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$\mathcal{N} = 2$ supersymmetry partial breaking and tadpole anomaly

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Abstract

We consider the $U(1)^n$ extension of the effective $\mathcal{N} = 2$ supersymmetric $U(1) \times U(1)$ model of arXiv:1204.2141; and study the explicit relationship between partial breaking of $\mathcal{N} = 2$ supersymmetry constraint and D3 brane tadpole anomaly of type IIB string on Calabi–Yau threefolds in presence of H^{RR} and H^{NS} fluxes. We also comment on supersymmetry breaking in the particular $\mathcal{N} = 2U(1)$ Maxwell theory; and study its interpretation in connection with the tadpole anomaly with extra localised flux sources.

1. Introduction

Breaking $\mathcal{N} = 2$ supersymmetric quantum field theories in 4d space time at two different mass scales has been subject of interest for many years [1–19]; and references therein. This scenario is possible in 4d $\mathcal{N} = 2$ supergravity theory; but not with $\mathcal{N} = 2$ global supersymmetry suspected to live at lower energies below Planck scale. If gravity is neglected, superalgebra relations require that once one of the two *global* supercharges $Q^{\pm}_{\alpha} = (Q_\alpha, \tilde{Q}_\alpha)$ is broken; say Q^{-}_{α}, the second Q^{+}_{α} has to be broken too. However this constraint can be bypassed in the presence of magnetic Fayet–Iliopoulos (FI) terms induced by non-perturbative BPS states such as D-branes of type II

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strings. With non zero magnetic FI couplings, the supercurrent algebra develops a constant term that violates the $SU_R(2)$ R-symmetry of the supercharges [6], offering as a consequence a way to break $\mathcal{N}=2$ supersymmetry partially via gauginos instead of gravitinos [4,20–23]. This idea has been approached in past by using non-linear realisation of half of the eight supersymmetric charges [24–30]; but further developed recently in [31–33] by using $\mathcal{N}=1$ superspace QFT_4 method where a simple, but instructive, effective $\mathcal{N}=2$ supersymmetric abelian $U(1)^2$ model, with two breaking scales Λ_1 and Λ_2, has been engineered.

In this paper, we consider the partial breaking of $\mathcal{N}=2$ supersymmetry in the effective 4d $U(1)\times U(1)$ model of [31,32], to which we refer from now on as ADJ effective gauge theory; and study explicitly its relationship with D3 brane tadpole anomaly of type IIB string compactified on local Calabi–Yau threefolds (CY3). To deal with brane realisation of the ADJ construction, we first relax the rank of the abelian group symmetry by considering the effective $U(1)^2$ model as the leading prototype in the family of 4d $\mathcal{N}=2$ $U(1)^n$ gauge models indexed by $n\geq 2$; and then think of this set of abelian gauge models in terms of an effective low energy theory following from D3 branes wrapping 3-cycles in type IIB string on local CY3 with an n-dimensional symplectic homology basis of 3-cycles (A^a, B_a), $a=1,\dots,n$. In this D3 brane realisation of 4d $\mathcal{N}=2$ $U(1)^n$ gauge theory, partial breaking of global $\mathcal{N}=2$ supersymmetry is induced by H_3^{RR} and H_3^{NS} fluxes; and the ADJ condition $\sum_a \frac{g_a}{\kappa_a}=0$ supporting the partial breakings is interpreted in terms of conservation of the total 3-forms flux Φ_{flux} in the Calabi–Yau threefolds; that is $\Phi_{flux}=\int_{CY3} H_3^{NS}\wedge H_3^{RR}=0$. We also study the missing $n=1$ term in the sequence of 4d $\mathcal{N}=2$ $U(1)^n$ gauge models with $n\geq 2$; this particular model, which corresponds type IIB string on conifold geometry, is anomalous in agreement with known results in literature; this anomaly may be directly learnt from the naive extension of ADJ condition which is given by the singular equation $\frac{g}{\kappa}=0$ requiring the vanishing of the gauge coupling constant $g=0$ for finite magnetic FI coupling κ. By trying to engineer a 4d $\mathcal{N}=2$ $U(1)$ model going beyond ADJ constraint by deforming the singularity like $\nu+\frac{g}{\kappa}=0$, we end with an explicit breaking of $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$. The brane interpretation of this deformation in terms of presence of D7 branes or O3 planes is also studied by using a result from [34].

The presentation is as follows: In section 2, we review the basis of the effective 4d $\mathcal{N}=2$ $U(1)^2$ model; and derive the $\mathcal{N}=2$ ADJ constraint equation and its $\mathcal{N}=1$ deformation. In section 3, we give the main lines of the $\mathcal{N}=1$ superfield formulation of the 4d $\mathcal{N}=2$ $U(1)^n$ gauge theory describing the gauge dynamics of n $\mathcal{N}=2$ gauge multiplets coupled to a single tensor multiplet. In section 4, we study the realisation of ADJ model in type IIB string on CY3 with non-trivial fluxes of the 3-form field strengths H_3^{NS} and H_3^{RR}.

In Section 5, we give conclusion and make some comments. In Section 6 we give two appendices; in the first appendix we collect some useful tools on type IIB string compactification to 4d space time; and in the second we describe the gauge and supersymmetric transformations of single-tensor and Maxwell multiplets in superspace.

2. $\mathcal{N}=2$ $U(1)^n$ theory and ADJ constraints

In this section, we review the main lines of the model of [31,32]; and study particular aspects of ADJ constraint equation supporting the partial supersymmetry breaking in this theory. Following [31–33], the ADJ model is a 4d $\mathcal{N}=2$ supersymmetric effective gauge theory where gravity is decoupled; but global $\mathcal{N}=2$ supersymmetry is broken at two different scales. The simplest version of the model realising the two partial breaking is given by the interacting dynamics of an $\mathcal{N}=2$ single tensor multiplet $\mathcal{T}^{(\mathcal{N}=2)}$ with two abelian $\mathcal{N}=2$ gauge supermultiplets $\mathcal{V}_1^{(\mathcal{N}=2)}$

and $\mathcal{V}_2^{(\mathcal{N}=2)}$. Besides special Kahler geometry of the gauge background, the model has a Chern–Simons type interaction between $\mathcal{T}^{(\mathcal{N}=2)}$ and the linear combination $g_1 \boldsymbol{V}_1^{(\mathcal{N}=2)} + g_2 \boldsymbol{V}_2^{(\mathcal{N}=2)}$.

2.1. ADJ theory in $\mathcal{N}=1$ superspace

Because of lack of a simple formulation of 4d supersymmetric gauge theories with *8* supercharges in $\mathcal{N}=2$ superspace, one is limited to use the standard $\mathcal{N}=1$ superspace method with the price that only half of supersymmetries is manifestly exhibited; the other half is hidden; but can be linearly realised in absence of magnetic FI couplings.

2.1.1. Fibering $\mathcal{N}=2$ chiral superspace

A way to deal with the *4* hidden supersymmetric charges is to use $\mathcal{N}=2$ chiral superspace and think about it in terms of fibration of two copies of $\mathcal{N}=1$ chiral superspaces; an $\mathcal{N}_{fiber}=1'$ chiral superspace, with odd coordinates $\tilde{\theta}^\alpha$, fibered on an $\mathcal{N}_{base}=1$ chiral superspace base with odd coordinates θ^α. Schematically, this fibration may be represented like

$$\begin{array}{ccc} \mathcal{N}_{fiber}=1' & \rightarrow & \mathcal{N}=2 \\ & & \downarrow \\ & & \mathcal{N}_{base}=1 \end{array} \tag{2.1}$$

In this chiral superspace fibration, typical $\mathcal{N}=2$ chiral superfields have the structure $\boldsymbol{\Phi}^{\mathcal{N}=2} = \boldsymbol{\Phi}(z,\theta,\tilde{\theta})$ with space time coordinate z^μ related to the real x^μ by two pure imaginary shifts $i\upsilon^\mu + i\tilde{\upsilon}^\mu$, one from $\mathcal{N}_{fiber}=1'$ fiber and the other from the $\mathcal{N}_{base}=1$ base as shown on the relation $z = y - i\tilde{\theta}\sigma\widetilde{\bar{\theta}}$ with $y = x - i\theta\sigma\bar{\theta}$. Viewed from fiber, $\boldsymbol{\Phi}^{\mathcal{N}=2}$ can be expanded in a finite series of $\tilde{\theta}$ as follows

$$\boldsymbol{\Phi}^{\mathcal{N}=2} = \Phi^{\mathcal{N}=1} + \sqrt{2}\,\tilde{\theta}^\alpha \Psi_\alpha^{\mathcal{N}=1} + \tilde{\theta}^2 F^{\mathcal{N}=1} + \ldots \tag{2.2}$$

with expansion modes given by $\mathcal{N}_{base}=1$ superfields: $\Phi^{\mathcal{N}=1} = \Phi\,(y,\theta)$ and similarly for $\Psi_\alpha^{\mathcal{N}=1}$ and $F^{\mathcal{N}=1}$. The extra dots stand for additional terms involving space time derivatives generated by $-i\tilde{\theta}\sigma^\mu\widetilde{\bar{\theta}}\partial_\mu = -i\tilde{\upsilon}^\mu\partial_\mu$. The expansion modes in (2.2) describe $\mathcal{N}=1$ chiral superfields in the base; and are related amongst others by those $\mathcal{N}_{fiber}=1'$ supersymmetric transformations in the fiber; for example

$$\begin{aligned} \tilde{\delta}\Phi^{\mathcal{N}=1} &= \sqrt{2}\tilde{\varepsilon}^\alpha \Psi_\alpha^{\mathcal{N}=1} \\ \tilde{\delta}\Psi_\alpha^{\mathcal{N}=1} &= \sqrt{2}\tilde{\varepsilon}^\alpha F^{\mathcal{N}=1} - \frac{i\sqrt{2}}{2}\sigma^\mu\widetilde{\bar{\varepsilon}}\partial_\mu\Phi^{\mathcal{N}=1} \\ \tilde{\delta}F^{\mathcal{N}=1} &= -\frac{i\sqrt{2}}{2}\,\partial_\mu\Psi_\alpha^{\mathcal{N}=1}\sigma^\mu\widetilde{\bar{\varepsilon}} \end{aligned} \tag{2.3}$$

By imposing appropriate constraint relations on $\boldsymbol{\Phi}^{\mathcal{N}=2}$, one obtains the desired $\mathcal{N}_{base}=1$ superfields to describe supersymmetric matter or radiation with *8* supercharges. In this way of doing, $\mathcal{N}=2$ supersymmetric gauge multiplet is then approached by using superfield strength $\mathcal{W}^{\mathcal{N}=2}$ with expansion as in (2.2); but satisfying moreover $D^\alpha\tilde{D}_\alpha\mathcal{W}^{\mathcal{N}=2} + hc = 0$. As this constraint involves both the chiral $\mathcal{W}^{\mathcal{N}=2}$ and its adjoint conjugate, one ends with a $\tilde{\theta}$-expansion involving both $\mathcal{N}=1$ chiral $\boldsymbol{X}$ and $\bar{D}^2\bar{\boldsymbol{X}}$ as follows

$$\begin{aligned} \mathcal{W}^{\mathcal{N}=2} &= \boldsymbol{X} + i\sqrt{2}\,\tilde{\theta}^\alpha \boldsymbol{W}_\alpha - \tilde{\theta}^2\left(\tfrac{1}{4}\bar{D}^2\bar{\boldsymbol{X}}\right) \\ \tilde{\mathcal{W}}^{\mathcal{N}=2} &= \mathcal{W}^{\mathcal{N}=2} + \tilde{\theta}^2\tfrac{1}{2\kappa} \end{aligned} \tag{2.4}$$

where the role of the extra constant coefficient $\frac{1}{2\kappa}$ will be discussed later on; it scales as $mass^2$ seen that $\left[\mathcal{W}^{\mathcal{N}=2}\right] = [\boldsymbol{X}] = mass^1$ and the $\mathcal{N}_{base} = 1$ chiral gauge superfield strength spinor $[\boldsymbol{W}_\alpha] = mass^{3/2}$; it may be generated by the particular and *asymmetric* shift of the $\tilde{\theta}^2$ component

$$\frac{1}{4}\bar{D}^2\bar{\boldsymbol{X}}\Big|_{\theta=0} = \bar{F}^{\bar{X}} \qquad \rightarrow \qquad \bar{F}^{\bar{X}} - \frac{1}{2\kappa} \tag{2.5}$$

By asymmetric we mean eq. (2.5) but without modifying the $\boldsymbol{X}$ superfield in (2.4). This property may be roughly interpreted as giving a non-zero VEV to the $\tilde{\theta}^2$-component field of expansion of (2.4) as

$$\left\langle \frac{1}{4}\bar{D}^2\bar{\boldsymbol{X}} \right\rangle = \frac{1}{\sqrt{2}}\langle A + iB\rangle = -\frac{1}{2\kappa} \tag{2.6}$$

breaking thus the $\mathcal{N}_{fiber} = 1'$ supersymmetry in fiber. For a stringy interpretation of the coupling constant $\frac{1}{\kappa}$ in terms of 3-form flux through non compact 3-cycles in local CY3; see eq. (4.30).

A quite similar expansion is valid for the $\mathcal{N} = 2$ tensor multiplet $\mathcal{T}^{(\mathcal{N}=2)}$ which is described as well by a constrained $\mathcal{N} = 2$ chiral superfield [31–33], see also Appendix A.2; it is given by

$$\mathcal{T}^{\mathcal{N}=2} = \boldsymbol{Y} + \sqrt{2}\,\tilde{\theta}^\alpha \boldsymbol{\chi}_\alpha - \tilde{\theta}^2\left(\frac{i}{2}\boldsymbol{\Phi} + \frac{1}{4}\bar{D}^2\bar{\boldsymbol{Y}}\right) \tag{2.7}$$

with $\mathcal{N}_{fiber} = 1'$ supersymmetric transformations in fiber as

$$\begin{aligned} \tilde{\delta}\boldsymbol{Y} &= +\sqrt{2}\,\tilde{\epsilon}^\alpha \boldsymbol{\chi}_\alpha \\ \tilde{\delta}\boldsymbol{\chi}_\alpha &= \sqrt{2}\tilde{\epsilon}_\alpha \mathcal{E} - \tfrac{i}{\sqrt{2}}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\tilde{\epsilon}}^{\dot{\alpha}}\partial_\mu \boldsymbol{Y} \\ \tilde{\delta}\mathcal{E} &= \tfrac{\sqrt{2}}{2i}\partial_\mu \boldsymbol{\chi}_\alpha \sigma^\mu_{\alpha\dot{\alpha}}\bar{\tilde{\epsilon}}^{\dot{\alpha}} \end{aligned} \tag{2.8}$$

where we have set

$$\mathcal{E} = -\frac{i}{2}\boldsymbol{\Phi} - \frac{1}{4}\bar{D}^2\bar{\boldsymbol{Y}} \tag{2.9}$$

With these tools at hand, we turn to describe useful features on superfield spectrum of $\mathcal{N} = 2$ supersymmetric ADJ model. For later use, we will give both the $\mathcal{N} = 2$ chiral superfields spectrum and their splitting in terms of $\mathcal{N}_{base} = 1$ superfields.

