Embedding Finite Planar Spaces into 3-Dimensional Projective Spaces

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1. INTRODUCTION

A linear space is a pair $L = (\mathcal{P}, \mathcal{L})$ of a set $\mathcal{P}$ of points and a family $\mathcal{L}$ of subsets of $\mathcal{P}$ called lines such that any two distinct points $p$ and $q$ are contained in a unique line $pq$ and every line contains at least two points. The degree $r_p$ of a point $p$ is the number of lines through $p$, and dually the degree $k_L$ of a line $L$ is the number of points on $L$. If every point and line has finite degree, then $L$ is called finite. A parallel class of $L$ is a set $\Pi$ of lines such that every point lies in exactly one of the lines of $\Pi$. A subspace of $L$ is a set $E$ of points such that every line which contains two points of $E$ is already contained in $E$.

Suppose $L$ is a finite linear space. If $n+1$ denotes the maximal point degree, then $n$ is called the order of $L$. A line $L$ is called projective, if $\Pi_1 \cap \Pi_2 = \{L\}$ for any two distinct parallel classes $\Pi_1$ and $\Pi_2$ with $L \in \Pi_j$ and $|\Pi_j| = n+1$.

A planar space is a linear space $S = (\mathcal{P}, \mathcal{L})$ together with a family $\mathcal{E}$ of subspaces called planes such that any three noncollinear points lie in a unique plane, every plane contains three noncollinear points, and there are at least two planes. In this paper we shall furthermore always assume that $S$ is not a degenerate projective space.

We want to study finite planar spaces with the following property

(E) Any two distinct planes intersect in a line.

Consider a finite planar space satisfying (E). It is well known that there is an integer $n$ such that for every point $p$ the lines through $p$, considered as points, and the planes through $p$, considered as lines, form a projective plane $S/p$ of order $n$. Also for each plane $E$ the structure $L(E) = (E, \mathcal{L}(E))$
with $\mathcal{L}(E) = \{ L \mid L$ is a line contained in $E \}$ is a linear space with constant point degree $n + 1$. Usually we identify $L(E)$ with $E$. We call this so defined integer $n$ the planar order of $S$. Finally, we call $S$ extendible, if it can be obtained from a planar space $S'$ by removing one point but no line or plane from $S'$. It is obvious that $S'$ has planar order $n$ and satisfies (E) in this case.

We shall prove the following two theorems.

**Theorem 1.** Let $S$ be a finite planar space which has property (E) and denote by $n$ the planar order of $S$. If $S$ possesses a line of degree at most $n$ which is projective in at least one of the planes containing it, then $S$ is extendible.

**Theorem 2.** Let $S$ be a finite planar space satisfying (E) and denote by $n$ the planar order of $S$. Then $S$ can be embedded into a 3-dimensional projective space, if one of the following conditions is satisfied:

(i) $S$ has at least $n^3 - 3n^2 + 9n + 12$ points.

(ii) $S$ has at least $n^3 - 3n^2 + 5n + 8$ points and $n \geq 13$.

**Remarks.** (1) Theorem 2 improves a result of A. Beutelspacher [1], who showed that a finite planar space of planar order $n$ which satisfies (E) is embeddable into a 3-dimensional projective space, provided it has at least $n^3$ points.

(2) In the proof of Theorem 1 we shall show that $S$ satisfies "locally" the bundle theorem. This will imply that $S$ can be extended.

## 2. Extending Finite Planar Spaces

In the rest of this paper, $S$ denotes a finite planar space satisfying (E) and $n$ denotes the planar order of $S$. For distinct planes $E$ and $F$ we denote by $E \cap F$ the line in which $E$ and $F$ meet. For a point $p$ outside a line $L$ we denote by $\langle L, p \rangle$ the unique plane containing $p$ and $L$. Finally, if $H$ and $L$ are distinct lines which are contained in a common plane, then $H$ and $L$ are called coplanar and $\langle H, L \rangle$ denotes the unique plane containing $H$ and $L$.

Our first proposition shows how $S$ can be extended.

**Lemma 2.1.** $S$ is extendible if and only if there is a parallel class $\Pi$ of $S$ satisfying the following condition: For every plane $E$, the set $\mathcal{L}(E) \cap \Pi$ is empty or a parallel class of $L(E)$. 
Proof. Suppose first that S can be extended to a planar space S' with one point more. If p is the point of S' not belonging to S, then the lines through p form a parallel class of S with the desired condition.

