \sum_{j=1}^{r} \dot{W}_{j,m} \int_{m2^{-N}}^{\tau} [D_{\tilde{V}_j} D_{\tilde{V}_i} T_{L_{m2^{-N}}} h](X_\sigma, W_\sigma)\, \mathrm{d}\sigma.$$

Since the conditional $\mathbb{P}^N$-expected value of

$$\dot{W}_{i,m} [D_{\tilde{V}_i} T_{L_{m2^{-N}}} h](X_{m2^{-N}}, W_{m2^{-N}})$$

given $\mathcal{B}_s$ is zero, the first term on the right does not appear in the computation. Moreover, After integrating the second two terms over $[m2^{-N}, (m+1)2^{-N}]$, multiplying by $\dot{W}_{i,m}$, and summing from $m = 2^N$ to $m = 2^N t$, one can easily check that the absolute values of the resulting quantities have $\mathbb{P}^N$-expected values which tend to 0 as $N \to \infty$.

Finally, again applying (22), one finds that

$$\int_{m2^{-N}}^{\tau}[D_{\tilde{V}_j}D_{\tilde{V}_i}T_{L_{m2^{-N}}}h](X_\sigma, W_\sigma)\,\mathrm{d}\sigma$$

can be replaced by

$$(\tau - m2^{-N})[D_{\tilde{V}_j}D_{\tilde{V}_i}T_{L_{m2^{-N}}}h](X_{m2^{-N}}, W_{m2^{-N}})$$

plus terms which make no contributions in the limit as $N \to \infty$. Hence, we are left with quantities of the form

$$\sum_{m=2^N s}^{2^N t} 2^{-2N-1}\dot{W}_{j,m}\dot{W}_{i,m}[D_{\tilde{V}_j}D_{\tilde{V}_i}T_{L_{m2^{-N}}}h](X_{m2^{-N}}, W_{m2^{-N}}).$$

Since the $\mathbb{P}^N$-conditional expected value of $2^{-2N}\dot{W}_{j,m}\dot{W}_{i,m}$ is $2^{-N}\delta_{i,j}$,

$$\mathbb{E}^{\mathbb{P}^N}\left[\left(\sum_{m=2^N s}^{2^N t} 2^{-2N-1}\dot{W}_{j,m}\dot{W}_{i,m}[D_{\tilde{V}_j}D_{\tilde{V}_i}T_{L_{m2^{-N}}}h](X_{m2^{-N}}, W_{m2^{-N}})\right)F\right]$$
$$= \frac{\delta_{i,j}}{2}\mathbb{E}^{\mathbb{P}^N}\left[2^{-N}\left(\sum_{m=2^N s}^{(m+1)2^N}[D_{\tilde{V}_j}D_{\tilde{V}_i}T_{L_{m2^{-N}}}h](X_{m2^{-N}}, W_{m2^{-N}})\right)F\right],$$

which, as $N \to \infty$, has that same limit as

$$\frac{\delta_{i,j}}{2}\mathbb{E}^{\mathbb{P}^N}\left[\left(\int_s^t [D_{\tilde{V}_j}D_{\tilde{V}_i}T_{L_s}h](X_s, W_s)\,\mathrm{d}s\right)F\right].$$

The proof of (25) is similar, but easier, and so we will skip the details. The only difference is that when we apply (22) to the difference $f(X_{(m+1)2^{-N}}) - f(X_{m2^{-N}})$, we throw away the $\mathrm{d}L_\tau$ integral since, under our hypotheses, it is non-negative.

5. Convergence

In this section, we complete our program of proving $\{\mathbb{P}^N : N \geq 0\}$ converges to the distribution of an appropriate Stratonovich reflected SDE. By the uniqueness result of Lions and Sznitman (Theorem 3.1 of [6]) and the tightness which we proved in 3.1, the convergence will follow as soon as we show that every limit $\mathbb{P}$ is the distribution of that reflected SDE.

Let $\mathbb{P}$ be any limit of $\{\mathbb{P}^N : N \geq 0\}$. By Theorem 4.1, we know that, for all $h \in C_{\mathrm{b}}^2(\mathbb{R}^d \times \mathbb{R}^r; \mathbb{R})$,

$$h(X_t - L_t, W_t) - h(x_0, 0) - \int_0^t \tilde{\mathcal{L}}h(s)\mathrm{d}s \text{ is a } \mathbb{P} \text{ martingale,} \tag{28}$$

relative to $\{\mathcal{B}_t : t \geq 0\}$, where

$$\tilde{\mathcal{L}}h(s) = \frac{1}{2}\sum_{i=1}^{r}[D_{\tilde{V}_i}^2 T_{L_s}h](X_s, W_s) + [D_{\tilde{V}_0}T_{L_s}h](X_s, W_s).$$

Using elementary stochastic calculus (cf. Theorem 8.1.1 in [11]), it follows from (28) that $\{W_t : t \geq 0\}$ is a $\mathbb{P}$-Brownian motion relative to $\{\mathcal{B}_t : t \geq 0\}$ and that, $\mathbb{P}$-almost surely,

$$X_t - x_0 - \int_0^t \left(\frac{1}{2} \sum_{i=1}^r [D_{V_i} V_i](X_s) + V_0(X_s) \right) \mathrm{d}s - L_t = \int_0^t \sigma(X_s) \mathrm{d}W_s,$$

which can be rewritten in Stratonovich form as

$$X_t - x_0 = \sum_{i=1}^r \int_0^t V_i(X_s) \circ \mathrm{d}W_s + \int_0^t V_0(X_s)\, \mathrm{d}s + L_t. \tag{29}$$

Thus, the only remaining question is whether $\{L_t : t \geq 0\}$ has the required properties. That is, whether, $\mathbb{P}$-almost surely, $|L|_t < \infty$ and $\int_0^t \mathbf{1}_{\mathcal{O}}(X_s)\, \mathrm{d}|L|_s = 0$ for all $t \geq 0$, and $\frac{\mathrm{d}L_t}{\mathrm{d}|L|_t} \in \nu(X_t)$ a.e.

Since the local variation norm is a lower semicontinuous function of local uniform convergence, Theorem 3.6 tells us that, $\mathbb{P}$-almost surely, $L_\cdot$ has locally bounded variation. In fact, by combining that theorem with the estimates in Theorem 3.5, one sees that, for all $t \geq 0$, $|L|_t$ has finite $\mathbb{P}$-moments of all orders.