More on matter sector The matter sector of ADJ model is quite simple; it involves one 4d $\mathcal{N} = 2$ matter multiplet having two dual realisations as given by eq. (2.11).

In the realisation we will be using in present study, $\mathcal{N} = 2$ matter is described by an $\mathcal{N} = 2$ chiral superfield $\mathcal{T}^{(\mathcal{N}=2)}$ with expansion along fiber direction as in (2.7). From the $\mathcal{N}_{base} = 1$ base view, this expansion has four chiral superfields: two bosonic $\boldsymbol{Y}$, $\boldsymbol{\Phi}$; and a fermionic superfield doublet $\boldsymbol{\chi}_\alpha = (\boldsymbol{\chi}_1, \boldsymbol{\chi}_2)$; altogether they capture $16 + 16$ off shell degrees of freedom. This number may be reduced down to $8 + 8$ by thinking about $\boldsymbol{Y}$ as an exotic auxiliary superfield playing the role of a Lagrange superfield parameter capturing the constraint on partial breaking of second supersymmetry; and about $\boldsymbol{\chi}_\alpha$ as a superfield prepotential of a hermitian linear multiplet $\boldsymbol{L}$ given by the relation

$$\boldsymbol{L} = D^\alpha \boldsymbol{\chi}_\alpha + \bar{D}_{\dot{\alpha}}\bar{\boldsymbol{\chi}}^{\dot{\alpha}} \tag{2.10}$$

Observe that $\boldsymbol{L}$ is invariant under the change $\boldsymbol{\chi}'_\alpha = \boldsymbol{\chi}_\alpha + \frac{i}{4}\bar{D}^2 D_\alpha \Omega$ with Ω an arbitrary real superfield; this symmetry together with footnote 1 allows to reduce the $16 + 16$ degrees of freedom down to $8 + 8$; for details see Appendix A.2; other features can be found in [31–33].

For completeness, notice that using a result on hypermultiplet duality on superfield representations of $\mathcal{N}=2$ matter multiplet [35–39], we can show that $\mathcal{N}=2$ superfield $\mathcal{T}^{(\mathcal{N}=2)}$ has two dual representations in terms of $\mathcal{N}_{base}=1$ superfields; the $(\boldsymbol{\Phi}, \boldsymbol{L})$ we will be using in this paper; and a second realisation based on two chiral superfields $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$;

$$\begin{array}{ll} a) : \mathcal{T}^{(\mathcal{N}=2)} \equiv \boldsymbol{\Phi}, \boldsymbol{L} & ; \boldsymbol{Y} \\ b) : \mathcal{T}^{(\mathcal{N}=2)} \equiv \boldsymbol{Q}_1, \boldsymbol{Q}_2 & ; \boldsymbol{Y} \end{array} \tag{2.11}$$

ADJ constraint First notice that in above (2.11), it looks like if we have three $\mathcal{N}_{base}=1$ superfields to describe $\mathcal{N}=2$ matter; this is not exact since $\boldsymbol{Y}$ is some how a "spurious superfield" carrying no physical degrees of freedom; it is a topological object exhibiting very special properties as shown by eqs. (2.19)–(2.21); this is our reason behind putting $\boldsymbol{Y}$ aside in eq. (2.11); it breaks $\mathcal{N}=2$ supersymmetry partially and is one of the nice observations in [32]; there it appears as a Lagrange superfield parameter capturing a constraint relation $f(g_a, \kappa_a)=0$ of the model, which to fix ideas may be thought of as

$$f(g_a, \kappa_a) = \sum_{a=1}^{n} \frac{g_a}{\kappa_a} = 0 \qquad , \qquad n \geq 2 \tag{2.12}$$

giving a relationship between the coupling constants g_a and the magnetic FI couplings $1/\kappa_a$ of the ADJ model; see also eq. (3.5) given below for explicit details. In the limit

$$\frac{1}{\kappa_a} \to 0 \tag{2.13}$$

the above constraint is trivially solved and then $\boldsymbol{Y}$ has no role to play in this $\mathcal{N}=2$ supersymmetric limit.

Notice also that the sum on integer n in eq. (2.12) rules out the particular case $n=1$; since the corresponding condition reads as

$$\frac{g_1}{\kappa_1} = 0 \tag{2.14}$$

leading to $g_1 = 0$ for finite $1/\kappa_1$; and then no ADJ theory with one $U(1)$ gauge factor [29–33]. By trying to overcome the constraint $\frac{g_1}{\kappa_1}=0$ by adding an extra term like

$$\nu + \frac{g_1}{\kappa_1} = 0 \tag{2.15}$$

with ν a real parameter having same scaling mass dimension as $\frac{g_a}{\kappa_a}$; one breaks *explicitly* $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$.

2.1.2. $\mathcal{N}=1$ *superfields in* $U(1)^2$ *ADJ model*

In our superspace description of $U(1) \times U(1)$ ADJ model, we will use a particular set of $\mathcal{N}=1$ superfields; these are the chiral $\boldsymbol{\Phi}$ and hermitian $\boldsymbol{L}$ for representing $\mathcal{T}^{(\mathcal{N}=2)}$; and 2 gauge superfields $(\boldsymbol{V}_1, \boldsymbol{V}_2)$, 2 chiral $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ for representing $\mathcal{V}_1^{(\mathcal{N}=2)} \oplus \mathcal{V}_2^{(\mathcal{N}=2)}$. Let us comment this system of superfields.

• $\mathcal{T}^{(\mathcal{N}=2)}$ *sector*

In the $\mathcal{N}=1$ superfield realisation given by the first relation of eq. (2.11), the dynamics of $\mathcal{T}^{(\mathcal{N}=2)}$ is described by two basic superfields and an auxiliary one; these are:

(i) the chiral superfield $\mathbf{\Phi}$ with the usual θ-expansion namely a leading scalar component ϕ; a Weyl fermions ψ_α and auxiliary field F_ϕ;

(ii) the standard hermitian linear multiplet $\boldsymbol{L}$ satisfying the superspace constraint relations $D^2\boldsymbol{L} = \bar{D}^2\boldsymbol{L} = \mathbf{0}$ following from (2.10); this is a particular superfield with θ-expansion in component fields as follows

$$\begin{aligned}\boldsymbol{L} \;=\; & C + i\theta.\eta - i\bar{\theta}.\bar{\eta} + \theta\sigma^\mu\bar{\theta}\varepsilon_{\mu\nu\rho\sigma}\partial^\nu B^{\rho\sigma} + \\ & \tfrac{1}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu\eta - \tfrac{1}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{\eta} - \tfrac{1}{4}\theta^2\bar{\theta}^2\Box C\end{aligned} \tag{2.16}$$

involving the propagating real field C and the field strength of the antisymmetric tensor field $B^{\rho\sigma}$; but no auxiliary field. The superfields $\mathbf{\Phi}$ and $\boldsymbol{L}$ are related under fiber $\mathcal{N}_{fiber} = 1'$ supersymmetric variations as follows

$$\begin{aligned}\tilde{\delta}_\epsilon \boldsymbol{L} &= \frac{\sqrt{2}}{2i}\left(\tilde{\epsilon}^\alpha D_\alpha \mathbf{\Phi} - \widetilde{\bar{\epsilon}}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\bar{\mathbf{\Phi}}\right) \\ \tilde{\delta}_\epsilon \mathbf{\Phi} &= -i\sqrt{2}\widetilde{\bar{\epsilon}}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\boldsymbol{L}\end{aligned} \tag{2.17}$$

with

$$\left[\tilde{\delta}_{\epsilon'}, \tilde{\delta}_\epsilon\right]\mathbf{\Psi} = -2i\left(\tilde{\epsilon}\sigma^\mu\widetilde{\bar{\epsilon}}' - \tilde{\epsilon}'\sigma^\mu\widetilde{\bar{\epsilon}}\right)\partial_\mu\mathbf{\Psi} \tag{2.18}$$

with $\mathbf{\Psi}$ standing for $\boldsymbol{L}$ and $\mathbf{\Phi}$.

(iii) an extra auxiliary superfield $\boldsymbol{Y}$ capturing information on non linear realisation of the $\mathcal{N}_{fiber} = 1'$ hidden supersymmetry; it is not needed for the closure of transformations (2.17); but will be used to approach partial supersymmetry breaking. Properties of this superfield have been explored in [31,32] where, using gauge fixing[1] method, it has been shown to have the following remarkable θ-expansion

$$\boldsymbol{Y}^{gauged} = \frac{i}{4!}\theta^2\varepsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} \tag{2.19}$$

for details see Appendix A.2. This relation shows that $\boldsymbol{Y}$ encodes data on the constant antisymmetric tensor $C_{\mu\nu\rho\sigma} = 4!\varepsilon_{\mu\nu\rho\sigma}\Delta$. Because of its special dependence in θ, $\boldsymbol{Y}$ has no physical degrees of freedom

$$\delta_\epsilon \boldsymbol{Y}^{gauged} = \mathbf{0} \tag{2.20}$$

and obeys moreover a nilpotency property

$$\boldsymbol{Y}^\dagger\boldsymbol{Y} = \theta^4\Delta^2 \qquad , \qquad \boldsymbol{Y}\boldsymbol{Y} = 0 = \boldsymbol{Y}^\dagger\boldsymbol{Y}^\dagger \tag{2.21}$$

In the $\mathcal{N} = 1$ superfield realisation given by the second relation of eq. (2.11), the role of $\mathbf{\Phi}$ and $\boldsymbol{L}$ gets played by the two chiral superfields $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ capturing opposite charge under a $U(1)$ gauge symmetry. The duality transformations between the two matter multiplet realisations are given by Legendre transform in superspace [35–39]; they may be written as follows

[1] Following analysis of appendix 8.2, the reduction of the $16+16$ degrees of $\mathcal{N} = 2$ chiral superfield $\mathcal{T}^{N=2}$ down to $8+8$ is achieved into two steps: a first reduction from $16+16$ down to $12+12$ ensured by gauge symmetry under $\chi'_\alpha = \chi_\alpha + \frac{i}{4}\bar{D}^2 D_\alpha\Omega$ (A.23); a second reduction from $12+12$ down to $8+8$ given by requiring symmetry under gauge transformation $Y \to Y - \frac{1}{2}\bar{D}^2\Upsilon$ with Υ a real superfield. Gauge fixing of this symmetry leads precisely to eq. (2.19); for more details see also [31].

$$
\begin{aligned}
\boldsymbol{Q}_1 &= 2^{\frac{-1}{4}}\sqrt{\boldsymbol{\Phi}}e^{+\boldsymbol{\Phi}'} \\
\boldsymbol{Q}_2 &= 2^{\frac{-1}{4}}\sqrt{\boldsymbol{\Phi}}e^{-\boldsymbol{\Phi}'}
\end{aligned}
\tag{2.22}
$$

with $\boldsymbol{\Phi}$ as in the first relation of eq. (2.11) and where $\boldsymbol{\Phi}'$ is another chiral superfield. We will not need this realisation in this paper; but to fix ideas we give some comments on their dynamics in §3.2; see eq. (3.18).

- $\mathcal{V}_a^{(\mathcal{N}=2)}$ *gauge sector*

The gauge sector of ADJ supersymmetric $U(1)^2$ model involves two $\mathcal{N}=2$ abelian Maxwell type multiplets described by the hermitian superfields $\boldsymbol{V}_1^{(\mathcal{N}=2)}$ and $\boldsymbol{V}_2^{(\mathcal{N}=2)}$ with superfields strength θ-expansions along fiber direction as in eq. (2.2). Following [31,32], the solution of constraint equations lead to the $\mathcal{N}=1$ superfields spectrum

$$
\begin{aligned}
\boldsymbol{V}_1^{(\mathcal{N}=2)} &\equiv \boldsymbol{V}_1, \boldsymbol{X}_1 \ , \ \boldsymbol{\kappa}_1 \\
\boldsymbol{V}_2^{(\mathcal{N}=2)} &\equiv \boldsymbol{V}_2, \boldsymbol{X}_2 \ , \ \boldsymbol{\kappa}_2
\end{aligned}
\tag{2.23}
$$

where the hermitian $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ are the usual $\mathcal{N}_{base}=1$ gauge superfield potentials; and where $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ are two chiral superfields. So the gauge symmetry of the model is $U_1(1)\times U_2(1)$. The extra $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ are constants and are as in (2.4); they may be put in correspondence with the auxiliary chiral superfield $\boldsymbol{Y}$ as it may be viewed by comparing (2.7) with (2.4); that is:

$$
\sum \frac{g_a}{\boldsymbol{\kappa}_a} \qquad \leftrightarrow \qquad \boldsymbol{Y}
\tag{2.24}
$$

The general form of the superspace lagrangian density $\mathcal{L}_{gauge}$ of the gauge superfields depending on the prepotential $\mathcal{F}(\boldsymbol{X}_1,\boldsymbol{X}_2)$ reads as follows

$$
\mathcal{L}_{gauge} = \mathcal{L}_{U_1(1)\times U_2(1)} + \mathcal{L}_{FI}
\tag{2.25}
$$

with gauge lagrangian density in $\mathcal{N}_{base}=1$ superspace given by

$$
\mathcal{L}_{U_1(1)\times U_2(1)} = \int d^4\theta \frac{i}{2}\left(\bar{\mathcal{F}}_a \boldsymbol{X}^a - \mathcal{F}_a \bar{\boldsymbol{X}}^a\right) + \int d^2\theta \left(-\frac{i}{4}\mathcal{F}_{ab} W^a.W^b\right) + hc
\tag{2.26}
$$

and Fayet–Iliopoulos part as

$$
\mathcal{L}_{FI} = \int d^4\theta \xi_a \boldsymbol{V}^a - \int d^2\theta \frac{e_a}{4}\boldsymbol{X}^a - \int d^2\theta \frac{i}{4\kappa_a}\mathcal{F}_a + hc
\tag{2.27}
$$

where the real ξ_a and complex (pure imaginary) e_a are constants and where the holomorphic $\mathcal{F}_a = \frac{\partial \mathcal{F}}{\partial X^a}$ and $\mathcal{F}_{ab} = \frac{\partial^2 \mathcal{F}}{\partial X^a \partial X^b}$.