Now suppose that such a parallel class II exists. If we add an infinite point \( \infty \) to S and every line of II, then we obtain a linear space S'. Adjoining \( \infty \) also to every plane E with \( \mathcal{P}(E) \cap II \neq \emptyset \) while leaving the other planes as they are, our condition on II yields that S' becomes a planar space. Hence, S is extendible.

**Definition 2.2.** (a) A bundle is a quadruple \((L_1, L_2, L_3, L_4)\) of four mutually disjoint lines such that \(L_j\) and \(L_k\) are coplanar for \(j \neq k\) and \(\{j, k\} \neq \{3, 4\}\) and that no three of the lines \(L_j\) lie in a common plane.

(b) S is said to satisfy the bundle theorem, if \(L_3\) and \(L_4\) are coplanar for every bundle \((L_1, L_2, L_3, L_4)\).

(c) We call a line \(L\) of S a good line, if it has degree at most \(n\) and if \(L\) and \(L_4\) are coplanar for every bundle \((L_1, L_2, L, L_4)\).

Remark. A theorem of Kahn [2] says that S can be embedded into a projective space, if it satisfies the bundle theorem. We just want to extend S and for this it will be enough to know that there is a good line.

**Theorem 2.3.** If S possesses a good line, then S is extendible.

Proof. Let G be a good line. Denote by \(E_1, \ldots, E_{n+1}\) the planes which contain G and let F be a plane with \(G \cap F = \emptyset\) (such a plane exists in view of \(k_\infty \leq n\)). Let \(L_j\) be the line in which F and \(E_j\) meet and denote by \(g\) the set of points which do not lie on G or one of the lines \(L_j\). In the further, we denote by \(j, k, r, s\) always elements of \(\{1, \ldots, n+1\}\). Set

\[L_{jk}(q) = \langle L_j, q \rangle \cap \langle L_k, q \rangle\quad \text{for } j \neq k \text{ and } q \in g,
\]

and

\[\Pi = \{G, L_1, \ldots, L_{n+1}\} \cup \{L_{jk}(q) \mid q \in g, j \neq k\}.
\]

First notice that the lines \(L_{jk}(q), q \in g\), are disjoint to every line of \(\{G, L_1, \ldots, L_{n+1}\}\). Furthermore, each line \(L_{jk}(q), q \in g\), is contained in \(\langle G, q \rangle\), for if \(L_{jk}(q)\) were not in \(\langle G, q \rangle\), then \((L_j, L_k, G, L_{jk}(q))\) would be a bundle in which the good line G is not coplanar to \(L_{jk}(q)\). In particular, \(L_{jk}(q)\) and G are coplanar.

(1) II is a parallel class of S. For: Let \(q\) be a point of \(g\) and suppose that \(L_{jk}(q_1)\) and \(L_{rs}(q_2)\) are lines through \(q\). W.l.o.g. we may assume that \(L_k\) and \(L_r\) are not contained in the plane \(\langle G, q \rangle\). Since \(L_{jk}(q_1)\) contains \(q\) and is coplanar to \(L_k\) and G, this implies \(L_{jk}(q_1) = \langle L_k, q \rangle \cap \langle G, q \rangle\). In
the same way, \( L_{rs}(q) = \langle L_r, q \rangle \cap \langle G, q \rangle \) follows. Thus \( L_{jk}(q_1) = L_{rs}(q_2) \), if \( k = r \). If \( k \neq r \), then the same argument shows \( L_{jk}(q_1) = L_{kr}(q) = L_{rs}(q_2) \), since \( L_{kr}(q) \) is also a line passing through \( q \). This shows that every point of \( q \) is contained in a unique line of \( \Pi \). The points outside of \( q \) are contained in a unique line of \( \{G, L_1, \ldots, L_{n+1}\} \) and not in any of the lines \( L_{jk}(q) \). Hence, \( \Pi \) is a parallel class of \( S \).

(2) If \( \mathcal{L}(E) \cap \Pi \neq \emptyset \) for a plane \( E \), then \( \mathcal{L}(E) \cap \Pi \) is a parallel class of \( E \). For: First suppose that \( E = F \). Then \( \mathcal{L}(E) \cap \Pi = \{L_1, \ldots, L_{n+1}\} \) so that \( \mathcal{L}(E) \cap \Pi \) is a parallel class of \( E \).