In order to prove the other properties of $L_\cdot$ we will use the second part of Theorem 4.1, which says that for every $f \in C_b^2(\mathbb{R}^d; \mathbb{R})$ satisfying $\frac{\partial f}{\partial \nu}(x) \geq 0$ for all $x \in \partial\mathcal{O}$ and $\nu \in \nu(x)$,

$$f(X_t) - \int_0^t \mathcal{L} f(X_s) \mathrm{d}s \quad \text{is a } \mathbb{P} \text{ submartingale} \tag{30}$$

relative to $\{\mathcal{B}_t : t \geq 0\}$, where

$$\mathcal{L} f(x) = \frac{1}{2} \sum_{i=1}^r D_{V_i}^2 f(x) + D_{V_0} f(x).$$

Now compare this to what one gets by applying Itô's formula to (29). Namely, his formula says that if $\xi_t^f = \int_0^t \nabla f(X_s) \cdot \mathrm{d}L_s$ then

$$f(X_t) - \int_0^t \mathcal{L} f(X_s) - \xi_t^f \quad \text{is a } \mathbb{P}\text{-martingale.}$$

Thus, $\xi_\cdot^f$ is $\mathbb{P}$-almost surely non-decreasing. Starting from this observation and using the arguments in Lemmas 2.3 and 2.5 of [12], one can prove the following lemma.

Lemma 5.1. *For $f \in C_b^2(\mathbb{R}^d; \mathbb{R})$, define $\xi_\cdot^f$ as above. Then, $\mathbb{P}$-almost surely, $\int_0^\infty \mathbf{1}_{\mathcal{O}}(X_s)\, \mathrm{d}|\xi^f|_s = 0$. Moreover, if $\frac{\partial f}{\partial \nu}(x) \geq 0$ for all x in an open set U and all $\nu \in \nu(x)$, then, $\mathbb{P}$-almost surely, $t \rightsquigarrow \int_0^t \mathbf{1}_U(X_s)\, \mathrm{d}\xi_s^f$ is non-decreasing.*

Because $e_i \cdot L_\cdot = \xi_\cdot^{x_i}$, it is obvious from the first part of Lemma 5.1 that $\int_0^\infty \mathbf{1}_{\mathcal{O}}(X_s)\, \mathrm{d}|L|_s = 0$ $\mathbb{P}$-almost surely, and so all that we have to do is to show that, $\mathbb{P}$-almost surely, $\frac{\mathrm{d}L_t}{\mathrm{d}|L|_t} \in \nu(X_t)$ a.e. To this end, let ϕ be the function in Part 2 of Definition 1.1, and define

$$a^f(x) = \inf_{\nu \in \nu(x)} \frac{\frac{\partial f}{\partial \nu}(x)}{\frac{\partial \phi}{\partial \nu}(x)} \quad \text{and} \quad b^f(x) = \sup_{\nu \in \nu(x)} \frac{\frac{\partial f}{\partial \nu}(x)}{\frac{\partial \phi}{\partial \nu}(x)}$$

for $x \in \partial\mathcal{O}$.

Lemma 5.2. *If* $\{x_n : n \geq 1\} \subseteq \partial\mathcal{O}$, $\nu_n \in \nu(x_n)$ *for each* $n \geq 1$, *and* $(x_n, \nu_n) \longrightarrow (x, \nu)$ *in* $\partial\mathcal{O} \times \mathbb{S}^{N-1}$, *then* $\nu \in \nu(x)$. *In particular, for each* $f \in C^2_{\mathrm{b}}(\mathbb{R}^d; \mathbb{R})$, a^f *is lower semicontinuous and* b^f *is upper semicontinuous on* $\partial\mathcal{O}$. *Furthermore, if* $(x, \ell) \in \partial\mathcal{O} \times \mathbb{S}^{N-1}$ *and there exists a* $\beta \geq 0$ *such that* $\nabla f(x) \cdot \ell \geq \beta a^f(x)$ *for a set* S *of* $f \in C^2_{\mathrm{b}}(\mathbb{R}^d; \mathbb{R})$ *with the property that* $\{\nabla f(x) : f \in S\}$ *is dense in* $\mathbb{R}^d$, *then* $\ell \in \nu(x)$.

Proof. The initial assertion is an easy consequence of Parts 1 and 2 of Definition 1.1. Next, suppose that $x_n \longrightarrow x$ in $\partial\mathcal{O}$. Because, by the first assertion, $\nu(y)$ is compact for each $y \in \partial\mathcal{O}$, for each $n \geq 1$ there is a $\nu_n \in \nu(x_n)$ such that $a_f(x_n) = \frac{\nabla f(x_n)\cdot\nu_n}{\nabla\phi(x_n)\cdot\nu_n}$. Now choose a subsequence $\{x_{n_m} : m \geq 1\}$ so that $\underline{\lim}_{n\to\infty} a^f(x_n) = \lim_{m\to\infty} a^f(x_{n_m})$ and $\nu_{n_m} \longrightarrow \nu$ in $\mathbb{S}^{N-1}$. Then $\nu \in \nu(x)$ and so

$$a^f(x) \leq \frac{\nabla f(x)\cdot\nu}{\nabla\phi(x)\cdot\nu} \leq \liminf_{n\to\infty} a^f(x_n).$$

The same argument shows that b^f is upper semicontinuous.

Next, let (x, ℓ) and β be as in the final assertion. Then, by Part 2 of Definition 1.1. By taking f to be linear in a neighborhood of $\bar{\mathcal{O}}$, one sees that for every $\upsilon \in \mathbb{R}^d$ there exists a $\nu \in \nu(x)$ such that $\upsilon \cdot \ell \geq \beta \frac{\upsilon\cdot\nu}{\nabla\phi(x)\cdot\nu}$. Hence, for each $x' \in \mathcal{O}$ there is a $\nu \in \nu(x)$ such that

$$(x' - x)\cdot\ell \geq \beta\frac{(x'-x)\cdot\nu}{\nabla\phi(x)\cdot\nu} \geq -\frac{\beta C_0}{\alpha}|x'-x|^2,$$

which, by (3), means that $\ell \in \nu(x)$. □

Lemma 5.3. *For each* $f \in C^2_{\mathrm{b}}(\mathbb{R}^d; \mathbb{R})$, $\mathbb{P}$*-almost surely* $\mathrm{d}\xi^f_{\cdot}$ *is absolutely continuous with respect to* $\mathrm{d}\xi^\phi_{\cdot}$ *and* $a^f(X_t) \leq \frac{\mathrm{d}\xi^f_{\cdot}}{\mathrm{d}\xi^\phi_{\cdot}}(t) \leq b^f(X_t)$ *for* $\mathrm{d}\xi^\phi_{\cdot}$*-almost every* $t \geq 0$.