2.2. *ADJ constraint and $\mathcal{N}=1$ deformation*

Here we use the $\mathcal{N}=1$ superfield spectrum of ADJ model to study the derivation of the constraint eq. (2.12) and its $\mathcal{N}=1$ deformation (2.15).

2.2.1. *Superfield $\boldsymbol{Y}$*

Viewed from $\mathcal{N}=2$ chiral superspace, the $U(1)\times U(1)$ ADJ supersymmetric model involves the $\mathcal{N}=2$ chiral superfields $\mathcal{W}_a^{\mathcal{N}=2}$ given by (2.4); and the $\mathcal{T}^{\mathcal{N}=2}$ of (2.7). These $\mathcal{N}=2$ chiral superfields are remarkable; they have the same scaling mass dimension and quite similar $\tilde{\theta}$-expansions which make them to share some general features. Indeed, though physically different objects, the resemblance between their $\tilde{\theta}$-expansions could serve as a guide to have more insight into the ADJ construction. This formal property has been used in [31,25] to study the interaction between an $\mathcal{N}=2$ Maxwell multiplet $\mathcal{W}^{\mathcal{N}=2}$ and a tensor $\mathcal{T}^{\mathcal{N}=2}$. There, the formal similarity between the two chiral superfields $\mathcal{W}^{\mathcal{N}=2}$ and $\mathcal{T}^{\mathcal{N}=2}$; in particular their scaling mass dimension and θ-expansions, has been used to build the linear combination of these $\mathcal{N}=2$ chiral superfield

$$\mathcal{W}^{\mathcal{N}=2}+2g\mathcal{T}^{\mathcal{N}=2} \tag{2.28}$$

to reach the gauge invariant quantity

$$\mathcal{F}_{\mu\nu}^{Max}-gB_{\mu\nu} \tag{2.29}$$

that plays a central role in the $\mathcal{N}=2$ Dirac–Born–Infeld $U_{\max}(1)$ theory; and also in studying electric–magnetic duality in $\mathcal{N}=2$ chiral superspace in presence of Chern–Simons coupling. In this relation, $\mathcal{F}_{\mu\nu}^{Max}$ is the usual field strength of the Maxwell gauge field potential; and $B_{\mu\nu}$ the antisymmetric gauge potential appearing in the tensor multiplet.

By exhibiting this formal similarity between $\mathcal{W}^{\mathcal{N}=2}$ and $\mathcal{T}^{\mathcal{N}=2}$; one finds that there exist a correspondence between their $\mathcal{N}=1$ superfields contents; by comparing the $\tilde{\theta}$-expansions (2.4) and (2.7); as well as topological relations reported in Appendix A (A.19) and (A.24), one ends with

Gauge multiplet $\tilde{\mathcal{W}}^{\mathcal{N}=2}$	Tensor multiplet $\mathcal{T}^{\mathcal{N}=2}$
$\boldsymbol{X}$	$\boldsymbol{Y}$
$i\boldsymbol{W}_\alpha$	$\boldsymbol{\chi}_\alpha$
$\frac{-1}{2\kappa}$	$\frac{i}{2}\boldsymbol{\Phi}$
$\bar{D}^2\bar{\boldsymbol{X}}$	$\bar{D}^2\bar{\boldsymbol{Y}}$

(2.30)

where $\boldsymbol{Y}$ occupies a place in $\mathcal{T}^{\mathcal{N}=2}$ that is similar to the place occupied by $\boldsymbol{X}$ in $\tilde{\mathcal{W}}^{\mathcal{N}=2}$. Obviously the superfields in left and right of table (2.30) have different meanings and carry different degrees of freedom; but as far as fibration of $\mathcal{N}=2$ supersymmetry is concerned; this correspondence may be used as an indication to get more insight into the general form of constraint equation captured by $\boldsymbol{Y}$.

2.2.2. *Deriving ADJ condition*

The ADJ constraint equation is obtained by from $\mathcal{N}=2$ Chern–Simons couplings between the linear combination of the gauge superfield strengths ($\sum_a g_a\tilde{\mathcal{W}}_a^{\mathcal{N}=2}$) and the tensor multiplet $\mathcal{T}^{\mathcal{N}=2}$. In $\mathcal{N}=2$ chiral superspace where $\mathcal{N}=2$ supersymmetry is manifest, this CS coupling reads in terms of $\tilde{\mathcal{W}}_a^{\mathcal{N}=2}$ and $\mathcal{T}^{\mathcal{N}=2}$ as follows

$$\mathcal{L}_{CS}=-2i\int d^2\theta d^2\tilde{\theta}\left(\sum_{a=1}^{n}g_a\tilde{\mathcal{W}}_a^{\mathcal{N}=2}\right)\mathcal{T}^{\mathcal{N}=2} \tag{2.31}$$

where $n=2$ for the case of $U(1)^2$ model; but can generally take any value $n\geq 2$ as the case of $U(1)^n$ models with n gauge $\tilde{\mathcal{W}}_a^{\mathcal{N}=2}$ coupled to $\mathcal{T}^{\mathcal{N}=2}$. By using (2.4) and (2.7) and performing integration with respect to $\tilde{\theta}$; one brings above CS coupling to the form

$$\mathcal{L}_{CS} = +2\int d^4\theta\left(\sum_{a=1}^{n} g_a \boldsymbol{V}^a\right)\boldsymbol{L} - \int d^2\theta\left(\sum_{a=1}^{n} g_a \boldsymbol{X}^a\right)\boldsymbol{\Phi}$$
$$- i\int d^2\theta\left(\sum_{a=1}^{n}\frac{g_a}{\kappa^a}\right)\boldsymbol{Y} \tag{2.32}$$

where $\mathcal{N}=1$ supersymmetry in the base of fibration is manifest and the fibered $\mathcal{N}=1'$ one becomes hidden. Because of linear dependence, the superfield equation of $\boldsymbol{Y}$ leads to the constraint

$$\sum_{a=1}^{n}\frac{g_a}{\kappa^a} = 0 \qquad , \qquad n \geq 2 \tag{2.33}$$

Notice that the deformation of the CS coupling (2.32) by adding the term $\nu\int d^2\theta\boldsymbol{Y}$ like

$$\mathcal{L}'_{CS} = +2\int d^4\theta\left(\sum_{a=1}^{n} g_a \boldsymbol{V}^a\right)\boldsymbol{L} - \int d^2\theta\left(\sum_{a=1}^{n} g_a \boldsymbol{X}^a\right)\boldsymbol{\Phi}$$
$$- i\int d^2\theta\left(\nu + \sum_{a=1}^{n}\frac{g_a}{\kappa^a}\right)\boldsymbol{Y} \tag{2.34}$$

preserves gauge symmetry as shown by (A.41) of Appendix A; but breaks explicitly $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$. Under this deformation, the ADJ constraint becomes

$$\nu + \sum_{a=1}^{n}\frac{g_a}{\kappa^a} = 0 \tag{2.35}$$

3. Superspace lagrangian

The $\mathcal{N}=1$ superspace expression of the lagrangian density $\mathcal{L}$ describing the interacting dynamics of the above $\mathcal{N}=2$ supersymmetric system $\left\{\mathcal{W}_a^{(\mathcal{N}=2)}, \mathcal{T}^{(\mathcal{N}=2)}\right\}$ can be approached in two manners depending on the $\mathcal{N}=1$ superfield realisation used to represent the $\mathcal{T}^{(\mathcal{N}=2)}$ single tensor multiplet.

3.1. Using $\mathcal{N}=1$ multiplets $(\boldsymbol{L}, \boldsymbol{\Phi})$

With the realisation of the tensor multiplet $\mathcal{T}^{(\mathcal{N}=2)}$ in terms of the superfields the chiral $\boldsymbol{\Phi}$, the hermitian $\boldsymbol{L}$ as well as the Lagrange chiral superfield $\boldsymbol{Y}$ carrying the ADJ constraint; and following [32], the superspace lagrangian density of the $\mathcal{N}=2$ supersymmetric $U(1)^2$ model describing coupled dynamics of $\mathcal{W}_a^{(\mathcal{N}=2)}$ and $\mathcal{T}^{(\mathcal{N}=2)}$ reads in $\mathcal{N}=1$ superspace as follows

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{ST} + \mathcal{L}_{CS} \tag{3.1}$$

with $\mathcal{L}_{gauge}$ as in eq. (2.26) and

$$\mathcal{L}_{ST} + \mathcal{L}_{CS} = \int d^4\theta\sqrt{\boldsymbol{L}^2 + 2\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi}} - \boldsymbol{L}\ln\left(\boldsymbol{L} + \sqrt{\boldsymbol{L}^2 + 2\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi}}\right)$$
$$+ 2\int d^4\theta g_a \boldsymbol{V}^a \boldsymbol{L} - \int d^2\theta\left(m + g_a \boldsymbol{X}^a\right)\boldsymbol{\Phi}$$
$$- i\int d^2\theta\left(\frac{g_1}{\kappa_1} + \frac{g_2}{\kappa_2}\right)\boldsymbol{Y} + hc \tag{3.2}$$

In what follows, we first make few comments useful for our later analysis; then we study scalar potential of the gauge model.

3.1.1. Properties of (3.2)

From the above expression of the superspace lagrangian density (3.2), we learn a set of special properties on the superfield realisation of matter using single tensor multiplet; in particular the following:

First, the mass constant m in (3.2) can be absorbed by shifting the linear combination $g_a \boldsymbol{X}^a$; it will be dropped out in what follows.

Second, the superfields $\boldsymbol{\Phi}$ and $\boldsymbol{L}$ are gauge invariant and scale as *mass*^{2}; their coupling to the gauge multiplet is of Chern–Simons type

$$2\int d^4\theta\,(g_1\boldsymbol{V}_1 + g_2\boldsymbol{V}_2)\,\boldsymbol{L} - \int d^2\theta\,(g_1\boldsymbol{X}_1 + g_2\boldsymbol{X}_2)\,\boldsymbol{\Phi} \tag{3.3}$$

they involve the remarkable linear combinations $g_1\boldsymbol{V}_1 + g_2\boldsymbol{V}_2$ and $g_1\boldsymbol{X}_1 + g_2\boldsymbol{X}_2$. Moreover, the contribution of the superfield $\boldsymbol{Y}$ in the full superspace lagrangian density (3.2) appears linearly as follows

$$i\int d^2\theta\left(\frac{g_1}{\kappa_1} + \frac{g_2}{\kappa_2}\right)\boldsymbol{Y} + hc \tag{3.4}$$

together with the particular linear combination $\frac{g_1}{\kappa_1} + \frac{g_2}{\kappa_2}$. So the auxiliary superfield $\boldsymbol{Y}$ in ADJ theory plays the role of a Lagrange superfield capturing the constraint relation

$$\frac{g_1}{\kappa_1} + \frac{g_2}{\kappa_2} = 0 \tag{3.5}$$

showing that the ratio $\frac{g_1}{g_2}$ of the two gauge couplings is fixed by the ratio $\frac{\kappa_1}{\kappa_2}$ of the magnetic FI coupling constants.

Third, the kinetic energy density of $\boldsymbol{\Phi}$ and $\boldsymbol{L}$ involves non-polynomial expressions, a square root term $\sqrt{\boldsymbol{L}^2 + 2\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi}}$ and a logarithm one namely $\boldsymbol{L}\ln\left(\boldsymbol{L} + \sqrt{\boldsymbol{L}^2 + 2\bar{\boldsymbol{\Phi}}\boldsymbol{\Phi}}\right)$; this non-linearity may be understood as due to the antisymmetric field $B_{\mu\nu}$. Self interactions of $(\boldsymbol{\Phi}, \boldsymbol{L})$ are also non-polynomial and are generally characterised by an arbitrary hermitian prepotential $H\left(\boldsymbol{\Phi}, \bar{\boldsymbol{\Phi}}; \boldsymbol{L}\right)$ with superspace lagrangian density as [35–37]

$$\mathcal{L}_{ST}^{(H)} = \int d^4\theta H\left(\boldsymbol{\Phi}, \bar{\boldsymbol{\Phi}}; \boldsymbol{L}\right) \tag{3.6}$$

3.1.2. Scalar potential

The scalar potential of the ADJ model (3.1)–(3.2) has two contributions as follows

$$\mathcal{V}_{sca} = \mathcal{V}_{gauge} + \mathcal{V}_{tens} \tag{3.7}$$

a contribution $\mathcal{V}_{gauge}$ coming from the auxiliary fields F^{X^a} and D^a of the gauge multiplets; and another contribution $\mathcal{V}_{tens}$ coming from the auxiliary field $\boldsymbol{\Phi}$; seen that $\boldsymbol{L}$ has no auxiliary field. The $\mathcal{V}_{gauge}$ contribution reads explicitly as

$$\mathcal{V}_{gauge} = H_{ab}\left(F^{X^a}\bar{F}^{\bar{X}^b} + \frac{1}{2}D^a D^b\right) \tag{3.8}$$

where $H_{ab} = \operatorname{Im}\mathcal{F}_{ab}$ is the metric of the special-Kähler manifold with inverse H^{ab}. For the $\mathcal{V}_{tens}$ contribution, we have

$$\mathcal{V}_{tens} = F^{\phi} G_{\phi\bar{\phi}} \bar{F}^{\bar{\phi}} \tag{3.9}$$

where $G_{\phi\bar{\phi}}$ is the analogue of metric H_{ab} for the matter sector.

