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Characterisation of Two-Sided PF-Rings*

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In this article we consider only associative rings with nonzero identity and unitary modules. A ring R is called a *two-sided* PF-*ring* or a *ring with perfect duality* if it is a left and right PF-ring or equivalently both ${}_RR$ and R_R are injective cogenerators. Two-sided PF-rings are natural generalizations of quasi-Frobenius rings. The aim of this work is to describe two-sided PF-rings, using the methods developed in [2]. This description is similar to the classical definition of Nakayama for quasi-Frobenius rings and clarifies the role of the descending chain condition in the theory of quasi-Frobenius rings.

Recall that an R-module M is *linearly compact in the discrete topology* or simply *linearly compact* if every finitely solvable system of congruences $m \equiv m_\alpha \pmod{M_\alpha}$ is solvable where the M_α are submodules of M and $m_\alpha \in M$. M is *finitely cogenerated* if M is an essential extension of a finite direct sum of simple modules. For every module ${}_RM(M_R)$ the dual M^* of M is a right (left) R-module $\mathrm{Hom}_R(M, R)$.

THEOREM 1. *A ring R is a two-sided PF-ring iff the modules ${}_RR$ and R_R are both finitely cogenerated and linearly compact and the duals of simple modules are also simple.*

Proof. The necessity is well known by [3, p. 291, 12.5.2 Satz and Exercise 4(b)]. We have to show only the sufficiency. We do it in several steps.

(1) For each subset A of R we denote by $l(A)$ and $r(A)$ the left and right annihilators of A, respectively. For every filter base $\{L_\alpha\}$ of left ideals L_α the equality $r(\bigcap L_\alpha) = \bigcup r(L_\alpha)$ holds. In fact, each $x \in R$ induces a continuous homomorphism $x: R \to R : r \to rx$ and hence the linear compact-

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ness of ${}_RR$ implies by [4, Satz 1] that $0=(\bigcap L_\alpha)x=\bigcap L_\alpha x$, provided $x\in r(\bigcap L_\alpha)$. This shows by [6, Prop. 3.19] that $x\in r(L_\alpha)$ for some index α because ${}_RR$ is finitely cogenerated. Therefore, $r(\bigcap L_\alpha)=\bigcup r(L_\alpha)$.

(2) The left and right socles $So({}_RR)$ and $So(R_R)$ are equal. Consider a minimal left ideal Ra. By $Ja=0$ we obtain $a\in r(J)$, where J is the radical of R. This shows that a can be considered as an element of the dual of the left R-module R/J. Since the left R-module R/J is a finite direct sum of simple modules and by the assumption finite direct sums of simple modules are reflexive, aR is clearly semisimple. Thus $aR\in So(R_R)$ and consequently $Ra\in So(R_R)$ because $So(R_R)$ is an ideal of R. Hence $So({}_RR)\subseteq So(R_R)$ and by symmetry $So(R_R)\subseteq So({}_RR)$ follows, i.e., $So({}_RR)=So(R_R)$.

(3) Let e be an arbitrary primitive idempotent of R. By the assumption eR is an essential extension of its socle $So(eR)$. Since $l(So(eR))$ contains clearly $R(1-e)+J$ and $e\notin l(So(eR))$, $l(So(eR))$ is obviously a maximal left ideal of R. Therefore, $So(eR)$ can be considered as a nonzero submodule of the dual of the simple module $R/l(So(eR))$ and thus $So(eR)$ is simple. By symmetry $So(Re)$ is also simple. The linear compactness of R ensures that R is a semiperfect ring and hence there is a complete system of pairwise orthogonal idempotents e_i $(i=1,\dots,n)$ such that $R=Re_1\oplus\cdots\oplus Re_n=eR_1\oplus\cdots\oplus eR_n$. Together with the above considerations we obtain that the dual of the left R-module R/J is $So(R_R)=r(J)$ and the dual of the right R-module R/J is $So({}_RR)=l(J)$.