Proof. First observe that $f \rightsquigarrow \xi^f_{\cdot}$ is linear. Now choose $\lambda > 0$ so that $\nabla(\lambda\phi - f)(x)\cdot\nu \geq 0$ for all $x \in \partial\mathcal{O}$ and $\nu \in \nu(x)$. Then, $\xi^{\lambda\phi-f}_{\cdot} = \lambda\xi^\phi_{\cdot} - \xi^f_{\cdot}$ is $\mathbb{P}$-almost surely non-decreasing, which proves that $\mathrm{d}\xi^f_{\cdot} \ll \mathrm{d}\xi^\phi_{\cdot}$ and that $\frac{\mathrm{d}\xi^f_{\cdot}}{\mathrm{d}\xi^\phi_{\cdot}} \leq \lambda$ $\mathbb{P}$-almost surely.

The proof that, $\mathbb{P}$-almost surely, $\alpha(t) \equiv \frac{\mathrm{d}\xi^f_{\cdot}}{\mathrm{d}\xi^\phi_{\cdot}}(t)$ lies between $a^f(X_t)$ and $b^f(X_t)$ for $\mathrm{d}\xi^\phi_{\cdot}$-almost every $t \geq 0$ is a simple localization of the preceding. For example, to prove the lower bound, use the lower semicontinuity of a^f to choose, for each $n \geq 1$, a finite cover of $\partial\mathcal{O}$ by open balls $B(x_{k,n}, r_n)$, $1 \leq k \leq k_n$ such that $x_{k,n} \in \partial\mathcal{O}$, $r_n \leq \frac{1}{n}$, and $a^f(y) \geq a^f(x_{k,n}) - \frac{1}{n}$ for all $y \in B(x_{k,n}, r_n) \cap \partial\mathcal{O}$. Then

$$\frac{\partial f}{\partial\nu}(y) \geq \left(a^f(x_{k,n}) - \frac{1}{n}\right)\frac{\partial\phi}{\partial\nu}(y) \quad \text{for all } 1 \leq k \leq k_n,\ y \in B(x_{k,n}, r_n),\ \text{and } \nu \in \nu(y).$$

Now let μ be the Borel measure on $C([0,\infty); \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^r) \times [0,\infty)$ determined by

$$\mu(\Gamma \times [a,b]) = \mathbb{E}^{\mathbb{P}}[\xi^\phi(b) - \xi^\phi(a),\ \Gamma]$$

for all Borel subsets Γ of $C([0,\infty); \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^r)$ and all $a < b$. Then, by Lemma 5.1, we can find a Borel measurable set $A \subseteq C([0,\infty); \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^r) \times [0,\infty)$ whose complement has μ-measure 0 and on which both

$$X_{\cdot} \in \partial\mathcal{O} \quad \text{and} \quad \mathbf{1}_{B(x_{k,n},r_n)}(X_{\cdot})\left(a^f(x_{k,n}) - \frac{1}{n}\right) \leq \mathbf{1}_{B(x_{k,n},r_n)}(X_{\cdot})\frac{\mathrm{d}\xi^f_{\cdot}}{\mathrm{d}\xi^\phi_{\cdot}}$$

hold for all $n \geq 1$ and $1 \leq k \leq k_n$. Hence, again by the lower semicontinuity of a^f, we see that $\frac{d\xi^f_\cdot}{d\xi^\phi_\cdot} \geq a(X_\cdot)$. The proof of the upper bound is the same. □

Theorem 5.4. *Let $\mathbb{P}^N$ be the distribution of (X^N, L^N) under Wiener measure. Then $\{\mathbb{P}^N : N \geq 0\}$ converges to the distribution $\mathbb{P}$ of the solution to the reflected stochastic differential equation (29).*

Proof. As we said earlier, everything comes down to showing that if $\mathbb{P}$ is a limit of $\{\mathbb{P}^N : N \geq 0\}$ then, $\mathbb{P}$-almost surely $\ell_\cdot \equiv \frac{dL_\cdot}{d|L|_\cdot} \in \nu(X_\cdot)$ $d|L|_\cdot$-almost everywhere. Thus, because, without loss of generality, we may assume that $|\ell_\cdot| \equiv 1$, the second part of Lemma 5.2 says that it suffices for us to show that, $\mathbb{P}$-almost surely, there exist a $\beta_\cdot \geq 0$ such that $\nabla f(X_\cdot) \cdot \ell_\cdot \geq \beta_\cdot a^f(X_\cdot)$ $d|L|_\cdot$-a.e. for sufficiently many f's. To this end, first note that, since $\xi^\phi_\cdot$ is $\mathbb{P}$-almost surely non-decreasing, $\beta_\cdot \equiv \nabla\phi(X_\cdot) \cdot \ell_\cdot \geq 0$ $d|L|_\cdot$-a.e. $\mathbb{P}$-almost surely. Second, because $L_\cdot = \sum_{i=1}^{d} \xi^{x_i}_\cdot$ $\mathbb{P}$-almost surely, we know that, $\mathbb{P}$-almost surely, $d|L|_\cdot \ll d\xi^\phi_\cdot$ and that, for each $f \in C^2_b(\mathbb{R}^d; \mathbb{R})$,

$$\nabla f(X_\cdot) \cdot \ell_\cdot = \frac{d\xi^f_\cdot}{d|L|_\cdot} = \frac{d\xi^f_\cdot}{d\xi^\phi_\cdot} \nabla\phi(X_\cdot) \cdot \ell_\cdot \geq \beta_\cdot a^f(X_\cdot) d|L|_\cdot\text{-a.e.} \tag{$*$}$$

Finally, let D be a countable, dense subset of $\mathbb{R}^d$, and for each $v \in D$ choose $f_v \in C^2_b(\mathbb{R}^d; \mathbb{R})$ so that $f_v(x) = v \cdot x$ in a neighborhood of $\bar{\mathcal{O}}$. Then, $\mathbb{P}$-almost surely, $(*)$ holds simultaneously with $f = f_v$ for every $v \in D$. □

Remark 5.5. In our derivation of Theorem 5.4 we used (30) to show that $L_\cdot$ has the required properties. However, using the ideas in Lemma 1.3 of [6], we could have based our proof on the fact that the approximating $L^N_\cdot$'s had these properties. Our choice of proof was dictated by two considerations. First, it seemed to us to be the simpler one. Second, and more important, it brings up an interesting question. Namely, does (30) by itself determine $\mathbb{P}$? In [12] it was shown that (30) determines $\mathbb{P}$ when $\mathcal{O}$ has a smooth boundary and $\mathcal{L}$ is strictly elliptic, even if the coefficients are not smooth. Thus, the question is whether the same result holds when $\mathcal{O}$ is only admissible and the coefficients of $\mathcal{L}$ are smooth but may be degenerate.