Substituting the various auxiliary fields by their field equations, we obtain the explicit expression of the full scalar potential of the model. For the contribution $\mathcal{V}_{gauge}$, we have

$$\mathcal{V}_{gauge} = H^{ab} \left(\frac{1}{2} r_a r_b + w_a \bar{w}_b \right) \tag{3.10}$$

with real r_a and complex w_a as follows

$$\begin{aligned} r_a &= g_a C + \tfrac{\xi_a}{2} \\ w_a &= g_a \phi + \tfrac{1}{4} e_a + \tfrac{i}{4\kappa_c} \mathcal{F}_{ac} \end{aligned} \tag{3.11}$$

Besides FI coupling constants, they depend on the degrees of freedom of the tensor multiplet namely C and ϕ. The other contribution is given by

$$\mathcal{V}_{tensor} = 2\varrho^2 \, |gX|^2 \tag{3.12}$$

where we have set

$$\varrho^2 = \sqrt{C^2 + 2\,|\phi|^2} \qquad , \qquad gX = g_1 X_1 + g_2 X_2 \tag{3.13}$$

So the total scalar potential reads as

$$\mathcal{V}_{sca} = H^{ab} \left(\frac{1}{2} r_a r_b + w_a \bar{w}_b \right) + 2\,|gX|^2 \varrho^2 \tag{3.14}$$

Observe the two following features: first for $\varrho^2 = 0$ and $w_a = 0$, the scalar potential $\mathcal{V}_{sca}$ has a non-zero value due to the non-vanishing ξ and hence $\mathcal{N} = 2$ supersymmetry breaks down. For $\varrho^2 = 0$ and $\xi_a = e_a = 0$, the scalar potential $\mathcal{V}_{sca}$ has as well a non-zero value proportional to the magnetic FI coupling as shown on the following expression

$$\mathcal{V}_{sca} = \sum \frac{1}{16\kappa_c \kappa_d} \mathcal{F}_{ca} H^{ab} \bar{\mathcal{F}}_{bd} \tag{3.15}$$

The stationarity condition of the scalar potential with respect to the various fields namely

$$\frac{\partial \mathcal{V}_{sca}}{\partial X} = 0 \qquad , \qquad \frac{\partial \mathcal{V}_{sca}}{\partial \phi} = 0 \qquad , \qquad \frac{\partial \mathcal{V}_{sca}}{\partial C} = 0 \tag{3.16}$$

leads, for the case $\left\langle C^2 + 2\,|\phi|^2 \right\rangle = 0$, to the following equation

$$\mathcal{F}_{abc} \left[F^{x^b} \left(\bar{F}^{\bar{x}^c} + \frac{1}{2\kappa_c} \right) + \frac{1}{2} D^b D^c \right] = 0 \tag{3.17}$$

leading to broken supersymmetric phase for the case where $\mathcal{F}_{abc} \neq 0$.

3.2. Using ($\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$) hypermultiplet

Using duality transformations (2.22), one can also express the $\mathcal{N} = 2$ tensor multiplet $\mathcal{T}^{(\mathcal{N}=2)}$ as a hypermultiplet described by two $\mathcal{N} = 1$ chiral superfields $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$. In this realisation of $\mathcal{T}^{(\mathcal{N}=2)}$, the previous superspace density (3.2) gets mapped to the equivalent expression

$$\mathcal{L}_{hyp} = \int d^4\theta \left(\bar{\boldsymbol{Q}}_1 e^{g_1 \boldsymbol{V}_1 + g_2 \boldsymbol{V}_2} \boldsymbol{Q}_1 + \bar{\boldsymbol{Q}}_2 e^{-g_1 \boldsymbol{V}_1 - g_2 \boldsymbol{V}_2} \boldsymbol{Q}_2 \right)$$
$$+ \int d^2\theta \left(m + i\sqrt{2} g_1 \boldsymbol{X}_1 + i\sqrt{2} g_2 \boldsymbol{X}_2 \right) \boldsymbol{Q}_1 \boldsymbol{Q}_2$$
$$- i \int d^2\theta \left(\frac{g_1}{\kappa_1} + \frac{g_2}{\kappa_2} \right) \boldsymbol{Y} + hc \tag{3.18}$$

where the superfields $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ carry opposite charges under the $U_1(1) \times U_2(1)$ gauge symmetry; but $\boldsymbol{Y}$ playing the same role.

The scalar potential of this superfield realisation of the gauge theory is given by

$$\mathcal{V}'_{sca} = H_{ab} \left[F^{x^a} \bar{F}^{\bar{x}^b} + \frac{1}{2} D^a D^b \right] + G_{u\bar{v}} F^{q^u} \bar{F}^{\bar{q}^{\bar{v}}} \tag{3.19}$$

It has the same form as (3.14)

$$\mathcal{V}'_{sca} = \frac{1}{2} r'_a H^{ab} r'_b + w'_a H^{ab} \bar{w}'_b + \varrho'^2 \left| m + \sqrt{2} i g_a x^a \right|^2 \tag{3.20}$$

but now with the dual expressions

$$\begin{aligned} r'_a &= -g_a \left(\left|q^1\right|^2 - \left|q^2\right|^2 \right) + \tfrac{\xi_a}{2} \\ w'_a &= \tfrac{\sqrt{2} g_a}{i} q^1 q^2 + \tfrac{1}{4} e_a + \tfrac{i}{4\kappa_c} \mathcal{F}_{ac} \end{aligned} \tag{3.21}$$

and

$$\varrho'^2 = \left|q^1\right|^2 + \left|q^2\right|^2 \tag{3.22}$$

The properties of the scalar potential (3.20), including the description of the two scale breakings of $\mathcal{N} = 2$ supersymmetry, have been explicitly studied in [32].

4. ADJ model and tadpole anomaly

In this section, we study a D3 brane realisation of ADJ theory and the interpretation of partial supersymmetric breaking in terms of 3-forms fluxes through 3-cycles in CY3. This brane realisation has been succinctly presented in the introduction section; here we use results on type IIB string on local CY3s to describe the underlying geometry and the nature of H_3^{RR}, H_3^{NS} fluxes behind $\mathcal{N} = 2$ ADJ model. To reach this goal, we first examine the geometric property the linear combinations of abelian gauge superfields; then we study the geometric derivation of the $\mathcal{N} = 2$ ADJ condition and its $\mathcal{N} = 1$ deformation given by (2.35); and after we give the explicit relationship between ADJ condition and D3 tadpole cancellation anomaly in type IIB.

4.1. $\mathcal{N} = 2$ ADJ model and 3-cycles in CY3

In the effective $\mathcal{N} = 2$ supersymmetric $U(1) \times U(1)$ gauge model, the superspace lagrangian density $\mathcal{L}_{U(1)^2}^{\mathcal{N}=2}$ depends, in addition to the single tensor multiplet $\mathcal{T}^{\mathcal{N}=2} = (\boldsymbol{L}, \boldsymbol{\Phi}, \boldsymbol{Y})$, on two $\mathcal{N} = 2$ abelian gauge multiplets $\mathcal{W}_{1,2}^{\mathcal{N}=2} = (\boldsymbol{X}_1, \boldsymbol{V}_1), (\boldsymbol{X}_2, \boldsymbol{V}_2)$.

4.1.1. Linear combinations

By an inspection of the superspace density (3.1), one notices that $\mathcal{L}_{U(1)^2}^{\mathcal{N}=2}$ depends on the following superfield linear combinations

$$\begin{aligned} \boldsymbol{V} &= g_1 \boldsymbol{V}_1 + g_2 \boldsymbol{V}_2 \\ \boldsymbol{V}' &= \xi_1 \boldsymbol{V}_1 + \xi_2 \boldsymbol{V}_2 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \boldsymbol{X} &= g_1 \boldsymbol{X}_1 + g_2 \boldsymbol{X}_2 \\ \boldsymbol{X}' &= e_1 \boldsymbol{X}_1 + e_2 \boldsymbol{X}_2 \\ \frac{\partial \mathcal{F}}{\partial X} &= \frac{1}{\boldsymbol{\kappa}_1} \frac{\partial \mathcal{F}}{\partial X_1} + \frac{1}{\boldsymbol{\kappa}_2} \frac{\partial \mathcal{F}}{\partial X_2} \end{aligned} \tag{4.2}$$

These superfield combinations may a priori be extended to any number n of $\mathcal{N}=2$ gauge multiplets $\mathcal{W}_a^{\mathcal{N}=2} = (\boldsymbol{X}_a, \boldsymbol{V}_a)$ as follows

$$\boldsymbol{V} = \sum_{a=1}^{n} g_a \boldsymbol{V}_a \tag{4.3}$$

$$\boldsymbol{X} = \sum_{a=1}^{n} g_a \boldsymbol{X}_a \tag{4.4}$$

where the g_a's are gauge coupling constants associated with each abelian $U_a(1)$ gauge multiplet $(\boldsymbol{X}_a, \boldsymbol{V}_a)$. Similar relations can be written down for $\boldsymbol{V}'$, $\boldsymbol{X}'$ and $\frac{\partial \mathcal{F}}{\partial X}$. However, because of ADJ constraint relation; the generalisation of the condition (3.5) to arbitrary $U(1)^n$ gauge symmetry is valid provided $n \geq 2$ as in (2.12). The restriction to the particular $n = 1$ case leads to singular relation

$$\frac{g_1}{\boldsymbol{\kappa}_1} = 0$$

requiring $g = 0$ for $\boldsymbol{\kappa} \neq 0$. To overcome this difficulty; one may resolve the $\frac{g_1}{\boldsymbol{\kappa}_1} = 0$ singularity by deforming it like $\nu + \frac{g}{\boldsymbol{\kappa}} = 0$; this leads to $g = -\boldsymbol{\kappa}\nu$; however remembering the property of eqs. (2.34)–(2.35), one learns that the deformation by ν breaks explicitly $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$.

To see the meaning of the linear combinations (4.1)–(4.4) as well as the interpretation of the deformation

$$\nu + \sum_{a=1}^{n} \frac{g_a}{\boldsymbol{\kappa}_a} = 0 \tag{4.5}$$

we need to go beyond 4d space time by thinking of:

- the $\mathcal{N}=2$ supersymmetric $U(1)^n$ gauge model as a part of an effective theory following from type IIB string compactified on a local CY3; and
- the constraint relation (4.5) as corresponding to the D3 tapole anomaly [34]

$$\frac{1}{2\kappa_{10}^2 T_3} \int_{CY3} \hat{H}_3^{RR} \wedge \hat{H}_3^{NS} + N_{D3} = 0 \tag{4.6}$$

where the 3-form gauge field strengths $\hat{H}_3^{RR}$, $\hat{H}_3^{NS}$; and the numbers T_3 and N_{D3} will be introduced later on.

To be explicit, we study in what follows the derivation of the linear combinations $\sum_a \xi_a \boldsymbol{V}^a$ and $\sum_a g_a \boldsymbol{V}^a$ from type IIB string compactification on a Calabi–Yau threefold $\mathcal{Z}_3$ with Kähler 2-form J_2 and complex holomorphic 3-form Ω_3. Then, we turn to the derivation of the linear combinations $\sum_a e_a \boldsymbol{X}^a$ and $\sum_a \frac{1}{\kappa_a} \frac{\partial \mathcal{F}}{\partial X^a}$ concerning the chiral superfields.

4.1.2. Kahler sector

To derive the two linear combinations involving the gauge multiplet namely the $\boldsymbol{V} = \sum_a \xi_a \boldsymbol{V}^a$, depending on FI coupling constants ξ_a, and the $\boldsymbol{V}' = \sum_a g_a \boldsymbol{V}^a$ involving gauge coupling constants g_a, it is interesting to start by describing the θ-expansions of these combinations. Focusing on the 4-vector υ_μ^a field components of the linear sum of gauge superfields $\boldsymbol{V}^a$ which expands in θ-series as

$$\begin{aligned} \boldsymbol{V} &= \theta\sigma^\mu\bar{\theta}\left(\upsilon_\mu\right) + \theta^2\bar{\theta}^2\left(\sum_{a=1}^n \chi_a D^a\right) \\ &\quad + \frac{i}{\sqrt{2}}\theta^2\bar{\theta}\left(\sum_{a=1}^n \chi_a \bar{\lambda}^a\right) - \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\left(\sum_{a=1}^n \chi_a \lambda^a\right) \end{aligned} \tag{4.7}$$

with

$$\upsilon_\mu = \sum_{a=1}^n \chi_a \upsilon_\mu^a \qquad , \qquad \chi_a = \xi_a,\ g_a \tag{4.8}$$

Obviously for the case $\chi_a = \xi_a$, the contribution to ADJ model is given by the D-term $\theta^2\bar{\theta}^2\left(\sum_{a=1}^n \xi_a D^a\right)$; and the interpretation of ξ_a's may be obtained by computing the field equations of the auxiliary D^a fields. However, we can reach the same result by looking for the derivation of this quantity from superstring compactification.

i) FI coupling constants

As a first step toward the $\xi_a \boldsymbol{V}^a$'s, we use the 4d space time language of 1-form gauge field potentials $V_1^a = \upsilon_\mu^a dx^\mu$ to rewrite the gauge component field linear combination $\sum \xi_a \upsilon_\mu^a$ as follows

$$\left(\sum_{a=1}^n \xi_a \upsilon_\mu^a\right) dx^\mu = \sum_{a=1}^n \xi_a V_1^a \equiv V_1 \tag{4.9}$$