Next suppose \( E = E_k \) for some \( k \). Then for each \( j \neq k \) and \( q \in E - (G \cup L_k) \), the line \( L_{jk}(q) \) is a line of \( \mathcal{L}(E) \cap \Pi \) through \( q \). Since \( G, L_k \in \Pi \), this shows that every point of \( E \) lies in at least one line of \( \mathcal{L}(E) \cap \Pi \). Now (1) shows that \( \mathcal{L}(E) \cap \Pi \) is a parallel class of \( E \).

Finally consider the case \( E \neq F \) and \( E \neq E_j \) for all \( j \). We assume that \( E \) contains a line \( L \) of \( \Pi \) so that we have to show that \( \mathcal{L}(E) \cap \Pi \) is a parallel class of \( E \). Since \( E \neq E_j \) for all \( j \), we have \( L \neq G \).

If \( L = L_k \) for some \( k \), let \( s \) be a value with \( s \neq k \). For each \( q \in E - L_k \), we have \( q \in q \) (since \( E \neq F, E_k \)) and \( L_{ks}(q) \) is a line of \( \Pi \) through \( q \). Since \( L_{ks}(q) \) is contained in \( \langle L_k, q \rangle = E \), it follows that each point of \( E \) is contained in a line of \( \mathcal{L}(E) \cap \Pi \). (1) implies again that \( \mathcal{L}(E) \cap \Pi \) is a parallel class of \( E \).

If \( L = L_{rs}(q) \) for some \( q \in q \) and indices \( r, s \), we can argue as follows. Let \( p \) be a point of \( E \cap F \) and denote by \( k \) the index with \( p \in L_k \). We may assume that \( r \neq k \). By (1), \( L_{rs}(q) = L_{rk}(q) \). Hence \( L_k \) is contained in \( \langle L_{rk}(q), p \rangle = E \) and it follows as before that \( \mathcal{L}(E) \cap \Pi \) is a parallel class of \( E \).

(1), (2), and Lemma 2.1 show that \( S \) can be extended. □

As a corollary, we obtain the theorem of Kahn [2] in a special case.

**Corollary 2.4.** If \( S \) satisfies the bundle theorem, then it can be embedded into a 3-dimensional projective space of order \( n \).

**Proof.** We proceed by induction on \( s = n^3 + n^2 + 1 - v \), where \( v \) denotes the number of points of \( S \). Obviously, \( s \geq 0 \) with equality if and only if \( S \) is a 3-dimensional projective space of order \( n \).

Suppose \( s > 0 \). Then \( S \) has a line \( L \) with \( k_L < n + 1 \). Since \( S \) satisfies the bundle theorem, \( L \) is a good line. Theorem 2.3 shows that \( S \) can be extended to a planar space \( S' \) with \( v' = v + 1 \) points. It is trivial that \( S' \) also satisfies (E) and the bundle theorem. The induction hypothesis shows that \( S' \) and therefore also \( S \) can be embedded into a 3-dimensional projective space. □
LEMMA 2.5. Let $G$ be a line of $S$ with $k_G < n + 1$. If $G$ is a projective line in the linear space $L(E)$ for at least one plane $E$ which contains $G$, then $G$ is a good line of $S$.

Proof. Assume by way of contradiction that $G$ is not a good line of $S$. Then there is a bundle $(L_1, L_2, G, X)$ in which $G$ and $X$ are not coplanar. Define the planes $F = \langle L_1, L_2 \rangle$, $E_1 = \langle G, L_1 \rangle$, and $E_2 = \langle G, L_2 \rangle$. We denote by $E_3, \ldots, E_{n+1}$ the planes of $S$ which contain $G$ and by $L_j$, $j = 3, \ldots, n+1$, the intersection line of $E_j$ and $F$. Let $p$ be any point of $X$ and let $k$ be the index with $p \in E_k$. Then $X = \langle L_1, p \rangle \cap \langle L_2, p \rangle$ (notice that $p \notin F = \langle L_1, L_2 \rangle$, since $(L_1, L_2, G, X)$ is a bundle) and $k \neq 1, 2$ (if $k$ were in $\{1, 2\}$, then $G$ and $X$ would be contained in $E_k - \langle L_k, p \rangle$ but $G$ and $X$ are not coplanar). W.l.o.g. we may assume $k = 3$. Finally, set $H_j = \langle L_j, p \rangle \cap E_3$, $j = 1, 2$. Then we have

1. $H_1 \neq H_2$. For: Assume to the contrary $H_1 = H_2$. Then $H_1 = H_2 = \langle L_1, p \rangle \cap \langle L_2, p \rangle = X$. Hence $X = H_1 = \langle L_1, p \rangle \cap E_3$ is contained in $E_3$, which is a contradiction since $G$ and $X$ are not coplanar.