6. Observations and applications

It should be noticed that although the approximating $L^N_\cdot$'s as well as limit $L_\cdot$ have locally bounded variation, we cannot replace our (X, L, W)-pathspace with one in which the middle component is the space of continuous paths of locally bounded variation. The reason is that although $L^N_\cdot$ will be absolutely continuous, $L_\cdot$ will not. Indeed, consider reflected Brownian motion on the halfline $[0, \infty)$. In this case $L^N_t = \sup_{0 \leq s \leq t}[-W^N_s]$ is piecewise constant and therefore absolutely continuous. On the other hand, $L_t = \sup_{0 \leq s \leq t}[-W_s]$, which is the local time at 0 of $W_\cdot$ and as such is singular.

The main application of our result that we consider is the following. Suppose that for each N, the paths X^N_t satisfy a certain geometric property almost surely and the set S of paths which satisfy this geometric property is closed in $C([0, \infty); \mathbb{R}^d)$. It then follows that the paths of X_t also satisfy this geometric property almost surely since

$$\mathbb{P}(S) \geq \limsup_{N \to \infty} \mathbb{P}^N(S) = 1 \tag{31}$$

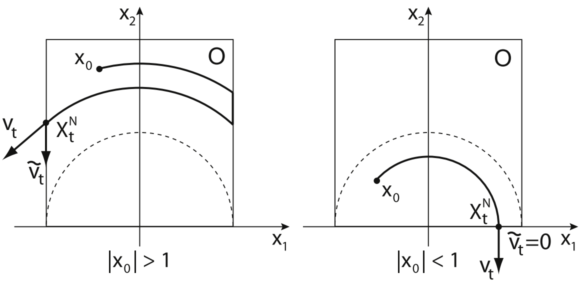

Fig. 1. The geometric behavior of the approximating process depends on the starting location.

where, abusing notation, we use $\mathbb{P}^N$ and $\mathbb{P}$ to denote the marginal distributions of $\mathbb{P}^N$ and $\mathbb{P}$ on X-pathspace. That is, $\mathbb{P}^N(A) = \mathbb{P}^N(A \times C([0,\infty);\mathbb{R}^d) \times C([0,\infty);\mathbb{R}^r))$ and $\mathbb{P}(A) = \mathbb{P}(A \times C([0,\infty);\mathbb{R}^d) \times C([0,\infty);\mathbb{R}^r))$. We conclude with several examples of the sort of application which we have in mind.

Example 6.1. In $\mathbb{R}^2$, let $\mathcal{O}$ be the rectangle $[-1,1] \times [0,2]$. Fix $x_0 \in \bar{\mathcal{O}}$ and consider the Stratonovich reflected SDE

$$\mathrm{d}X_t = \sigma(X_t) \circ \mathrm{d}W_t + \mathrm{d}L_t, \qquad X_0 = x_0,$$

where $\sigma(x) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$. Then

$$\text{if } |x_0| > 1, \qquad |X_t| \le |x_0| \quad \text{for } t > 0 \ \mathbb{P}\text{-a.s.} \tag{32}$$

and

$$\text{if } |x_0| < 1, \qquad |X_t| = |x_0| \quad \text{for } t > 0 \ \mathbb{P}\text{-a.s.} \tag{33}$$

Proof. In view of (31), it suffices to prove that (32) and (33) hold $\mathbb{P}^N$-a.s. The distribution of $X_\cdot$ under $\mathbb{P}^N$ is, in view of Theorem 2.3, the same as the distribution of $X_\cdot^N$ under $\mathbb{W}$, where $X_\cdot^N$ solves the ODE

$$\dot{X}_t^N = \mathrm{proj}_{T_{\bar{\mathcal{O}}(X_t^N)}}(\sigma(X_t^N)\dot{W}_t^N), \qquad X_0^N = x_0.$$

It is easy to check that $\forall x \in \bar{\mathcal{O}}, a \in \mathbb{R}$, $x \cdot \mathrm{proj}_{T_{\bar{\mathcal{O}}(x)}}(\sigma(x)a)$ is non-negative or non-positive according as $|x| \ge 1$ or $|x| \le 1$. Hence, because, for each W_t, $\frac{\mathrm{d}}{\mathrm{d}t}(|X_t^N|^2) = 2X_t^N \cdot \mathrm{proj}_{T_{\bar{\mathcal{O}}(X_t^N)}}(\sigma(X_t^N)\dot{W}_t^N)$ $\mathrm{d}t$-a.e., (32) and (33) for $X_\cdot^N$ are obvious. Fig. 1 shows a sample path of X_t^N under $\mathbb{W}$ (to save space, we denote the "intended velocity" $\sigma(X_t^N)\dot{W}_t^N$ by v_t and the "actual velocity" $\mathrm{proj}_{T_{\bar{\mathcal{O}}(X_t^N)}}(\sigma(X_t^N)\dot{W}_t^N)$ by $\tilde{v}_t$). □

We next consider coupled reflected Brownian motion, for which we will need the following lemmas.

Lemma 6.2. *Suppose $\mathcal{O}$ is bounded and admissible. Then $\mathcal{O} \times \mathcal{O}$ is bounded and admissible as well. Furthermore, for each $(x, y) \in \partial(\mathcal{O} \times \mathcal{O})$, the set of normal vectors $\nu(x, y)$ defined by*

(2) *has the representation*

$$\nu(x,y) = \left\{ \begin{pmatrix} a_1\nu_x \\ a_2\nu_y \end{pmatrix} : \nu_x \in \nu(x), \nu_y \in \nu(y), a_1^2 + a_2^2 = 1, a_1, a_2 > 0 \right\}, \quad \text{when } (x,y) \in \partial\mathcal{O} \times \partial\mathcal{O}, \tag{34}$$

$$\nu(x,y) = \left\{ \begin{pmatrix} \nu_x \\ 0 \end{pmatrix} : \nu_x \in \nu(x) \right\}, \quad \text{when } (x,y) \in \partial\mathcal{O} \times \mathcal{O},$$

and

$$\nu(x,y) = \left\{ \begin{pmatrix} 0 \\ \nu_y \end{pmatrix} : \nu_y \in \nu(y) \right\}, \quad \text{when } (x,y) \in \mathcal{O} \times \partial\mathcal{O}.$$

Proof. The representation formulas are a straightforward consequence of the definition of inward pointing unit proximal normal vectors in (2). That $\mathcal{O} \times \mathcal{O}$ satisfies Part 1 of Definition 1.1 follows from the representation formulas and the fact that $\mathcal{O}$ satisfies Part 1 of Definition 1.1.

We next show that $\mathcal{O} \times \mathcal{O}$ satisfies Part 2 of Definition 1.1. Since $\mathcal{O}$ is bounded, ϕ is bounded in $\mathcal{O}$ and so after adding a constant to ϕ if necessary, we may assume that $\phi \geq 1$ in $\bar{\mathcal{O}}$.