So $\xi_a \upsilon_\mu^a$ can be also viewed in terms of a linear combination of the 1-form gauge field potentials V_1^a. The next step is to transform above (4.9) into an integral over full dimensions of the Calabi–Yau threefolds; this is achieved by thinking about the 1-form gauge field V_1^a in 4d space time as due to a 4-form gauge potential $\hat{C}_4$ of a D3 brane living in 10d space time

$$\hat{C}_4 = \frac{1}{4!} C_{MNPQ}\, d\hat{x}^M \wedge d\hat{x}^N \wedge d\hat{x}^P \wedge d\hat{x}^Q \tag{4.10}$$

but with three directions wrapping the compact 3-cycles $[A_a]$ of the local CY3 as follows

$$V_1^a = \frac{1}{2\pi\alpha'} \int_{CY3} \hat{C}_4 \wedge \beta^a \tag{4.11}$$

with harmonic 3-form β^a belonging to $H^{(2,1)}(CY3, R) \oplus H^{(1,2)}(CY3, R)$. Putting this relation back into (4.9), we end with

$$\sum_{a=1}^{n} \xi_a V_1^a = \frac{1}{2\pi\alpha'} \int_{CY3} \hat{C}_4 \wedge dJ_2 \tag{4.12}$$

with dJ_2 standing for 3-form obtained by complex deformation of Kähler 2-form J_2 of the CY3 [40–43]

$$dJ_2 = \sum_{a=1}^{n} \xi_a \beta^a \quad , \quad \xi_a = \frac{1}{2}\left(\int_{[B^a]} dJ_2 + hc\right) \tag{4.13}$$

where 3-cycle $\left[B^a\right]$ is the dual of $[A_a]$ in the CY3. Recall that the pair $[A_a]$ and $\left[B^a\right]$ form a symplectic basis of 3-cycles in the homology group of the CY3; they are in 1:1 correspondence with the 3-form harmonic basis (α_a, β^a) of the cohomology group.

ii) Gauge coupling constants

To derive the linear combination $\sum_{a=1}^{n} g_a \upsilon_\mu^a$ and the expression of the gauge coupling constants g_a, we use the 4d Chern–Simons interaction $\mathcal{L}_{CS}^{4d}$ between the gauge potentials υ_μ^a and the antisymmetric field strength $H_{\nu\rho\sigma}$,

$$\mathcal{L}_{CS}^{4d} = \frac{1}{3!}\left(\sum_{a=1}^{n} g_a \upsilon_\mu^a H_{\nu\rho\sigma}\right) \varepsilon^{\mu\nu\rho\sigma} \tag{4.14}$$

that we rewrite, by using wedge product $V_1^a \wedge H_3$ of 4d space time 1- and 3-forms like

$$\mathcal{S}_{CS}^{4d} = \int_{M^4} \left(\sum_{a=1}^{n} g_a V_1^a\right) \wedge H_3 \tag{4.15}$$

But seen that in type IIB string, we have two kinds of 3-forms H_3^{RR} and $\hat{H}_3^{NS}$; then we can think of the 4d Chern–Simons action $\mathcal{S}_{CS}^{4d}$ as resulting from the following 10d expression

$$\mathcal{S}_{CS}^{10d} = \int_{M^{10}} \hat{C}_4 \wedge \hat{H}_3^{NS} \wedge \hat{H}_3^{RR} \tag{4.16}$$

This relation leads in general to two kinds of 4d space time contributions; one involving 4d space time 3-form H_3^{RR} and the other the 4d 3-form H_3^{NS} as follows

$$\int_{M^4}\left(\int_{CY3} \hat{C}_4 \wedge \hat{H}_3^{NS}\right) \wedge H_3^{RR} - \int_{M^4}\left(\int_{CY3} \hat{C}_4 \wedge \hat{H}_3^{RR}\right) \wedge H_3^{NS} \tag{4.17}$$

If we restrict to first contribution and comparing with (4.11), we end with the following expression for the gauge coupling constants

$$g_a = \frac{1}{2\pi\alpha'} \int_{CY3} \alpha_a \wedge H_3^{NS} \tag{4.18}$$

that reads also as follows

$$g_a = \frac{1}{2\pi\alpha'} \int_{B^a}^{\Lambda_0} H_3^{NS} \tag{4.19}$$

with α' the string constant and where Λ_0 is a cut off playing the role of running scale of the well known renormalisation group equation.

To conclude, the linear combinations of the $\mathcal{N}=2$ gauge multiplets used in ADJ model have the following geometric interpretation in type IIB string on local CY3

$$\sum_{a=1}^{n} \xi_a V_1^a = \frac{1}{2\pi\alpha'} \int_{CY3} \hat{C}_4 \wedge dJ_2 \tag{4.20}$$

$$\sum_{a\geq 1} g_a V_1^a = \frac{1}{4\pi^2\alpha'^2} \int_{CY3} \hat{C}_4 \wedge H_3^{NS} \tag{4.21}$$

4.1.3. Chiral sector

The $\boldsymbol{X}^a$'s are chiral superfields with θ-components given by complex scalar fields $\boldsymbol{X}^a|_{\theta=0} = X^a$; Weyl spinors $D_\alpha \boldsymbol{X}^a|_{\theta=0} = \psi_\alpha^a$ and auxiliary fields $\frac{1}{2} D^\alpha D_\alpha \boldsymbol{X}^a\Big|_{\theta=0} = F^a$. The linear combination $\sum e_a \boldsymbol{X}^a$ is a chiral superfield

$$\sum_{a\geq 1} e_a \boldsymbol{X}^a = X + \theta.\left(\sum_{a\geq 1} e_a \psi^a\right) + \theta^2 \left(\sum_{a\geq 1} e_a F^a\right) \tag{4.22}$$

with leading θ-component given by a similar relation to the superfield one namely

$$X = \sum_{a\geq 1} e_a X^a \tag{4.23}$$

A similar relation is valid for the magnetic FI combination

$$\frac{1}{\kappa}\frac{\partial \mathcal{F}}{\partial X} = \sum_{a\geq 1} \frac{1}{\kappa_a} \frac{\partial \mathcal{F}}{\partial X^a} \tag{4.24}$$

In the embedding of ADJ model into 10d type IIB string compactification on CY3, the complex scalars X^a and $\mathcal{F}_a$ describe the expansion modes of the holomorphic 3-form Ω_3 over the 3-form harmonic basis (α_a, β^a) of the local CY3

$$\Omega_3 = X^a \alpha_a - \mathcal{F}_a \beta^a \tag{4.25}$$

with

$$X^a = \int_{CY3} \Omega_3 \wedge \beta^a \qquad , \qquad \mathcal{F}_a = \int_{CY3} \Omega_3 \wedge \alpha_a \tag{4.26}$$

The electric FI e_a and magnetic $\frac{1}{\kappa_a}$ coupling constants are obtained by equating the superpotential

$$W(X) = e_a X^a - \frac{1}{\kappa_a} \mathcal{F}_a \tag{4.27}$$

with the expression of $W(X)$, build out of $G_3 = H_3^{RR} - \tau H_3^{NS}$ and the complex holomorphic 3-form namely [46,34],

$$W(X) \sim \int_{CY3} \Omega_3 \wedge G_3 \tag{4.28}$$

We obtain

$$G_3 = e_a \beta^a - \frac{1}{\kappa_a} \alpha_a \tag{4.29}$$

So, we have

$$e_a = \int_{CY3} \alpha_a \wedge G_3 \quad , \quad \frac{1}{\kappa_a} = \int_{CY3} G_3 \wedge \beta^a \tag{4.30}$$

4.2. Deriving ADJ constraint from type IIB string

In ADJ model, the condition (2.12) is intimately related with the real 4-form C_4 with antisymmetric gauge potential field $C_{\mu\nu\rho\sigma}$ as in (2.19) and directions filling the 4d space time dimensions

$$\begin{aligned} &C_4 = C_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &F_5 = 0 \end{aligned} \tag{4.31}$$

This 4-form gauge field potential cannot have field strength $F_5 = dC_4$ in 4d space time; and thus it should be treated as an auxiliary field as done by ADJ theory. Notice that this is a constant field that may be ignored by setting it to zero; but this corresponds to a particular solution since in general it reads like

$$\begin{aligned} &C_{\mu\nu\rho\sigma} = \Delta\varepsilon_{\mu\nu\rho\sigma} \quad , \quad \Delta = cst \\ &d\Delta = 0 \end{aligned} \tag{4.32}$$

and captures a constraint relation that we study below. Observe that Δ scales as mass^2; and so the scaling mass dimension of C_4 is mass^{-2}.

4.2.1. Tadpole anomaly

To get more insight into the meaning of the $\mathcal{N} = 2$ ADJ condition $\sum_a \frac{g_a}{\kappa_a} = 0$ and its $\mathcal{N} = 1$ deformation $\nu + \sum_a \frac{g_a}{\kappa_a} = 0$, one has to go beyond 4d space time where the gauge potential C_4 is no longer constant

$$F_5|_{4d} = dC_4|_{4d} = 0 \quad , \quad \hat{F}_5\Big|_{10d} = d\hat{C}_4\Big|_{10d} \neq 0 \tag{4.33}$$

In a higher dimension D space time compactified on real $(D-4)$ manifold, $M^D = M^{1,3} \times M^{(D-4)}$, some of the directions of the extended 4-form field $\hat{C}_4$ can wrap dimensions in the internal space allowing as consequence Chern–Simons couplings between $\hat{C}_4$ and other p_i-forms $\hat{F}_{p_i}$ living in M^D; for example $\mathcal{S}^D_{CS} \sim \int_{M^D} \hat{C}_4 \wedge \hat{F}_{D-4}$. This is exactly what happens in the case of type IIB strings compactified on Calabi–Yau threefolds where the 4-form gauge potential $\hat{C}_4$, sourced by D3 brane, appears naturally and where couplings with other p-form gauge field potentials are possible. In 10d type IIB theory, the Chern–Simons coupling reads as follows

$$\mathcal{S}^{10d}_{CS} \sim \int\limits_{M^{1,9}} \hat{C}_4 \wedge \hat{H}^{RR}_3 \wedge \hat{H}^{NS}_3 \tag{4.34}$$

where the 3-forms $\hat{H}^{NS}_3$ and $\hat{H}^{RR}_3$ are the gauge field strengths of the antisymmetric $\hat{B}^{NS}_{\mu\nu}$ and $\hat{B}^{RR}_{\mu\nu}$ gauge potentials of type IIB strings. The field equation of the 5-form field strength $\hat{F}_5$ of the $\hat{C}_4$ gauge potential is given by

$$dF_5 = \hat{H}^{NS}_3 \wedge \hat{H}^{RR}_3 + 2\kappa^2_{10} T_3 \varrho^{loc}_3 \tag{4.35}$$

with $T_3 = \frac{1}{(2\pi)^3 \alpha'^2}$ the D3-brane tension, $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$; and where ϱ_3^{loc}-form stands for the D3 charge density due to localised sources including D7-branes or O3 planes and also of mobile D3-branes [34]; see also [44,45]

$$\int_{CY3} \varrho_3^{loc} = N_{D3} \tag{4.36}$$

From 4d space time view, the integration of the Chern–Simons coupling (4.34) on the internal coordinates

$$\mathcal{S}_{CS}^{4d} = \int\limits_{M^{1,3}} \left(\int_{CY3} \hat{C}_4 \wedge \hat{H}_3^{RR} \wedge \hat{H}_3^{NS} \right) \tag{4.37}$$

leads to various terms that can be organised into three block terms

$$\mathcal{S}_{CS}^{4d} = \int\limits_{M^{1,3}} \left(\mathcal{L}_0^{4d} + \mathcal{L}_1^{4d} + \mathcal{L}_2^{4d} \right) \tag{4.38}$$

with

$$\mathcal{L}_0^{4d} \sim \left(\int_{CY3} \hat{H}_3^{RR} \wedge \hat{H}_3^{NS} \right) \wedge C_4 \tag{4.39}$$

$$\mathcal{L}_1^{4d} \sim \left(\int_{CY3} \hat{C}_4 \wedge \hat{H}_3^{RR} \right) \wedge H_3^{NS} \tag{4.40}$$

$$\mathcal{L}_2^{4d} \sim -\left(\int_{CY3} \hat{C}_4 \wedge \hat{H}_3^{NS} \right) \wedge H_3^{RR} \tag{4.41}$$

where un-hatted fields refer to 4d space time fields. Notice that the two last terms are precisely the ones given by eq. (4.17), they contribute to Chern–Simons couplings in 4d space time.

The remaining term (4.39), involving the integral of $\hat{H}_3^{RR} \wedge \hat{H}_3^{NS}$ through full CY3; is a topological term describing the flux Φ_{flux} of the 6-form $\hat{H}_3^{RR} \wedge \hat{H}_3^{NS}$ through the CY3

$$\Phi_{flux} \sim \int_{CY3} \hat{H}_3^{RR} \wedge \hat{H}_3^{NS} \tag{4.42}$$

For compact CY3, this flux has no where to go and so has to vanish; but as we will see in a moment it may be compensated by D3-brane charges coming from local sources. So the contribution of this term to 4d space time lagrangian density reads as

$$\frac{\Phi_{flux}}{24} \varepsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} = \frac{\Phi_{flux}}{24} \Delta \tag{4.43}$$

where we have used (4.32). This term appears linearly in the ADJ lagrangian density; so it captures a constraint equation requiring the flux Φ_{flux} to vanish as noticed above. This constraint can be then interpreted as nothing but the vanishing condition given by the integral of (4.35) on CY3 for the particular case $\varrho_3^{loc} = 0$. For a non-zero ϱ_3^{loc} density of D3-brane charges as in eq. (4.36), the tadpole vanishing condition reads as

$$\frac{1}{2\kappa_{10}^2 T_3} \int\limits_{CY3} \hat{H}_3^{RR} \wedge \hat{H}_3^{NS} + N_{D3} = 0 \tag{4.44}$$

and the contribution to the 4d space time lagrangian density gets modified like

$$\mathcal{L}_0^{4d} = \frac{\Delta}{24}\left(\Phi_{flux} + N_{D3}\right) \tag{4.45}$$

The presence of the N_{D3} flux breaks explicitly $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$; N_{D3} flux plays the same role as the parameter ν in the deformed constraint eq. (4.5).