(4) $r(L)\neq r(JL)$ for each finitely generated left ideal $L\neq 0$. In the case $L=Ra$ there is $b\in R$ such that abR is a minimal right ideal because R_R is finitely cogenerated. Consequently, $Jab=0$ by $So({}_RR)=So(R_R)$ and hence $b\in r(Ja)\backslash r(a)$, i.e., $r(Ja)\neq r(a)$. Suppose now by induction that $L=Ra_1+\cdots+Ra_n$ and for $L_1=Ra_1+\cdots+Ra_{n-1}$ we have $r(JL_1)\neq r(L_1)$. Choose $b\in r(JL_1)\backslash r(L_1)$. If $a_nb=0$, then we are done. In the case $a_nb\neq 0$ there is $c\in R$ such that a_nbcR is a minimal right ideal and thus $a_nbc\in So(R_R)=So({}_RR)$, i.e., $Ja_nbc=0$ and then it follows clearly $bc\in r(JL)\backslash r(L)$. Thus we have always $r(JL)\neq r(L)$.

(5) For each left ideal L and $a\in R$ with $r(L)\subseteq r(a)$ there is a minimal finitely generated left ideal $K\subseteq L$ satisfying $r(K)\subseteq r(a)$. In fact, the equality $r(L)=\bigcap_{x\in L} r(x)$ implies by [5, Lemma 2] the existence of finitely many elements $x_1,\dots,x_n$ with $\bigcap_i r(x_i)\subset r(a)$ because $aR\approx R/r(a)$ is finitely cogenerated. This shows that the set of finitely generated left ideals $K\subseteq L$ with $r(K)\subseteq r(a)$ is nonempty. If $\{K_\alpha\}$ is a descending chain of finitely generated left ideals satisfying $r(K_\alpha)\subseteq r(a)$, then $r(\bigcap K_\alpha)=\bigcup r(K_\alpha)\subseteq r(a)$. Therefore, we obtain a finitely generated left ideal $T\subseteq\bigcap K_\alpha$ with $r(T)\subseteq r(a)$. Henceforth, Zorn's Lemma ensures existence of a minimal finitely generated left ideal K with $r(K)\subseteq r(a)$. If now T is any left ideal contained in K with $r(T)\subseteq r(a)$, then the above result ensures a finitely

generated left ideal T' with $T' \subseteq T$ and $r(T') \subseteq r(a)$. This means that $T = T' = K$ and hence we obtain that K is a minimal left ideal contained in L satisfying $r(K) \subseteq r(a)$.

(6) For each left ideal L and $a \in R$ with $r(L) \subseteq r(a)$ there is $b \in L$ with $r(a) = r(b)$. Indeed, if $a = 0$ there is nothing to prove. Therefore, we may suppose $a \neq 0$. Then by Step (5) there is a minimal finitely generated left ideal $K \subseteq L$ with $r(K) \subseteq r(a)$. In particular $K \neq 0$. Let φ_1 denote the following mapping defined by

$$\varphi_1 : r(JK)/r(K) \to (K/JK)^* : (y + JK)\, \varphi_1(x + r(K)) = yx, \qquad x \in r(JK),\ y \in K.$$

One can easily see that φ_1 is a monomorphism. Let $N = \varphi_1(r(JK)/r(K))$. Then N is a nonzero submodule of $M = (K/JK)^*$. Assume now indirectly that $N \neq M$. In this case there is a nonzero submodule $\bar{P}$ properly contained in K/JK such that for each $0 \neq n \in N$ there is $\bar{y} \in \bar{P}$ with $\bar{y}n \neq 0$. Henceforth for the inverse image P of $\bar{P}$ under the epimorphism $K \to K/JK$ we obtain $JK \subset P \subset K$ and for each $x \in r(JK) \setminus r(K)$ there exists $y \in P$ with $yx \neq 0$. Thus $r(P) = r(K) \subseteq r(a)$, so that by Step (5) there exists a finitely generated left ideal $P' \subseteq P$ satisfying $r(P') \subseteq r(a)$, which contradicts to the minimality of K. Hence $N = M$ and φ_1 is an isomorphism.