2. $G$ is not a projective line of $E_3$. For: Set $\Pi_j = \{E \cap E_j \mid E$ is a plane containing $L_j\}$, $j = 1, 2$. In view of the property $(E)$, $\Pi_1$ and $\Pi_2$ are parallel classes of $E_3$ with $n + 1$ elements. For $j \in \{1, 2\}$ we have $p \in H_j$ and $H_j \in \Pi_j$. Since $H_1 \neq H_2$, we obtain $\Pi_1 \neq \Pi_2$. However, $G, L_3 \in \Pi_1, \Pi_2$. By definition, $G$ is therefore not a projective line of $E_3$.

3. $G$ is not a projective line of $E_1$ or $E_2$. For: Assume to the contrary $G$ is a projective line of $E_1$. Define $\Pi_1 = \{E_1 \cap E \mid E$ contains $L_2\}$ and $\Pi_2 = \{E_1 \cap E \mid E$ contains $H_2\}$. Then $\Pi_1$ and $\Pi_2$ are parallel classes of $E_1$ which contain $G$ and have $n + 1$ elements. Since $L_2$ and $H_2$ are contained in $\langle L_2, p \rangle$ the line $H := \langle L_2, p \rangle \cap E_1$ lies in $\Pi_1$ and $\Pi_2$. Since $G$ is a projective line of $E_1$ and in view of $G \neq H$, we conclude $\Pi_1 = \Pi_2$. It follows $L_1 = E_1 \cap F \in \Pi_1 = \Pi_2$ so that $L_1$ and $H_2$ are coplanar. Now the lines $H_1$ and $H_2$ of the plane $E_3$ are both coplanar to $L_1$ and contain $p$. Since $L_1$ is not a line of $E_3$, this yields $H_1 = H_2$, which contradicts (1). Consequently, $G$ is also not a projective line of $E_2$.

By our hypothesis, $G$ is a projective line in $E_s$ for some $s$. By (2) and (3), $s \geq 4$. Set $H = \langle L_s, p \rangle \cap E_3$. In view of (1), we may assume that $H \neq H_1$. Set $X' = \langle L_1, p \rangle \cap \langle L_s, p \rangle$. Since $p \notin F, E_1, E_s$, the line $X'$ is not contained in $F = \langle L_1, L_2 \rangle$, $E_1 = \langle L_1, G \rangle$, or $E_s = \langle L_s, G \rangle$. Thus $(L_1, L_s, G, X')$ is a bundle.

If $X'$ is contained in $E_3$, then $H = \langle L_s, p \rangle \cap E_3 = X' = \langle L_1, p \rangle \cap E_3 = H_1$, which is a contradiction.

If $X'$ is not contained in the plane $E_3$, then $p \in X' \cap E_3$ shows that $G$ and $X'$ are not coplanar. The existence of the bundle $(L_1, L_2, G, X)$ with $G$ and $X$ not coplanar implied that $G$ is not a projective line of $E_2 = \langle L_2, G \rangle$ (see
The same argument used for the bundle \((L_1, L_2, G, X')\) shows that \(G\) is not projective in \(E_\gamma = \langle L_\gamma, G \rangle\). But \(E_\gamma\) is by definition the plane through \(G\) in which \(G\) is projective.

This final contradiction shows that \(G\) and \(X\) are coplanar. 

Lemma 2.5 and Theorem 2.3 prove Theorem 1. An immediate consequence is the following.

**Corollary 2.6.** If \(S\) possesses a line of degree \(n\), then it is extendible.

### 3. Embedding Finite Planar Spaces

Our next lemma will ensure the existence of lines being projective in one of the planes containing it, provided that \(S\) has “enough” points.

**Lemma 3.1.** Let \(a\) be a positive integer and denote by \(v\) the number of points of \(S\). If \(n^3 - (a - 1)(n^2 + n + 1) < v < n^3 + n^2 + n + 1\), then \(S\) has a line \(G\) with \(n + 1 - a \leq k_G < n + 1\).