4.2.2. Revisiting $\mathcal{N}=2$ ADJ condition and its deformation

Here, we give the general expression of the condition of tadpole cancellation in terms of the H_3^{RR} and H_3^{NS} fluxes. If assuming that the CY3 is compact, then we have

$$\begin{aligned} &\frac{1}{2\pi\alpha'}\int_{CY3} H_3^{RR}\wedge\beta^a = p_a \\ &\frac{1}{2\pi\alpha'}\int_{CY3} \alpha_a\wedge H_3^{RR} = p_{n+a} \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} &\frac{1}{2\pi\alpha'}\int_{CY3} H_3^{NS}\wedge\beta^a = q_a \\ &\frac{1}{2\pi\alpha'}\int_{CY3} \alpha_a\wedge H_3^{NS} = q_{n+a} \end{aligned} \tag{4.47}$$

From these relations, we learn

$$\frac{1}{4\pi^2\alpha'^2}\int_{CY3} \hat{H}_3^{NS}\wedge\hat{H}_3^{RR} = q_A\Omega^{AB}p_B \tag{4.48}$$

where the quantised vectors are as

$$p_A = \begin{pmatrix} p_a \\ p_{a+n}\end{pmatrix} \quad , \quad q_A = \begin{pmatrix} q_b \\ q_{b+n}\end{pmatrix} \tag{4.49}$$

and where Ω^{AB} is the usual $2n\times 2n$ symplectic matrix

$$\Omega^{AB} = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0\end{pmatrix} \tag{4.50}$$

Putting back into (4.44), the tadpole condition becomes

$$\nu + 4\pi^2\alpha'^2\, q_A\Omega^{AB}p_B = 0 \tag{4.51}$$

with

$$\nu = 2\kappa_{10}^2 T_3 N_{D3} = (2\pi)^4\alpha'^2 N_{D3} \tag{4.52}$$

The $\mathcal{N}=2$ ADJ constraint (2.12) may be recovered by requiring $N_{D3}=0$; and choosing the vectors p_A and q_A like

$$\frac{1}{\kappa_a} = 2\pi\alpha'^2 q_a \quad , \quad g_a = 2\pi p_{a+n} \tag{4.53}$$

the other p_a and q_{a+n} are set to zero.

5. Conclusion and comments

In this paper, we have studied a D3 brane realisation of partial breaking of $\mathcal{N} = 2$ supersymmetry in ADJ model of Ref. [32]. This is a particular 4d $\mathcal{N} = 2$ supersymmetric $U(1) \times U(1)$ gauge model describing the coupling of the $\mathcal{N} = 2$ gauge multiplets $\mathcal{W}_1^{(\mathcal{N}=2)}$, $\mathcal{W}_2^{(\mathcal{N}=2)}$ with a single tensor $\mathcal{T}^{(\mathcal{N}=2)}$; it is also the leading model in the family of effective 4d $\mathcal{N} = 2$ $U(1)^n$ gauge theory describing the dynamics of the multiplet $\mathcal{T}^{(\mathcal{N}=2)}$ coupled to n $\mathcal{N} = 2$ Maxwell gauge superfield strengths $\mathcal{W}_1^{(\mathcal{N}=2)}, \ldots, \mathcal{W}_n^{(\mathcal{N}=2)}$ with $n \geq 2$. The coupling between gauge and matter superfields is of Chern–Simons type; it reads in $\mathcal{N} = 2$ chiral superspace as follows

$$\mathcal{L}_{CS} = \int d^8\theta \mathcal{T}^{(\mathcal{N}=2)} \left(\sum_{a=1}^{n} g_a \mathcal{W}_a^{(\mathcal{N}=2)} \right) \tag{5.1}$$

One of the basic constraint equations in this 4d effective $\mathcal{N} = 2$ supersymmetric $U(1)^n$ gauge theory, formulated in $\mathcal{N} = 1$ superspace with lagrangian density $\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{ST} + \mathcal{L}_{CS}$, is the one given by $\frac{\delta \mathcal{L}}{\delta Y} = 0$; the superfield equation of the auxiliary $\boldsymbol{Y}$ whose full contribution comes from the $\mathcal{L}_{CS}$ term namely

$$\frac{\delta \mathcal{L}_{CS}}{\delta Y} = 0 \tag{5.2}$$

Because of linear dependence of $\mathcal{L}_{CS}$ into the superfield $\boldsymbol{Y}$, the above superfield equation turns into the constraint $\sum_{a=1}^{n} \frac{g_a}{\kappa_a} = 0$. This condition relates the gauge coupling constants g_a to magnetic FI couplings $\frac{1}{\kappa_a}$; and, upon computing energy of the ground state of the model, one also obtains links between partial supersymmetry breaking scales and the g_a, κ_a constant parameters as done in [32].

In our brane realisation, the ADJ model is represented by D3 branes wrapping 3-cycles in type IIB on local CY3 in presence of non-trivial fluxes of the 3-forms gauge field strengths H_3^{RR} and H_3^{NS}. In this picture, the Chern–Simons coupling (5.1) is associated with a particular term in the Calabi–Yau compactification of 10-dim field action

$$\mathcal{S}_{CS}^{10d} \sim \int_{M^{1,9}} \hat{C}_4 \wedge \hat{H}_3^{RR} \wedge \hat{H}_3^{NS} \tag{5.3}$$

down to 4d space time; for details see eqs. (4.34)–(4.41). In this brane representation of ADJ theory, the constraint relation $\sum_{a=1}^{n} \frac{g_a}{\kappa_a} = 0$ is a particular realisation of the total flux conservation condition $\Phi_{flux} = \frac{1}{2\kappa_{10}^2 T_3} \int_{CY3} \hat{H}_3^{NS} \wedge \hat{H}_3^{RR} = 0$ which, by using the n-dimensional symplectic homology basis of 3-cycles (A^a, B_a) of the CY3, reads in general like $\Phi_{flux} = q_A \Omega^{AB} p_B$ with p_A and q_B as in eqs. (4.46)–(4.48).

$$\Phi_{flux} = (q_a, q_{a+n}) \begin{pmatrix} 0 & \delta_{ab} \\ -\delta_{ab} & 0 \end{pmatrix} \begin{pmatrix} p_b \\ p_{b+n} \end{pmatrix} \tag{5.4}$$

By expanding the above relation as $\Phi_{flux} = q_a p_{a+n} - q_{a+n} p_a$; it follows that the condition $\sum_{a=1}^{n} \frac{g_a}{\kappa_a} = 0$ is indeed a particular solution of vanishing $\Phi_{flux} = 0$ given the choice of fluxes as in eq. (4.53). This flux choice corresponds to expanding gauge field strengths H_3 on the basis (α_a, β^a) of 3-forms, dual to 3-cycles basis, like $H_3^{NS} = q^a \alpha_a$ and $H_3^{RR} = p_{a+n} \beta^a$. Geometrically, these values correspond to the local Calabi–Yau picture where the 3-cycles B_a are taken in the non-compact space approximation. In the case $n = 1$; describing the special $\mathcal{N} = 2$ $U(1)$ gauge model, the CY3 is the T^*S^3 conifold with compact A given by the 3-sphere S^3 and non compact $B \simeq \mathbb{R}^3$; and the above expression of the Φ_{flux} flux reduces to

$$(q_1, 0)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ p_2 \end{pmatrix} = q_1 p_2 \tag{5.5}$$

The relationship between ADJ constraint and the flux $q_A \Omega^{AB} p_B$ shows that $\sum_{a=1}^{n} \frac{g_a}{\kappa_a} = 0$ is nothing but the vanishing condition of the D3 tadpole anomaly in absence of D7 branes or O3 planes; the particular $n = 1$ case is therefore anomalous.

By taking into account D3 charge density due to localised sources D7-branes or O3 planes; and by using a result from the study of [34], the previous total flux conservation condition gets promoted to the following relation

$$\tilde{\Phi}_{flux} = \Phi_{flux} + N_{D3} = 0 \tag{5.6}$$

with N_{D3} as in (4.36). In this picture, the tadpole anomaly can be lifted; but with the price of breaking explicitly $\mathcal{N} = 2$ supersymmetry in the underlying effective field theory down to $\mathcal{N} = 1$. In this situation, the ADJ constraint $\sum_a \frac{g_a}{\kappa_a} = 0$ becomes deformed like $\nu + \sum_a \frac{g_a}{\kappa_a} = 0$ as shown by eq. (4.45).

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Appendix A

Here we give two appendices; in Appendix A.1, we collect useful tools on type IIB on CY3; and in Appendix A.2, we study properties shared by supersymmetric and gauge transformations of the $\mathcal{N} = 2$ gauge superfield $\mathcal{W}^{\mathcal{N}=2}$ and the $\mathcal{N} = 2$ tensor multiplet $\mathcal{T}^{\mathcal{N}=2}$ used in this paper.

A.1. Useful tools on type IIB on CY3

Effective $\mathcal{N} = 2$ supersymmetric QFT models in 4d space-time can be embedded in string compactifications; they are constructed in various manners; in particular by compactifying type II strings on CY3s or heterotic string on $K3 \times \mathbb{T}^2$. These compactifications to 4d $\mathcal{N} = 2$ low energy theories are related by duality symmetries. By decoupling massive modes which are of order compactification scale, one can build the structure of the effective $\mathcal{N} = 2$ supersymmetric theory with decoupled gravity. The massless states following from the CY3 compactification can be organised into $\mathcal{N} = 2$ multiplets as follows:

(1) vectors $\boldsymbol{V}^{\mathcal{N}=2}$; each contains a complex scalar, a vector and two Weyl fermions.

(2) hypermultiplets $\boldsymbol{H}^{\mathcal{N}=2}$; each contains four real scalars and two Weyl fermions.

(3) two other kinds of non-standard $\mathcal{N} = 2$ multiplets having an antisymmetric tensor $B_{\mu\nu}$; these are: (a) the tensor multiplet $\boldsymbol{T}^{\mathcal{N}=2}$ having: 3 scalars, $B_{\mu\nu}$ and 2 Weyl fermions; and (b) the vector-tensor $\boldsymbol{R}^{\mathcal{N}=2}$ containing: a scalar, a vector, $B_{\mu\nu}$ and 2 Weyl fermions.

Type IIB in 10d The *10d* type IIB supergravity multiplet has *128* + *128* on shell degrees of freedom with bosonic sector as follows

$$\begin{array}{ll} \text{NS–NS} & : \hat{\phi}, \hat{G}_{MN}, \hat{B}_{MN} \\ \text{RR} & : \hat{C}_0, \hat{C}_{MN}, \hat{C}_{MNPQ} \end{array} \tag{A.1}$$

In addition to the metric $\hat{G}_{MN}$, we have a real axion $\hat{C}_0$ and the dilation $\hat{\phi} = \ln g_{IIB}$ with g_{IIB} the string coupling. We also have p-forms namely the NS–NS $\hat{B}_2$ and the RR p-form gauge field potentials $\hat{C}_p$ together with the corresponding gauge invariant field strengths

$$\begin{array}{ll} \hat{F}_{p+1} & = d\hat{C}_p \\ \hat{H}_3 & = d\hat{B}_2 \end{array} \tag{A.2}$$

This theory has an S-duality symmetry that allows to combine these fields into $SL(2, Z)$ representations; in particular as

$$\begin{array}{ll} \tau & = \hat{C}_0 - ie^{-\hat{\phi}} \\ \hat{G}_3 & = \hat{F}_3 - \tau\hat{H}_3 \\ \tilde{F}_5 & = \hat{F}_5 - \frac{1}{2}\hat{C}_2 \wedge \hat{H}_3 + \frac{1}{2}\hat{B}_2 \wedge \hat{F}_3 \end{array} \tag{A.3}$$

where τ may be interpreted as the complex structure of 2-torus as in the embedding of type IIB in F-theory. The 5-form $\tilde{F}_5$ gauge field strength span in 5 of the 10d space time dimensions; it is a self dual form $*\tilde{F}_5 = \tilde{F}_5$ with $*$ defined as

$$*\hat{F}^{M_0M_1\ldots M_n} = \frac{1}{n!}\,\varepsilon^{M_0M_1\ldots M_nM_{n+1}\ldots.M_8M_9}\ \hat{F}_{M_{n+1}\ldots M_9} \tag{A.4}$$

To avoid confusion between p-form of same rank and also their descendent after compactification, we shall use the notations $\hat{B}_2 = \hat{B}_2^{NS}$ for NS 2-form gauge potential sourced by the elementary string F1; and $\hat{C}_2 = \hat{B}_2^{RR}$ for the RR 2-form gauge potential sourced by the solitonic D1 string.

Type IIB on CY3 To descend to 4d space time, we have to factorise the 10d fields as products of parts; one depending on 4d space time and the other on the internal coordinates. This is achieved by decomposing the 2- and 4-forms on a harmonic basis of form of the local CY3 as follows

$$\begin{array}{ll} \hat{B}_2^{NS} & = B_2^{NS} + b_{NS}^I\omega_I \\ \hat{B}_2^{RR} & = B_2^{RR} + b_{RR}^I\omega_I \\ \hat{C}_4 & = C_4 + A_2^I \wedge \omega_I + V_1^a \wedge \alpha_a - U_{1a}\beta^a + \varrho_I \wedge \tilde{\omega}^I \end{array} \tag{A.5}$$

Here the set $\omega_I, \alpha_a, \beta^a, \tilde{\omega}^I$ stand for a real harmonic basis of p-forms generating the cohomology of the CY3 obeying amongst others the following useful relations

$$\int_{CY3} \omega_I \wedge \tilde{\omega}^J = \delta_I^J \qquad , \qquad \int_{CY3} \alpha_a \wedge \beta^b = \delta_a^b \tag{A.6}$$

Notice that the 10d self duality condition of $\hat{F}_5 = d\hat{C}_4$ implies that the 2-form $A_2^I = \frac{1}{2!}A_{\mu\nu}^I dx^\mu \wedge dx^\nu$ and the scalars ϱ_I in eq. (A.5) are related as $dA_2^I = *d\varrho_I$; and so only one of them should be kept; for our concern we have kept A_2^I. The same feature holds for the 1-form gauge field potentials V_1^a and U_{1a}; in other words the decomposition of $\hat{C}_4$ capturing the right number of degrees of freedom is reduced to

$$\hat{C}_4 = C_4 + A_2^I \wedge \omega_I + V_1^a \wedge \alpha_a \tag{A.7}$$

Combining altogether, we learn from above decomposition that the 4d space time spectrum that we obtain, after compactification on CY3, the following multiplets where only bosonic fields are reported

Multiplets	Type IIB$/\mathcal{Z}_3$	Number
Gravity	$\left(\mathcal{G}_{\mu\nu}, V_\mu^0\right)$	1
Vector	$\left(V_\mu^a, X^a, \bar{X}_a\right)$	$h^{2,1}$
Tensor	$\left(A_{\mu\nu}^I, b_{RR}^I, t^I, \bar{t}^I\right)$	$h^{1,1}$
Bi-tensor	$\left(B_{\mu\nu}^{NS}, B_{\mu\nu}^{RR}, \xi, S\right)$	1

(A.8)

and where the 4d scalars ξ and S stand respectively the 4d space time dilaton and 4d axion.