Consider a nonzero element f of K^*. Since Kf is a nonzero finitely generated left ideal, there is $c \in r(JKf) \setminus r(Kf)$ by Step (4). This shows that fc can be considered as an element of $(K/JK)^*$, and hence fcR is a semisimple. Consequently, we obtain that K^* is an essential extension of its socle $So(K^*)$. On the other hand, it is routine to verify that $So(K^*)$ is the dual of K/JK which is a finite direct sum of simple modules and hence K^* is finitely cogenerated and $(K/JK)^*$ is the socle of K^*.

For $A = K + Ra$ the equality $r(A) = r(K) \cap r(a) = r(K)$ holds and thus we can define a mapping φ_2 by putting

$$\varphi_2 : r(a)/r(K) \to (A/Ra)^* : (y + Ra)\, \varphi_2(x + r(K)) = yx, \qquad x \in r(a),\ y \in A.$$

One can verify that φ_2 is a monomorphism. By the isomorphism $A/Ra = (K + Ra)/Ra \approx K/(K \cap Ra)$ the module $r(a)/r(K)$ can be considered as a submodule of $(K/(K \cap Ra))^*$ and hence of K^*. Since K^* is finitely cogenerated, we have the nonzero intersection $H = (r(a)/r(K)) \cap (r(JK)/r(K))$ in the case $r(a) \neq r(K)$. H is properly contained in the scale of K^*, otherwise we have $r(JK) \subseteq r(a)$ which is a contradiction to the minimality of K. Suppose indirectly $r(a) \neq r(K)$. Then φ_1 implies the nontrivial direct decomposition $K/JK = \bar{P} \oplus \bar{Q}$ such that the dual of $\bar{Q}$ is H and $\bar{P}$ is the annihilator of H. This shows for the inverse image P of $\bar{P}$ under the epimorphism $K \to K/JK$ that P is properly contained in K and

$r(P)/r(K) = H$. Thus $r(P) \subseteq r(a)$ and hence there is a finitely generated left ideal properly contained in K with $r(T) \subseteq r(a)$. This contradicts to the minimality of K. Therefore, we have $r(K) = r(a)$. Since $aR \simeq R/r(a)$ and $r(JK)/r(K) = (K/JK)^*$, we obtain that $(K/JK)^*$ can be considered as a submodule of $So(R_R)$. Consequently, K/JK is isomorphic to a direct summand of a left R-module $R/J = (So(R_R))^*$ and hence K/JK can be generated by one element. Since K is finitely generated, we deduce immediately that K is a principal left ideal, i.e., $K = Rb$ for some $b \in R$ and hence $r(K) = r(b) = r(a)$.