Let $\Phi(x,y) \equiv \phi(x)\phi(y)$. Then for all $(x,y) \in \partial(\mathcal{O} \times \mathcal{O})$, $\nu \in \nu(x,y)$, we have, by our representation formulas, that

$$\begin{aligned} \nabla\Phi(x,y) \cdot \nu &= a_1\phi(y)\nabla\phi(x) \cdot \nu_x + a_2\phi(x)\nabla\phi(y) \cdot \nu_y \\ &\geq a_1\phi(y)\alpha + a_2\phi(x)\alpha \geq \alpha(a_1 + a_2) \geq \alpha \end{aligned}$$

(where $(a_1, a_2) = (1, 0)$ and $(0, 1)$ for the cases $(x,y) \in (\partial\mathcal{O} \times \mathcal{O}) \cup (\mathcal{O} \times \partial\mathcal{O})$), and so Part 2 holds with the function $\Phi(x,y)$. Finally, as $\mathcal{O} \times \mathcal{O}$ is bounded, Part 3 follows immediately from Part 2. □

Lemma 6.3. *Let $\mathcal{O}$ be bounded and admissible. Then for $(x,y) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}}$,*

$$T_{\bar{\mathcal{O}} \times \bar{\mathcal{O}}}(x,y) = T_{\bar{\mathcal{O}}}(x) \times T_{\bar{\mathcal{O}}}(y).$$

Furthermore,

$$proj_{T_{\bar{\mathcal{O}} \times \bar{\mathcal{O}}}(x,y)} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} proj_{T_{\bar{\mathcal{O}}}(x)}(\xi) \\ proj_{T_{\bar{\mathcal{O}}}(x)}(\eta) \end{pmatrix}.$$

Proof. When $D \subset \mathbb{R}^d$ is admissible, it follows from Part (3) of Lemma 2.5 that

$$T_{\bar{D}}(z) = \left\{ v \in \mathbb{R}^d : \lim_{h \searrow 0} \frac{d_{\bar{D}}(z + hv)}{h} = 0 \right\}$$

(i.e. lim replaces lim inf). Since $\mathcal{O}$, and by Lemma 6.2, $\mathcal{O} \times \mathcal{O}$, are bounded and admissible, the first statement then follows immediately from the relation

$$d^2_{\bar{\mathcal{O}} \times \bar{\mathcal{O}}} \left(\begin{pmatrix} x \\ y \end{pmatrix} + h \begin{pmatrix} v \\ w \end{pmatrix} \right) = d^2_{\bar{\mathcal{O}}}(x + hv) + d^2_{\bar{\mathcal{O}}}(y + hw).$$

The second statement then follows from the first by a similar argument. □

6.1. *Synchronously coupled reflected Brownian motion*

We now discuss synchronously coupled reflected Brownian motion. A d-dimensional synchronously coupled reflected Brownian motion is a $2d$-dimensional process $Z_t = (X_t, Y_t)$ in a product domain $\bar{\mathcal{O}} \times \bar{\mathcal{O}}$ which satisfies the reflected SDE

$$\mathrm{d}Z_t = \sigma(Z_t)\mathrm{d}W_t + \mathrm{d}L_t,$$

where

$$\sigma(z) \equiv \begin{pmatrix} I \\ I \end{pmatrix}.$$

Note that, because σ is constant, there is no difference between the Stratonovich and Itô versions of the above SDE. We will express this reflected SDE in a more convenient form as the pair of reflected SDEs

$$\mathrm{d}X_t = \mathrm{d}W_t + \mathrm{d}L_t, \qquad X_0 = x_0 \quad \text{and} \quad \mathrm{d}Y_t = \mathrm{d}W_t + \mathrm{d}M_t, \qquad Y_0 = y_0.$$

We think of X_t and Y_t as being two d-dimensional processes which are driven by the same Brownian motion W_t and which are constrained to lie in the same domain $\bar{\mathcal{O}}$. The two processes move in sync except for when one or the other is bumps against the boundary and gets nudged.

We now consider the geometric properties of synchronously coupled reflected Brownian motion in two domains. Such properties were used to prove the "hot spots conjecture" for these domains (see [2,1] for more details).

Example 6.4. Let $\mathcal{O} \subset \mathbb{R}^2$ be the obtuse triangle lying with its longest face on the horizontal axis, and denote its left and right acute angles by α and β. Suppose $x_0 \neq y_0$, and for $x \neq y$, let $\angle(x, y) = \arg(y - x)$. Then, $\mathbb{P}$-almost surely,

$$-\beta \le \angle(x_0, y_0) \le \alpha \Longrightarrow \text{for all } t \text{ either } -\beta \le \angle(X_t, Y_t) \le \alpha \quad \text{or} \quad X_t = Y_t. \tag{35}$$

Proof. By (31), it suffices to show that (35) holds $\mathbb{P}^N$-a.s. Fix N and $W_t \in \Omega$. In view of Theorem 2.3 and Lemma 6.3 it will suffice to show that X_t^N and Y_t^N satisfy (35) where X_t^N and Y_t^N satisfy the ODE

$$\begin{aligned} \dot{X}_t^N &= \mathrm{proj}_{T_{\bar{\mathcal{O}}}(X_t^N)}(\dot{W}_t^N), \qquad X_0^N = x_0 \\ \dot{Y}_t^N &= \mathrm{proj}_{T_{\bar{\mathcal{O}}}(Y_t^N)}(\dot{W}_t^N), \qquad Y_0^N = y_0. \end{aligned} \tag{36}$$

It is straightforward to check that the functions X_t^N, Y_t^N starting at $X_0^N = x_0$, $Y_0^N = y_0$ and defined inductively for $t \in [m2^{-N}, (m+1)2^{-N}]$ by $X_t^N = \mathrm{proj}_{\bar{\mathcal{O}}}(X_{m2^{-N}}^N + (t - m2^{-N})\dot{W}_t^N)$ and $Y_t^N = \mathrm{proj}_{\bar{\mathcal{O}}}(Y_{m2^{-N}}^N + (t - m2^{-N})\dot{W}_t^N)$ satisfy (36). A simple geometric argument shows that if $\angle(x, y) \in [-\beta, \alpha]$ then $\angle(\mathrm{proj}_{\bar{\mathcal{O}}}(x), \mathrm{proj}_{\bar{\mathcal{O}}}(y)) \in [-\beta, \alpha]$ or $\mathrm{proj}_{\bar{\mathcal{O}}}(x) = \mathrm{proj}_{\bar{\mathcal{O}}}(y)$. From this it follows by induction that X_t^N and Y_t^N satisfy (35) as desired. Fig. 2 shows a pair of sample paths X_t^N and Y_t^N in the interval $m2^{-N} \le t \le (m+1)2^{-N}$ where we use v to denote the constant vector $2^N(W_{(m+1)2^{-N}} - W_{m2^{-N}})$. □