To embed the effective ADJ theory in type IIB string on local CY3, we have to think about the degrees of freedom of ADJ model to belong to a subsector of type IIB/CY3 namely

$$\begin{aligned} \text{vector} &: \left(V_\mu^a, X^a, \bar{X}_a\right) \\ \text{tensor} &: \left(B_{\mu\nu}^{NS}, B_{\mu\nu}^{RR}, \xi, S\right) \end{aligned} \tag{A.9}$$

with $V_1^a = V_\mu^a dx^\mu$ standing for 1-form gauge potential in 4d space time. The fields of (A.9) are obtained by inverting eq. (A.5) by using properties of the harmonic basis of the homology cycles of the CY3; they are given by

$$\begin{aligned} V_1^a &= \int_{CY3} \hat{C}_4 \wedge \beta^a \\ X^a &= \int_{CY3} \Omega_3 \wedge \beta^a \end{aligned} \tag{A.10}$$

A.2. *More on $\mathcal{W}^{\mathcal{N}=2}$ and $\mathcal{T}^{\mathcal{N}=2}$ multiplets*

In this appendix, we shed light on those features behind formal similarities between Maxwell $\mathcal{W}^{\mathcal{N}=2}$ and single tensor $\mathcal{T}^{\mathcal{N}=2}$ superfields; this is done by studying their behaviour under the second supersymmetric charge in the fibration (2.1); as well as their link through gauge symmetry.

A.2.1. *Realising $\mathcal{N}=2$ by two $\mathcal{N}=1$ gauge superfields*

There are different ways to deal with off shell representations of $\mathcal{N}=2$ gauge and tensor multiplets; here we want to focus on those aspects shared by their formulation in terms of the $\mathcal{N}=2$ chiral superfields $\mathcal{W}^{\mathcal{N}=2}$ and $\mathcal{T}^{\mathcal{N}=2}$ introduced in section 2. These superfields carry $8_B + 8_F$ degrees of freedom; they may be obtained by first considering two $\mathcal{N}=1$ superfields having $16_B + 16_F$ degrees of freedom; and then reducing this number down to $8_B + 8_F$ degrees by imposing appropriate constraints. The study of the reduction

$$16_B + 16_F \qquad \rightarrow \qquad 8_B + 8_F \tag{A.11}$$

for both gauge $\mathcal{W}^{\mathcal{N}=2}$ and tensor $\mathcal{T}^{\mathcal{N}=2}$ multiplets is interesting in the sense it allows to get more information on: (i) the link between the second supersymmetry and gauge symmetry; and (ii) the feature behind similarities between $\mathcal{W}^{\mathcal{N}=2}$ and $\mathcal{T}^{\mathcal{N}=2}$. Explicitly, we proceed as follows:

First, we use $\mathcal{N}=1$ formalism, where the first supersymmetry is manifest; to study a simple realisation of $\mathcal{W}^{\mathcal{N}=2}$ and $\mathcal{T}^{\mathcal{N}=2}$ engineered from two $\mathcal{N}=1$ hermitian gauge superfields V_1 and V_2; but handled in different manners.

After that, we draw the line on how this construction extends to the $\mathcal{N}=2$ chiral superspace; this extension is also important for $\mathcal{W}^{\mathcal{N}=2}$ and $\mathcal{T}^{\mathcal{N}=2}$ because it constitutes the starting point for studying electric–magnetic duality in $\mathcal{N}=2$ superspace in presence of the Chern–Simons term given by the coupling

$$L_{CS}^{\mathcal{N}=2} = \int d^2\theta d^2\tilde{\theta}\ ig\mathcal{W}^{\mathcal{N}=2}\mathcal{T}^{\mathcal{N}=2} + hc \tag{A.12}$$

Actually, this duality gives another facet on the relationship between $\mathcal{W}^{\mathcal{N}=2}$ and $\mathcal{T}^{\mathcal{N}=2}$ as shown by eq. (2.28); it will not be developed here; but for details see [31] and references therein.

$\mathcal{N}=2$ supersymmetry in $\mathcal{N}=1$ superspace

In $\mathcal{N}=1$ superspace, the hermitian gauge superfields V_1 and V_2 describe two representations of $\mathcal{N}=1$ supersymmetry; and roughly speaking a $\mathcal{N}=2$ multiplet. Altogether, V_1 and V_2 carry 16_B+16_F off shell degrees of freedom; half of them coming from V_1 and the other half from V_2. With these two superfields, the second supersymmetry is generated by the transformations

$$\begin{aligned}\tilde{\delta}V_1 &= \tfrac{-i}{\sqrt{2}}\left(\epsilon D+\bar{\epsilon}\bar{D}\right)V_2\\ \tilde{\delta}V_2 &= i\sqrt{2}\left(\epsilon D+\bar{\epsilon}\bar{D}\right)V_1\end{aligned} \tag{A.13}$$

with supersymmetric parameter $\epsilon=\delta\tilde{\theta}$. These 16_B+16_F degrees can be reduced down to 8_B+8_F by imposing constraint equations on V_1 and V_2; it happens that this can be done in two different manners; one leading to $(\boldsymbol{X},\boldsymbol{W}_\alpha)$, the $\mathcal{N}=1$ superfields representation of $\mathcal{W}^{\mathcal{N}=2}$; and the other to the $(\boldsymbol{\Phi},\boldsymbol{L})$ realisation of $\mathcal{T}^{\mathcal{N}=2}$; or up to a gauge symmetry, to $\left(\boldsymbol{Y},\boldsymbol{\chi}_\alpha,\boldsymbol{\Phi}\right)$ with $\boldsymbol{L}=D^\alpha\boldsymbol{\chi}_\alpha+\bar{D}_{\dot{\alpha}}\bar{\boldsymbol{\chi}}^{\dot{\alpha}}$ and $\boldsymbol{Y}$ as in (2.19).

From (V_1,V_2) to $\mathcal{W}^{\mathcal{N}=2}$

In this realisation, the $\mathcal{N}=1$ hermitian superfields V_1 and V_2 are interpreted as gauge superfield potentials obeying the following $U(1)$ gauge transformations

$$\begin{aligned}V_1' &\equiv V_1+\boldsymbol{\Lambda}_l\\ V_2' &\equiv V_2+\left(\boldsymbol{\Lambda}_c+\bar{\boldsymbol{\Lambda}}_c\right)\end{aligned} \tag{A.14}$$

They involve two $\mathcal{N}=1$ superfield gauge parameters; the hermitian $\boldsymbol{\Lambda}_l=\bar{\boldsymbol{\Lambda}}_l$ and the chiral $\boldsymbol{\Lambda}_c$ solving the superspace conditions

$$\begin{aligned}D^2\Lambda_l &= 0 = \bar{D}^2\Lambda_l\\ \bar{D}_{\dot{\alpha}}\Lambda_c &= 0 = D_\alpha\bar{\Lambda}_c\end{aligned} \tag{A.15}$$

Each one of these superparameters reduces by 4 the number of degrees of freedom in the corresponding gauge superfield. The $\mathcal{N}=2$ Maxwell chiral superfield strength $\mathcal{W}^{\mathcal{N}=2}\equiv(\boldsymbol{X},\boldsymbol{W}_\alpha)$ is related to (V_1,V_2) through the following gauge invariant quantities

$$\boldsymbol{X}=\frac{1}{2}\bar{D}^2V_1 \qquad , \qquad \boldsymbol{W}_\alpha=-\frac{1}{4}\bar{D}^2D_\alpha V_2 \tag{A.16}$$

they can be combined into an $\mathcal{N}=2$ chiral superfield strength as in (2.4) namely

$$\mathcal{W}^{\mathcal{N}=2} = \boldsymbol{X} + i\sqrt{2}\,\tilde{\theta}^{\alpha}\boldsymbol{W}_{\alpha} - \tilde{\theta}^{2}\left(\frac{1}{4}\bar{D}^{2}\bar{\boldsymbol{X}}\right)$$

which by substituting can be also expressed like $\mathcal{W}^{\mathcal{N}=2} = \frac{1}{4}\bar{D}^{2}\Gamma$ with

$$\Gamma = 2V_1 - i\sqrt{2}\tilde{\theta}^{\alpha}\left(D_{\alpha}V_2\right) - \tilde{\theta}^{2}\left(D^{2}V_1\right) \tag{A.17}$$

Notice that in analogy with the Wess–Zumino gauge commonly used for V_2, the vector superfield V_1 in (A.16) has also a Wess–Zumino like expansion leading, after substituting in $\boldsymbol{X} = \frac{1}{2}\bar{D}^{2}V_1$, to the following θ-expansion

$$\boldsymbol{X} = X + \sqrt{2}\theta\psi_1 + \theta^{2}\bar{\theta}^{2}F_X \tag{A.18}$$

with X standing for the complex scalar of the $\mathcal{N}=2$ gauge multiplet and $\psi_{\alpha 1}$ for one of the two gauginos; the other gaugino $\psi_{\alpha 2}$ comes from $\boldsymbol{W}_{\alpha}$. However, the non-propagating complex auxiliary field F_X is realised here as $F_X = -d_1 - i\partial_{\mu}\upsilon_1^{\mu}$ with imaginary part $\operatorname{Im} F_X$ given by the topological quantity

$$\partial_{\mu}\upsilon_1^{\mu} = \frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{[\mu\nu\rho\sigma]} \tag{A.19}$$

with $\mathcal{F}_{[\mu\nu\rho\sigma]}$ interpreted as the field strength of a 3-form gauge potential $\mathcal{A}_{[\nu\rho\sigma]}$ describing a non-propagating component field. Being completely antisymmetric, this 4-tensor $\mathcal{F}_{[\mu\nu\rho\sigma]}$ can be realised like $\frac{1}{\kappa}\varepsilon_{\mu\nu\rho\sigma}$ where $\frac{1}{\kappa}$ is precisely the deformation term appearing in the second line of (2.4).

A.2.2. From (V_1, V_2) to tensor multiplet $\mathcal{T}^{\mathcal{N}=2}$

First recall that in this paper, we have considered two kinds of realisations of the tensor multiplet $\mathcal{T}^{\mathcal{N}=2}$: (i) a short representation where $\mathcal{T}^{\mathcal{N}=2} \equiv (\boldsymbol{L}, \boldsymbol{\Phi})$ having $8_B + 8_F$ degrees of freedom; and (ii) a long representation $\mathcal{T}^{\mathcal{N}=2} \equiv \left(\boldsymbol{Y}, \boldsymbol{\chi}_{\alpha}, \boldsymbol{\Phi}\right)$ involving $16_B + 16_F$ degrees of freedom obeying the symmetry (A.23). The first one is obtained from the second by gauging away half of the degrees of freedom as in eq. (A.11). In fact both short and long representation of $\mathcal{T}^{\mathcal{N}=2}$ may be imagined as solutions of the following linear constraint relations

$$\begin{aligned} \bar{D}^{2}U_1 &= D^{2}U_1 &&= 0 \\ \bar{D}^{2}D_{\alpha}U_2 &= D^{2}\bar{D}_{\dot{\alpha}}U_2 &&= 0 \end{aligned} \tag{A.20}$$

where we have used the notation (U_1, U_2) to avoid confusion with the hermitian multiplets (V_1, V_2) used in eq. (A.16). Notice that eq. (A.20) may thought of as the complement of eq. (A.16) in the space parameterised by (V_1, V_2).

A first type of solution of these constraint relations is easily identified by remembering that the $\mathcal{N}=1$ linear multiplet $\boldsymbol{L}$ satisfies also the conditions $D^{2}\boldsymbol{L} = \bar{D}^{2}\boldsymbol{L} = 0$ exactly as U_1. Moreover because of the chirality properties $\bar{D}_{\dot{\alpha}}\boldsymbol{\Phi} = D_{\alpha}\boldsymbol{\Phi} = 0$ as well as the relations $\left[D_{\alpha}, \partial_{\mu}\right] = \left[\bar{D}_{\dot{\alpha}}, \partial_{\mu}\right] = 0$; it follows that (U_1, U_2) are nothing but

$$\begin{aligned} U_1 &= \boldsymbol{L} \\ U_2 &= \boldsymbol{\Phi} + \bar{\boldsymbol{\Phi}} \end{aligned} \tag{A.21}$$

where we have dropped out the spurious superfield $\boldsymbol{Y}$.

The second type of solution of (A.20) is given by $\mathcal{T}^{\mathcal{N}=2} \equiv \left(\boldsymbol{Y}, \boldsymbol{\chi}_{\alpha}, \boldsymbol{\Phi}\right)$; it is convenient for the use of the $\mathcal{N}=2$ chiral superspace and reads, roughly speaking, in terms of the four $\mathcal{N}=1$ chiral superfields as follows

$$\begin{aligned} U_1 &= D^{\alpha}\boldsymbol{\chi}_{\alpha} + \bar{D}_{\dot{\alpha}}\bar{\boldsymbol{\chi}}^{\dot{\alpha}} \\ U_2 &= \left(\boldsymbol{\Phi}+\bar{\boldsymbol{\Phi}}\right) + \tfrac{i}{2}\left(D^2\boldsymbol{Y} - \bar{D}^2\bar{\boldsymbol{Y}}\right) \end{aligned} \tag{A.22}$$

with the gauge symmetry property

$$\begin{aligned} \boldsymbol{\chi}'_{\alpha} &= \boldsymbol{\chi}_{\alpha} + \tfrac{i}{4}\bar{D}^2 D_{\alpha}\Omega \\ \boldsymbol{Y}' &= \boldsymbol{Y} - \tfrac{1}{2}\bar{D}^2\Upsilon \\ \boldsymbol{\Phi}' &= \boldsymbol{\Phi} \end{aligned} \tag{A.23}$$

where Ω and Υ are two $\mathcal{N}=1$ hermitian gauge superfield parameters. Notice that the implementation of the superfield $\boldsymbol{Y}$ is required by off shell closure of $\mathcal{N}=2$ supersymmetry as in eq. (2.8). Notice also the three following features:

(i) first in the gauge where $\boldsymbol{Y}'=0$, the chiral superfield $\boldsymbol{Y}$ is given by $\frac{1}{2}\bar{D}^2\Upsilon$; by comparing this expression with the first relation of (A.16), one learns that $\boldsymbol{Y}$ has same form as the chiral superfield $\boldsymbol{X}=\frac{1}{2}\bar{D}^2V_1$. Obviously $\boldsymbol{X}$ and $\boldsymbol{Y}$ are different things; the first one carries propagating physical degrees of freedom; while the second is a pure auxiliary superfield with no physical field.