**Proof.** Since \(v < n^3 + n^2 + n + 1\), not every line has degree \(n + 1\). Of all the lines of degree at most \(n\) let \(G\) be a line of maximal degree. Set \(d = n + 1 - k_G\), let \(E_1, \ldots, E_{n+1}\) be the planes which contain \(G\), and let \(M\) be the set of lines which are disjoint to \(G\) and which are contained in one of the planes \(E_j\). Since each plane has \(n^2 + n + 1\) lines, each \(E_j\) contains \(dn\) lines of \(M\). Since every line of \(M\) has degree at most \(n\), we have \(k_L \leq k_G\) for all lines \(L\) of \(M\). Because every point outside of \(G\) lies on exactly \(d\) lines of \(M\), we obtain

\[
(v - k_G)d = \sum_{L \in M} k_L \leq |M| k_G = (n + 1) dn k_G
\]

and consequently \(k_G(n^2 + n + 1) \geq v\), i.e., \(k_G > n - a\).

**Remark.** If \(S\) has at least \(n^3\) points but not \(n^3 + n^2 + n + 1\) points, then it has a line of degree \(n\). By Corollary 2.6, \(S\) is therefore extendible. An inductive argument shows that \(S\) can be embedded into a 3-dimensional projective space. This is the theorem of A. Beutelspacher mentioned in the Introduction.

**Lemma 3.2.** Let \(L\) be a finite linear space which is embeddable into a projective plane \(P\) of finite order \(n\). Suppose \(L \neq P\) and every point of \(L\) has degree \(n + 1\). Let \(G\) be a line of degree \(k\), where \(k = \max\{k_L | L\ is\ a\ line\ of\ degree\ at\ most\ n\}\), and denote by \(w\) the number of points of \(L\) outside of \(G\). Then \(G\) is a projective line of \(L\), if one of the following conditions is satisfied:
(a) \( w \geq n^2 - 4n + 9 \) and \( n > 3(n+1-k) \).

(b) \( w \geq n^2 - 4n + 13 \) and \( k \geq n - 3 \). Furthermore \( k \neq n - 3 \), if \( n = 7 \) or \( n = 8 \).

**Proof.** In view of \( L \neq P \), \( L \) has a line of degree at most \( n \) and \( k \) is therefore defined. Suppose (a) or (b) is fulfilled.

Let \( p_1, \ldots, p_d, d := n + 1 - k \), be the points of \( P \) on \( G \) which are not points of \( L \). For \( j = 1, \ldots, d \) denote by \( \Pi_j \) the set of lines of \( L \) which pass in \( P \) through \( p_j \). Since every point of \( L \) has degree \( n + 1 \), the \( \Pi_j \) are parallel classes of \( L \).

Let \( \Pi \) be any parallel class of \( L \) containing \( G \). In order to show that \( G \) is a projective line of \( L \) it suffices to show that \( \Pi = \Pi_j \) for some \( j \), if \( |\Pi| = n + 1 \).

Assume to the contrary that \( |\Pi| = n + 1 \) and \( \Pi \neq \Pi_j \) for all \( j \). Let \( M_j \) be the set of lines \( \neq G \) of \( \Pi \) passing in \( P \) through \( p_j \) and set \( m_j = |M_j| \). Then \( n = m_1 + \cdots + m_d \).

W.l.o.g. we may assume \( m_1 \geq m_2 \geq \cdots \geq m_d \). In the view of \( \Pi \neq \Pi_1 \), we have \( m_2 \geq 0 \) and \( d \geq 2 \). For \( j \in \{1, \ldots, d\} \) and all lines \( L \) of \( \Pi - M_j \), we have \( k_L \leq |\Pi_j| - 1 - m_j \leq n - m_j \), because \( L \) is parallel to \( G \) and every line of \( M_j \). Since \( M_2 \neq \emptyset \), this implies \( m_1 \leq n - 2 \).

If \( L \) is a line of \( M_1 \), then \( k_L \leq n - 2 \) (For: Assume \( k_L \geq n - 1 \). Since \( L \) is parallel to \( G \), we have \( k_L \leq k_G = n + 1 - d \leq n - 1 \). It follows that \( d = 2 \) so that \( k_L \leq n - m_2 = m_1 \leq n - 2 \), a contradiction.) We obtain

\[
\begin{align*}
\sum_{L \in \Pi, L \neq G} k_L &\leq m_1(n-2) + (n-m_1)(n-m_1) = f(m_1). \\
\end{align*}
\]

Now \( f(4) = f(n-2) = n^2 - 4n + 8 \) shows \( m_1 < 4 \). In particular, \( n = m_1 + \cdots + m_d \leq dm_1 = (n+1-k)m_1 \leq 3(n+1-k) \). This shows that (a) is not satisfied. Hence (b) is satisfied so that \( d \leq 4 \) and \( w \geq n^2 - 4n + 13 \).