Example 6.5 (*Proposition 2 in [1]*)**.** We now consider synchronously coupled reflected Brownian motion in a lip domain. A lip domain is a domain in $\mathbb{R}^2$ which is bounded below by a function $f_1(x)$ and above by another function $f_2(x)$ each of which is Lipschitz continuous

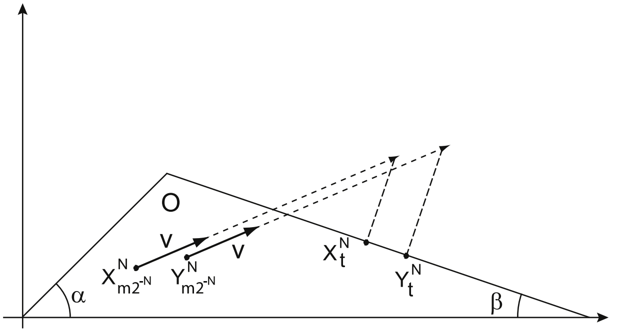

Fig. 2. The effect of the boundary can be interpreted as a projection.

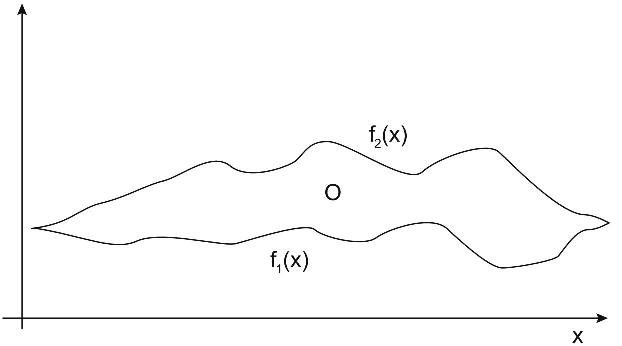

Fig. 3. A lip domain.

with constant bounded by 1. The domains are so named because they look like a pair of lips (see Fig. 3).

Consider synchronously coupled reflected Brownian motion in a lip domain $\mathcal{O}$ where the defining functions $f_1(x)$ and $f_2(x)$ are smooth and have Lipschitz constants bounded by $\lambda < 1$. Then $\mathcal{O}$ is a bounded admissible domain. Recall the definition of $\angle(x, y)$ from the previous example, and let $x_0, y_0 \in \mathbb{R}^2$ be such that $x_0 \neq y_0$ and $\angle(x_0, y_0) \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. We have the following geometric property for the paths X_t and Y_t:

$$\forall t, \quad \text{either } \angle(X_t, Y_t) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \quad \text{or} \quad X_t = Y_t \quad \mathbb{P}\text{-a.s.} \tag{37}$$

Proof. In view of (31) and Theorem 2.3 it suffices to show that for every $W_t \in \Omega$, X_t^N and Y_t^N satisfy (37) when X_t^N and Y_t^N solve the ODE (36). Let $\Theta_t^N \equiv \angle(X_t^N, Y_t^N)$ or 0 according to whether $X_t^N \neq Y_t^N$ or $X_t^N = Y_t^N$. It is enough to show that, dt-almost everywhere, $\dot{\Theta}_t^N \leq 0$ when $\Theta_t^N \in [\frac{\pi}{4}, \frac{\pi}{2} - \tan^{-1}(\lambda)]$ and $\dot{\Theta}_t^N \geq 0$ when $-\Theta_t^N \in [\frac{\pi}{4}, \frac{\pi}{2} - \tan^{-1}(\lambda)]$. By symmetry it will suffice to prove the first statement.

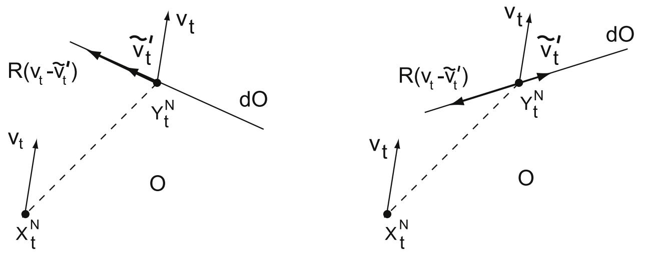

Fig. 4. The effect of the boundary in two cases.

Let $v_t = \dot{W}^N_t$, $\tilde{v}_t = \mathrm{proj}_{T_{\mathcal{O}}(X^N_t)}(v_t)$, and $\tilde{v}'_t = \mathrm{proj}_{T_{\mathcal{O}}(Y^N_t)}(v_t)$. We compute:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\Theta^N_t\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\tan^{-1}\left(\frac{(Y^N_t - X^N_t)_2}{(Y^N_t - X^N_t)_1}\right) = \frac{(Y^N_t - X^N_t)\cdot R(\tilde{v}_t - \tilde{v}'_t)}{|Y^N_t - X^N_t|^2}\\ &= \frac{(Y^N_t - X^N_t)\cdot R(\tilde{v}_t - v_t)}{|Y^N_t - X^N_t|^2} + \frac{(Y^N_t - X^N_t)\cdot R(v_t - \tilde{v}'_t)}{|Y^N_t - X^N_t|^2}\end{aligned}$$

where $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the matrix which rotates vectors in $\mathbb{R}^2$ by 90° counter-clockwise. Suppose $\Theta^N_t \in [\frac{\pi}{4}, \frac{\pi}{2} - \tan^{-1}(\lambda)]$. Then since the Lipschitz constants of f_1 and f_2 are strictly less than 1, X^N_t cannot be on the f_2-boundary and Y^N_t cannot be on the f_1-boundary. For each t, it follows that either $v_t = \tilde{v}_t$ or $\arg(R(\tilde{v}_t - v_t)) \in [\pi - \tan^{-1}(\lambda), \pi + \tan^{-1}(\lambda)]$ and either $v_t = \tilde{v}'_t$ or $\arg(R(v_t - \tilde{v}'_t)) \in [\pi - \tan^{-1}(\lambda), \pi + \tan^{-1}(\lambda)]$. And so each of the terms in the sum above is ≤ 0. We depict in Fig. 4, the case where $X^N_t \in \mathcal{O}$ and $Y^N_t \in \partial\mathcal{O}$. □