(ii) Second alike for the relations (A.18)–(A.19) satisfied by the $\boldsymbol{X}$ superfield and allowing adjunction of a topological term to the lagrangian density implemented in superspace formulation by the $\frac{\tilde{\theta}^2}{2\kappa}$ deformation in the second line of (2.4), one has as well a quite similar feature for the superfield $\boldsymbol{Y}$. Indeed, following [31], there is a gauge where $\boldsymbol{Y}$ reads as

$$\boldsymbol{Y} = \frac{i}{4!}\theta^2\varepsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} \tag{A.24}$$

with $C_{\mu\nu\rho\sigma}$ a 4-form field with no propagating degree of freedom.

(iii) Third if denoting by

$$\begin{aligned} \hat{w}_{\alpha} &= -\tfrac{1}{4}\bar{D}^2 D_{\alpha}\Omega \\ \hat{x} &= \tfrac{1}{2}\bar{D}^2\Upsilon \end{aligned} \tag{A.25}$$

then the gauge transformations (A.23) read as

$$\begin{aligned} \boldsymbol{\chi}'_{\alpha} &= \boldsymbol{\chi}_{\alpha} - i\hat{w}_{\alpha} \\ \boldsymbol{Y}' &= \boldsymbol{Y} - \hat{x} \\ \boldsymbol{\Phi}' &= \boldsymbol{\Phi} \end{aligned} \tag{A.26}$$

Putting this change into

$$\mathcal{T}^{\mathcal{N}=2} = \boldsymbol{Y} + \sqrt{2}\tilde{\theta}^{\alpha}\boldsymbol{\chi}_{\alpha} - \tilde{\theta}^2\left(\frac{i}{2}\boldsymbol{\Phi} + \frac{1}{4}\bar{D}^2\bar{\boldsymbol{Y}}\right)$$

we find that it transforms like

$$\mathcal{T}'^{\mathcal{N}=2} = \mathcal{T}^{\mathcal{N}=2} - \hat{w} \tag{A.27}$$

with $\mathcal{N}=2$ superfield gauge parameter

$$\hat{w} = \hat{x} + i\sqrt{2}\tilde{\theta}^{\alpha}\hat{w}_{\alpha} - \frac{1}{4}\tilde{\theta}^{2}\bar{D}^{2}\overline{\hat{x}} \tag{A.28}$$

having the same structure as the Maxwell superfield $\mathcal{W}^{\mathcal{N}=2}$. From this view, it follows that a single-tensor superfield $\mathcal{T}^{\mathcal{N}=2}$ is a chiral superfield

$$\mathcal{Z} = Z(y,\theta) + \sqrt{2}\tilde{\theta}^{\alpha}\Upsilon_{\alpha}(y,\theta) - \tilde{\theta}^{2}F(y,\theta)\left[\frac{i}{2}\Phi_{Z}(y,\theta) + \frac{1}{4}\bar{D}^{2}\bar{Z}(y,\theta)\right] \tag{A.29}$$

obeying the gauge symmetry (A.27) with superfield parameter as in (A.28).

A.2.3. Comment on gauge symmetry

Here we comment on the gauge symmetry of the field action term $\mathcal{S}(Y) = \int d^4x\mathcal{L}(Y)$ with lagrangian density $\mathcal{L}(Y) \equiv \mathcal{L}$ given by

$$\mathcal{L} = \eta\int d^2\theta\mathbf{Y} + hc \tag{A.30}$$

and coupling constant a pure imaginary number $\eta = i\nu$. This field action $\mathcal{S}(Y)$ is an extra term that has been used in this paper to induce a deformation of the ADJ constraint $\sum_a \frac{g_a}{\kappa_a} = 0$ into $\nu + \sum_a \frac{g_a}{\kappa_a} = 0$; it lifts the singularity of the particular equation $\frac{g_1}{\kappa_1} = 0$; but breaks $\mathcal{N} = 2$ supersymmetry explicitly.

Gauge invariance of $\mathcal{S}(Y)$ can be studied either directly by using the superspace method taking into account that the superfield $\mathbf{Y}$ is a chiral multiplet; or more explicitly by working with component fields. In the first way, the gauge transformation of $\mathbf{Y}$ can be learnt from (A.23); it reads as $\mathbf{Y}' = \boldsymbol{Y} + \delta_{gauge}\mathbf{Y}$ with $\delta_{gauge}\mathbf{Y} = -\frac{1}{2}\bar{D}^2\Upsilon$; where Υ is an arbitrary $\mathcal{N} = 1$ real superfield. To shed light on this gauge transformation and its effect on the action $\mathcal{S}(Y)$; we will use the component field language to first build the explicit θ-expansions of Υ and $\bar{D}^2\Upsilon$; then turn back to study gauge symmetry of (A.30).

• *Component field analysis*

We begin by describing the θ-expansion of the superfield Y which can be expressed into two manners: (i) either by using the complex chiral superspace coordinate basis $Z_c = \left(x - i\theta\sigma\bar{\theta}, \theta\right)$ where Y is expanded as $y + \sqrt{2}\theta\Psi + \theta^2\mathrm{F}$; or ($ii$) by working in the real superspace coordinate basis $Z_R = \left(x^{\mu}, \theta^{a}, \bar{\theta}_{\dot{a}}\right)$ where the superfield Y has also an explicit $\bar{\theta}$ dependence as shown below[2]

$$Y = y - i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}y + \frac{1}{4}\theta^2\bar{\theta}^2\Box y + \sqrt{2}\theta\Psi + \frac{i}{\sqrt{2}}\theta^2\partial_{\mu}\Psi\sigma^{\mu}\bar{\theta} + \theta^2F_Y \tag{A.31}$$

From this expansion, we learn

$$\mathcal{L} = i\nu\left(F_Y - \bar{F}_Y\right) \tag{A.32}$$

In the gauge transformation $\delta_{gauge}\mathbf{Y} = -\frac{1}{2}\bar{D}^2\Upsilon$, the real superfield parameter Υ has a θ-expansion involving as usual several component field parameters which, for later use can be expressed like

$$\Upsilon = \gamma - \frac{1}{2}\theta^2\bar{M} - \frac{1}{2}\bar{\theta}^2M - \theta\sigma^{\mu}\bar{\theta}w_{\mu} + \theta^2\bar{\theta}^2\left(d_{\Upsilon} + \frac{1}{4}\Box\gamma\right) + \textit{fermionic} \tag{A.33}$$

[2] Our notations are as in Wess–Bagger's book: supersymmetry and supergravity [47].

where "*fermionic*" stands for those monomials with odd powers in Grassman variables involving fermionic fields. The components γ, d_Υ are real scalar components; the complex scalars M, $\bar{M}$ are related by complex conjugation; and the real 4-vector w_μ is parameterised like $w_\mu = \frac{1}{3!}\varepsilon_{\mu\nu\rho\sigma}A^{[\nu\rho\sigma]}$ with $A^{[\nu\rho\sigma]}$ a completely antisymmetric real rank 3-tensor. This dual parametrisation of w_μ is dictated by an underlying arbitrariness in dealing with the superfield Υ; by shifting the real superparameter Υ by a hermitian superfield Γ like $\Upsilon' = \Upsilon + \Gamma$, the quantity $\bar{D}^2\Upsilon$ remains invariant provided Γ satisfying the constraints

$$\bar{D}^2\Gamma = D^2\Gamma = 0 \qquad , \qquad \Gamma^\dagger = \Gamma \tag{A.34}$$

But these conditions are same as the ones defining linear multiplet $\boldsymbol{L}$ introduced in Section 2 and which we recall here for comparison $\bar{D}^2\boldsymbol{L} = D^2\boldsymbol{L} = 0$ with $\boldsymbol{L}^\dagger = \boldsymbol{L}$; the real superfield Γ has therefore a similar θ-expansion as eq. (2.16); and so it can be expressed like

$$\Gamma = \gamma + \theta\sigma^\mu\bar{\theta}\varepsilon_{\mu\nu\rho\sigma}\partial^\nu\omega^{[\rho\sigma]} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\gamma + \textit{fermionic} \tag{A.35}$$

showing that the term $\varepsilon_{\mu\nu\rho\sigma}\partial^\nu\omega^{\rho\sigma}$ is nothing but a gauge transformation of $w_\mu = \frac{1}{3!}\varepsilon_{\mu\nu\rho\sigma}A^{[\nu\rho\sigma]}$ with gauge parameter given by the rank 2-antisymmetric $\omega^{\rho\sigma} = -\omega^{\sigma\rho}$. Taking advantage of the arbitrariness (A.34), one can make convenient choices; in particular we can put the real superfield Υ into the following form; see also [31],

$$\Upsilon = -\frac{1}{2}\theta^2\bar{M} - \frac{1}{2}\bar{\theta}^2 M - \theta\sigma^\mu\bar{\theta}w_\mu + \frac{1}{\sqrt{2}}\theta^2\bar{\theta}\bar{\lambda} + \frac{1}{\sqrt{2}}\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 d_\Upsilon \tag{A.36}$$

With this choice; the initial gauge symmetry (A.23) gets now reduced to gauge symmetry of the 4-vector $w_\mu = \frac{1}{3!}\varepsilon_{\mu\nu\rho\sigma}A^{[\nu\rho\sigma]}$ namely

$$w_\mu \qquad \to \qquad w_\mu + \varepsilon_{\mu\nu\rho\sigma}\partial^\nu\omega^{\rho\sigma} \tag{A.37}$$

• *More on gauge invariance*

From the gauge fixed expression (A.36), we can make two useful computations namely the explicit expressions of $\frac{1}{2}\bar{D}^2\Upsilon$ and the integral over Grassman variables $i\nu\int d^2\theta\ \boldsymbol{Y} + hc$. For the θ-expansion of $\frac{1}{2}\bar{D}^2\Upsilon$; we find

$$\begin{aligned} \tfrac{1}{2}\bar{D}^2\Upsilon \ =\ & M + \sqrt{2}\theta\lambda - \theta^2\left[d_\Upsilon + i\partial_\mu w^\mu\right] + i\theta\sigma^\nu\bar{\theta}\partial_\nu M \\ & - \tfrac{i}{\sqrt{2}}\theta^2\partial_\nu\lambda\sigma^\nu\bar{\theta} + \tfrac{1}{4}\theta^2\bar{\theta}^2\Box M \end{aligned} \tag{A.38}$$

where

$$\begin{aligned} \partial_\mu w^\mu \ &=\ \tfrac{1}{3!}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu A_{[\nu\rho\sigma]} \\ &=\ \tfrac{1}{4!}\varepsilon^{\mu\nu\rho\sigma}F_{[\mu\nu\rho\sigma]} \equiv \Delta \end{aligned} \tag{A.39}$$

has an interpretation in term of rank 4-antisymmetric field strength $F_{[\mu\nu\rho\sigma]}$ of the 3-form potential field $A_{[\nu\rho\sigma]}$. Under the residual gauge (A.37), the term $\partial_\mu w^\mu$ is therefore manifestly gauge invariant; this property can be explicitly checked on the transformation

$$\partial_\mu w^\mu \qquad \to \qquad \partial_\mu w^\mu + \varepsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu\omega_{\rho\sigma} \tag{A.40}$$

where the extra term $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu\omega_{\rho\sigma}$ vanishes identically due to a tensor calculus feature.

Comparing (A.31) and (A.38), one remarks, that by using a Wess–Zumino-like gauge, $\mathbf{Y}$ can be expressed as $\boldsymbol{Y}_{gauged} = i\theta^2\partial_\mu w^\mu$. Moreover, using this gauge fixed expression, the field action $\mathcal{S} = \int d^4x\mathcal{L}$ with the lagrangian density $\mathcal{L}$ as in eq. (A.30) reads therefore like,

$$\mathcal{S} = -2\nu \int d^4x\ \partial_\mu w^\mu \tag{A.41}$$

and, because of (A.40), is manifestly gauge invariant $\delta_{gauge}\mathcal{S} = 0$.

In the end of this comment, we would like to add that even if thinking of the scalar $\partial^\mu w_\mu = \Delta$ as the dual of a generic rank 4-tensor gauge potential like $\Delta = \frac{1}{4!}\varepsilon^{\mu\nu\rho\sigma}A_{[\mu\nu\rho\sigma]}$ where the completely antisymmetric $A_{[\mu\nu\rho\sigma]}$ obeys the gauge transformation $\delta A_{[\mu\nu\rho\sigma]} = \partial_{[\mu}\Lambda_{\nu\rho\sigma]}$, the variation of the quantity $\frac{1}{4!}\varepsilon^{\mu\nu\rho\sigma}A_{[\mu\nu\rho\sigma]}$ behaves as a divergence term $\frac{1}{3!}\partial_\mu\left(\varepsilon^{\mu\nu\rho\sigma}\Lambda_{\nu\rho\sigma}\right)$; and, up on ignoring boundary effects, this variation does not contribute at the level of the action $\mathcal{S} = -2\nu\int d^4x\ \frac{1}{4!}\varepsilon^{\mu\nu\rho\sigma}A_{[\mu\nu\rho\sigma]}$. However, in this way of doing, the field strength $F_{[\mu\nu\rho\sigma\tau]} \equiv F_{(5)}$, associated to the potential field $A_{[\mu\nu\rho\sigma]} \equiv A_{(4)}$, would be a rank 5-antisymmetric tensor field which is invariant under the gauge transformation $A_{(4)} \to A_{(4)} + d\Lambda_{(3)}$; but because of space time dimension constraint, the rank 5-tensor should be equal to zero, $F_{(5)} = dA_{(4)} = 0$; this means that the 4-form $A_{(4)}$ is a pure gauge potential without curvature which may be thought of as $A_{(4)} = dA_{(3)}$; and then the gauge parameter $\Lambda_{(3)}$ as just a shift of the origin of $A_{(3)}$.

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