We have, furthermore, \( w \leq f(m_1) \) and \( n = m_1 + \cdots + m_d \leq m_1 + (d-1)m_2 \leq m_1 + 3m_2 \leq 4m_1 \).

If \( m_1 = 3 \), then \( n^2 - 4n + 13 \leq w \leq f(3) = n^2 - 3n + 3 \) so that \( n \geq 10 \). \( n \leq m_1 + 3m_2 \) yields \( m_2 = 3 \). Hence every line of \( M_1 \) has degree at most \( n - m_2 = n - 3 \). Now we can improve (a) and obtain \( w \leq m_1(n-3) + (n-m_1)(n-m_1) = n^2 - 3n \) so that \( n \geq 13 \). But this is not possible, since \( n \leq 4m_1 = 12 \). Hence \( m_1 \neq 3 \).

If \( m_1 = 2 \), then \( w \leq f(2) = n^2 - 2n \) so that \( n \geq 7 \). \( n \leq m_1 + 3m_2 \leq 4m_1 \) shows \( m_2 = 2 \) and \( n \leq 8 \). Now condition (b) yields \( d \leq 3 \) so that \( n \leq m_1 + (d-1)m_2 \leq 6 \), a contradiction.

Consequently \( m_1 = 1 \) so that \( n \leq dm_1 = d \). But \( n + 1 - d \) as the degree of the line \( G \) must be at least 2, a contradiction.

**Proof of Theorem 2.** Suppose that the hypotheses of (i) or (ii) of Theorem 2 is fulfilled. Denote by \( v \) the number of points of \( S \) and set
s = n^3 + n^2 + n + 1 - v. As in the proof of Corollary 2.4, it suffices to show that S is extendible if s > 0. Suppose therefore that s > 0. Then M := \{k_L | L is a line of degree at most n\} is not empty. Let k be the maximal element of M and denote by G a line of degree k. From Lemma 3.1 and the conditions on v, we get k \geq n - 3. Let E_1, ..., E_{n+1} be the planes which contain G. We may assume that |E_j| \geq |E_1| for all j \in \{1, ..., n + 1\}. If we set \(w = |E_1| - k_G\), we have

\[v = k_G + \sum_{j=1}^{n+1} (|E_j| - k_G) \leq k_G + (n + 1)w.\]

Let q be any point outside of E_1. Then the map \(x \rightarrow p^x = pq\) for each point p of E_1, and \(L^x = \langle L, q \rangle\) for every line L of E_1, is a homomorphism of the linear space E_1 into the projective plane S/q. Hence, the linear space E_1 can be embedded into a projective plane of order n. Since S/p is a projective plane of order n for every point p, every point p of E_1 has degree \(n + 1\) in the linear space E_1.

Assume by way of contradiction that G is not a projective line of E_1. Then \(k_G \leq n - 1\), since a line of degree n of E_1 lies in a unique parallel class of E_1. If \(n \geq 13\), then Lemma 3.2(a) shows \(w \leq n^2 - 4n + 8\) and we conclude

\[v \leq n - 1 + (n + 1)(n^2 - 4n + 8) = n^3 - 3n^2 + 5n + 7,\]

which is not possible. Thus \(n \leq 12\). This shows that condition (ii) of Theorem 2 is not satisfied. Hence, condition (i) is fulfilled, i.e.,

\[v > n^3 - 3n^2 + 9n + 11 = n - 1 + (n + 1)(n^2 - 4n + 12).\]

In particular, \(w \geq n^2 - 4n + 13\). Since G is not a projective line of E_1, Lemma 3.2(b) implies \(k = n - 3\) and \(n \in \{7, 8\}\). However, if \(n \leq 8\), then

\[v \geq n^3 - 3n^2 + 9n + 12 \geq n^2 - 2n^2.\]

and Lemma 3.1 implies \(k \geq n - 2\).

This contradiction proves that G is a projective line of E_1. By Lemma 2.5, G is a good line of S so that Theorem 2.3 shows that S can be extended. This completes the proof of Theorem 2.

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