6.2. *Mirror coupled reflected Brownian motion*

Our final example involves mirror coupled reflected Brownian motion. A d-dimensional mirror coupled reflected Brownian motion is a $2d$-dimensional process $Z_t = (X_t, Y_t)$ in a product domain $\bar{\mathcal{O}} \times \bar{\mathcal{O}}$ which satisfies the reflected SDE

$$\mathrm{d}Z_t = \sigma(Z_t)\mathrm{d}W_t + \mathrm{d}L_t, \tag{38}$$

where

$$\sigma(z) = \sigma(x, y) \equiv \begin{pmatrix} I \\ I - 2\dfrac{(y-x)(y-x)^\top}{|y-x|^2} \end{pmatrix},$$

defined up until the first time τ that Z_t hits the diagonal of $\bar{\mathcal{O}} \times \bar{\mathcal{O}}$, at which point we stop our process (i.e. $Z_t \equiv Z_\tau$ for $t \geq \tau$). We will express this reflected SDE in a more convenient form as the pair of reflected SDEs

$$\begin{aligned}&\mathrm{d}X_t = \mathrm{d}W_t + \mathrm{d}L_t, \qquad X_0 = x_0\\ &\mathrm{d}Y_t = \left(I - 2\frac{(Y_t - X_t)(Y_t - X_t)^\top}{|Y_t - X_t|^2}\right)\mathrm{d}W_t + \mathrm{d}M_t, \qquad Y_0 = y_0.\end{aligned} \tag{39}$$

We think of X_t and Y_t as being two d-dimensional processes which are "mirror coupled" with respect to the driving Brownian motion W_t and which are constrained to lie in the same domain

$\bar{\mathcal{O}}$. That is, if you consider the hyperplane which perpendicularly bisects the line segment connecting X_t and Y_t to be a "mirror", then the two processes move in such a way that they are mirror images of each other until either process bumps into the boundary and is nudged (which causes the mirror to shift). We refer the reader to the papers [2,1] for a more thorough overview.

We will prove the same geometric property we considered for synchronously coupled reflected Brownian motion in Example 6.5, but now for mirror coupled reflected Brownian motion. The point is that (38) can be viewed as a Stratonovich reflected SDE and so again it suffices to prove the geometric property for the approximating processes.

We make this rigorous with the following lemma which shows that, off the diagonal of $\mathcal{O}\times\mathcal{O}$, the Stratonovich correction factor for (38) is 0.

Lemma 6.6. *For $t < \tau$,*

$$\sum_{j=1}^{d} \frac{1}{2} \mathrm{d} \left\langle \left(I - 2\frac{(Y_t - X_t)(Y_t - X_t)^\top}{|Y_t - X_t|^2} \right)_{ij}, (W_t)_j \right\rangle = 0. \tag{40}$$

In fact,

$$\mathrm{d} \left\langle \left(I - 2\frac{(Y_t - X_t)(Y_t - X_t)^\top}{|Y_t - X_t|^2} \right)_{ij}, (W_t)_j \right\rangle = 0, \quad \textit{for each } j. \tag{41}$$

Proof. It suffices to prove (41). Let $V_i \equiv (Y_t - X_t)_i$, where we have suppressed the dependence of V_i on t. An easy calculation shows that

$$d\langle V_\ell, (W_t)_j \rangle = \frac{-2V_j V_\ell}{\sum_k V_k^2} \mathrm{d}t$$

and

$$\frac{\partial}{\partial V_i} \left(\frac{V_i V_j}{\sum_k V_k^2} \right) = \frac{V_j \left(\sum_k V_k^2\right) - 2V_i^2 V_j}{\left(\sum_k V_k^2\right)^2}$$

$$\frac{\partial}{\partial V_j} \left(\frac{V_i V_j}{\sum_k V_k^2} \right) = \frac{V_i \left(\sum_k V_k^2\right) - 2V_i V_j^2}{\left(\sum_k V_k^2\right)^2}$$

$$\frac{\partial}{\partial V_\ell} \left(\frac{V_i V_j}{\sum_k V_k^2} \right) = \frac{-2V_i V_j V_\ell}{\left(\sum_k V_k^2\right)^2}, \quad \text{for } \ell \neq i, j.$$

Putting these together, we have that

$$
\begin{aligned}
d\langle \frac{V_i V_j}{\sum\limits_k V_k^2}, (W_t)_j \rangle &= \left(\frac{V_j \left(\sum\limits_k V_k^2 \right) - 2V_i^2 V_j}{\left(\sum\limits_k V_k^2 \right)^2} \right) \left(\frac{-2V_j V_i}{\sum\limits_k V_k^2} \right) \\
&\quad + \left(\frac{V_i \left(\sum\limits_k V_k^2 \right) - 2V_i V_j^2}{\left(\sum\limits_k V_k^2 \right)^2} \right) \left(\frac{-2V_j^2}{\sum\limits_k V_k^2} \right) \\
&\quad + \sum_{\ell \neq i,j} \left(\frac{-2V_i V_j V_\ell}{\left(\sum\limits_k V_k^2 \right)^2} \right) \left(\frac{-2V_j V_\ell}{\sum\limits_k V_k^2} \right) \\
&= 0.
\end{aligned}
$$

From this, (41) immediately follows. □

We now prove a geometric property.

Example 6.7 (Example 6.5 *for Mirror Coupling*)**.** Let $\mathcal{O}$ be the same lip domain defined by smooth functions considered in Example 6.5 and consider the mirror coupled reflected Brownian motion starting from x_0 and y_0 where $x_0 \neq y_0$. Then (37) holds where X_t and Y_t are given by (39).

Proof. Let $D_\varepsilon = \{z = (x, y) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}} : |x - y| < \varepsilon\}$ be the "ε-diagonal" of $\bar{\mathcal{O}} \times \bar{\mathcal{O}}$. Consider a sequence of smooth functions $\rho_k : \bar{\mathcal{O}} \times \bar{\mathcal{O}} \to [0, 1]$ such that $\rho(z) \equiv 0$ on $D_{\frac{1}{2k}}$ and $\rho(z) \equiv 1$ off $D_{\frac{1}{k}}$. Let $\sigma_k(z) = \rho_k(z)\sigma(z)$. Then $\sigma_k \in C^2(\bar{\mathcal{O}} \times \bar{\mathcal{O}})$.

Let $\mathbb{P}^k$ be the measure on Z-pathspace induced by the solutions to the reflected SDE

$$
\mathrm{d}Z_t^k = \sigma_k(Z_t^k) \circ \mathrm{d}W_t + \mathrm{d}L_t^k
$$

and define $\mathbb{P}^{k,N}$ to be the measures on Z-pathspace induced by solutions to the approximating reflected ODE

$$
\mathrm{d}Z_t^{k,N} = \sigma_k(Z_t^{k,N})\mathrm{d}W_t^N + \mathrm{d}L_t^{k,N}. \tag{42}
$$

Recall that the stopping time τ corresponds to the first time X_t equals Y_t and define $\tau_k \equiv \inf\{t : |X_t - Y_t| < \frac{1}{k}\}$. Let $S = \{Z_t \in C([0, \infty); \mathbb{R}^{2d}) : -\frac{\pi}{4} \leq \angle(X_t, Y_t) \leq \frac{\pi}{4},\ \forall t < \tau\}$ and let $S_k = \{Z_t \in C([0, \infty); \mathbb{R}^{2d}) : -\frac{\pi}{4} \leq \angle(X_t, Y_t) \leq \frac{\pi}{4},\ \forall t < \tau_k\}$.