(7) For any two elements $a, b \in R$ satisfying $r(a) = r(b)$ there is an element $s \in R$ with $b = sa$. If $a = 0$ there is nothing to prove. Suppose $a \neq 0$. The equality $r(a) = r(b)$ implies that the mapping $aR \to bR : ar \to br$ is well defined and it is an isomorphism. Let $t \in R$ such that atR is a minimal right ideal of R. The above isomorphism ensures that the minimal right ideals atR and btR are isomorphic. Thus $r(at) = r(bt)$ is a maximal right ideal, and the dual of the simple module $R/r(at)$ is $lr(at)$ as it is easy to check. Consequently, at and bt generate the same simple module $lr(at)$. Henceforth, there is an element $s \in R$ with $bt = sat$. From this fact it follows that $r(b - sa)$ contains properly $r(a)$. Now consider the family of all right ideals I_α such that there exists $s_\alpha \in R$ with $(b - s_\alpha a) I_\alpha = 0$ and $I_\alpha \supset r(a)$. This family of right ideals is partially ordered by inclusion. Note that if I_1, I_2 are in the family, $I_1 \subseteq I_2$ and $s_1, s_2 \in R$ are such that $(b - s_1 a) I_1 = 0$ and $(b - s_2 a) I_2 = 0$, then $(s_1 - s_2) a I_1 = ((b - s_2 a) - (b - s_1 a)) I_1 \subseteq (b - s_2 a) I_1 + (b - s_1 a) I_1 \subseteq (b - s_2 a) I_2 + (b - s_1 a) I_1 = 0$, i.e., $s_1 - s_2 \in \ell(aI_1)$. Now consider an ascending chain I_α in the family. For each α let $s_\alpha \in R$ be such that $(b - s_\alpha a) I_\alpha = 0$. Then the system of congruences $s \equiv s_\alpha \pmod{\ell(aI_\alpha)}$ is finitely solvable, because if $I_1 \subseteq \cdots \subseteq I_n$ are in the family, then $s_i - s_n \in \ell(aI_i)$ for $i = 1, ..., n$ that is $s_n \equiv s_i \pmod{\ell(aI_i)}$ for $i = 1, ..., n$. Therefore, the linear compactness ensures the existence of a solution s for this system. If $I = \bigcup I_\alpha$, then $(s - s_\alpha) a I_\alpha = 0$ for all α, so that $(b - sa) I_\alpha = 0$ for all α, i.e., $(b - sa) I = 0$ and I is in the family. Therefore, Zorn's Lemma implies the existence of a right ideal I maximal in the family. Let $s \in R$ be such that $(b - sa) I = 0$. We claim $I = R$. Suppose indirectly $I \neq R$. Then by Step (6) there is an element u of R with $r(b - sa) = r(ua)$. Consequently, by the similar way to the case $r(a) = r(b)$ there is $v \in R$ with $r(b - sa) \subset r(b - sa - vua) = r(b - (s + vu)a)$. This shows I is properly contained in $r(b - (s + vu)a)$ which contradicts to the maximality of the I. Henceforth, $I = R$ and thus $b = sa$.

(8) For any left ideal L we have $lr(L) = L$. In fact, if $a \in lr(L)$, then $r(L) \subseteq r(a)$ and hence there is $b \in L$ with $r(a) = r(b)$ and thus $a = sb$ for some $s \in R$, i.e., $a \in L$. This shows $L = lr(L)$. Since our conditions are symmetrical, the above results are true for left annihilators and right ideals. In

fact, for all right ideals A we have $A = rl(A)$ and hence R is a ring with annihilator condition.

By [1, Corollary 5.9] we have that ${}_RR$ and R_R are injective and hence they are cogenerators, too, i.e., R is a two-sided PF-ring.

COROLLARY 1. *A ring R is a two-sided PF-ring iff the modules ${}_RR$ and R_R are both linearly compact and finitely congenerated and for any module M of finite length we have $lg(M) = lg(M^*)$, where lg means the length of the module.*

The proof of the following consequence is the same as in [3, Satz 13.4.2] hence we omit it.

COROLLARY 2. *Let R be a ring such that the modules ${}_RR$ and R_R are both linearly compact and finitely congenerated. Then the following statements are equivalent.*

(1) *R is a two-sided PF-ring.*

(2) *For each primitive idempotent e the socles $So(Re)$ and $So(eR)$ of Re and eR, respectively are simple and all simple left and right modules have nonzero homomorphic images in the socles $So({}_RR)$ and $So(R_R)$, respectively.*

(3) *For each primitive idempotent e the socles $So(Re)$ and $So(eR)$ are simple and $So({}_RR) = So(R_R)$ holds.*

(4) *For a maximal set $\{e_1, ..., e_n\}$ of pairwise orthogonal primitive idempotents such that Re_i and Re_j are not isomorphic for all $i \neq j$, there is a permutation π of the set $\{1, ..., n\}$ satisfying*

$$So(e_i R) \approx \bar{e}_{\pi(i)} \bar{R}, \qquad So(Re_{\pi(i)}) \approx \bar{R} \bar{e}_i,$$

where $\bar{R}$ denotes the factor ring of R by its radical.

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