Our goal is to show that $\mathbb{P}(S) = 1$, where $\mathbb{P}$ is the measure induced on Z-pathspace by (38). It is clear that the subsets S_k decrease monotonically to S, and so it suffices to prove that $\mathbb{P}(S_k) = 1, \forall k$.

We first claim that $\mathbb{P}(S_k) = \mathbb{P}^k(S_k)$. This is true because S_k is $\mathcal{F}_{\tau_k}$-measurable, and, in view of Lemma 6.6 and the equality $\sigma = \sigma_k$ on $D_{\frac{1}{k}}$, it is clear that $\mathbb{P}(A) = \mathbb{P}^k(A)$ for $A \in \mathcal{F}_{\tau_k}$. So we need only show that $\mathbb{P}^k(S_k) = 1$, and for this it will suffice to show that $\mathbb{P}^{k,N}(S_k) = 1$. We argue this as we did in Example 6.5.

Fix N and $W_t \in \Omega$ and let $\Theta_t^{k,N} \equiv \angle(X_t^{k,N}, Y_t^{k,N})$. By symmetry, it is enough to show that $\dot{\Theta}_t^{k,N} \le 0$ for $\Theta_t^{k,N} \in [\frac{\pi}{4}, \frac{\pi}{2} - \tan^{-1}(\lambda)]$ for almost every $t < \tau_k$. Let $v_t = \dot{W}_t^N$ and

$$w_t = \left(I - \frac{(Y_t^{k,N} - X_t^{k,N})(Y_t^{k,N} - X_t^{k,N})^\top}{|Y_t^{k,N} - X_t^{k,N}|^2} \right) v_t.$$

Then, in view of Theorem 2.3 and Lemma 6.3, $\dot{X}_t^{k,N} = \tilde{v}_t \equiv \text{proj}_{T_{\bar{\mathcal{O}}}(X_t^N)}(v_t)$ and $\dot{Y}_t^{k,N} = \tilde{w}_t \equiv \text{proj}_{T_{\bar{\mathcal{O}}}(Y_t^N)}(w_t)$ (recall that $\rho_k(X_t^{k,N}, Y_t^{k,N}) = 1$ for $t < \tau_k$).

We compute:

$$\begin{aligned}
\dot{\Theta}_t^{k,N} &= \frac{(Y_t^{N,k} - X_t^{N,k}) \cdot R(\tilde{v}_t - \tilde{w}_t)}{|Y_t^{N,k} - X_t^{N,k}|^2} \\
&= \frac{(Y_t^{N,k} - X_t^{N,k}) \cdot R(\tilde{v}_t - v_t)}{|Y_t^{N,k} - X_t^{N,k}|^2} + \frac{(Y_t^{N,k} - X_t^{N,k}) \cdot R(v_t - w_t)}{|Y_t^{N,k} - X_t^{N,k}|^2} \\
&\quad + \frac{(Y_t^{N,k} - X_t^{N,k}) \cdot R(w_t - \tilde{w}_t)}{|Y_t^{N,k} - X_t^{N,k}|^2}.
\end{aligned}$$

The argument in the Proof 6.1 again shows that the first and third terms are non-positive. That the second term is non-positive follows from the fact that either $v_t = w_t$ or $\arg(v_t - w_t) = \pm\Theta_t^{k,N}$. □

References

[1] Rami Atar, Krzysztof Burdzy, On Neumann eigenfunctions in lip domains, J. Amer. Math. Soc. 17 (2) (2004) 243–265. (electronic).

[2] Rodrigo Bañuelos, Krzysztof Burdzy, On the “hot spots” conjecture of J. Rauch, J. Funct. Anal. 164 (1) (1999) 1–33.

[3] Piernicola Bettiol, A deterministic approach to the Skorokhod problem, Control Cybernet. 35 (4) (2006) 787–802.

[4] Bernard Cornet, Existence of slow solutions for a class of differential inclusions, J. Math. Anal. Appl. 96 (1) (1983) 130–147.

[5] Arturo Kohatsu-Higa, Stratonovich type SDE’s with normal reflection driven by semimartingales, Sankhyā Ser. A 63 (2) (2001) 194–228.

[6] P.-L. Lions, A.-S. Sznitman, Stochastic differential equations with reflecting boundary conditions, Comm. Pure Appl. Math. 37 (4) (1984) 511–537.

[7] Roger Petterson, Wong-zakai approximations for reflecting stochastic differential equations, Stoch. Anal. Appl. 17 (4) (1999) 609–617.

[8] R.A. Poliquin, R.T. Rockafellar, L. Thibault, Local differentiability of distance functions, Trans. Amer. Math. Soc. 352 (11) (2000) 5231–5249.

[9] Yasumasa Saisho, Stochastic differential equations for multidimensional domain with reflecting boundary, Probab. Theory Related Fields 74 (3) (1987) 455–477.

[10] Daniel W. Stroock, Probability Theory, an Analytic View, 2nd edition, Cambridge University Press, Cambridge, 1993.

[11] Daniel W. Stroock, S.R. Srinivasa Varadhan, Multidimensional Diffusion Processes, in: Classics in Mathematics, Springer-Verlag, Berlin, 2006, Reprint of the 1997 edition.

[12] Daniel W. Stroock, S.R.S. Varadhan, Diffusion processes with boundary conditions, Comm. Pure Appl. Math. 24 (1971) 147–225.

[13] Hiroshi Tanaka, Stochastic differential equations with reflecting boundary condition in convex regions, Hiroshima Math. J. 9 (1) (1979) 163–177.

[14] R. Tyrrell Rockafellar, Roger J.-B. Wets, Variational Analysis, in: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 317, Springer-Verlag, Berlin, 1998.

[15] Richard Vinter, Optimal Control, in: Systems & Control: Foundations & Applications, Birkhäuser Boston Inc., Boston, MA, 2000.

[16] Eugene Wong, Moshe Zakai, On the convergence of ordinary integrals to stochastic integrals, Ann. Math. Statist. 36 (1965) 1560–1564.

[17] Eugene Wong, Moshe Zakai, On the relation between ordinary and stochastic differential equations, Internat. J. Engrg. Sci. 3 (1965) 213–229.