Explicit Hopf–Lax type formulas for Hamilton–Jacobi equations and conservation laws with discontinuous coefficients

Adimurthi a, Siddhartha Mishra a,b, G.D. Veerappa Gowda a,*

a TIFR center, PO Box 1234, Bangalore-560012, India
b Department of Mathematics, Indian Institute of Science, Bangalore, India

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Abstract

We deal with a Hamilton–Jacobi equation with a Hamiltonian that is discontinuous in the space variable. This is closely related to a conservation law with discontinuous flux. Recently, an entropy framework for single conservation laws with discontinuous flux has been developed which is based on the existence of infinitely many stable semigroups of entropy solutions based on an interface connection. In this paper, we characterize these infinite classes of solutions in terms of explicit Hopf–Lax type formulas which are obtained from the viscosity solutions of the corresponding Hamilton–Jacobi equation with discontinuous Hamiltonian. This also allows us to extend the framework of infinitely many classes of solutions to the Hamilton–Jacobi equation and obtain an alternative representation of the entropy solutions for the conservation law. We have considered the case where both the Hamiltonians are convex (concave). Furthermore, we also deal with the less explored case of sign changing coefficients in which one of the Hamiltonians is convex and the other concave. In fact in convex–concave case we cannot expect always an existence of a solution satisfying Rankine–Hugoniot condition across the interface. Therefore the concept of generalised Rankine–Hugoniot condition is introduced and prove existence and uniqueness.

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* Corresponding author.
E-mail addresses: aditi@math.tifrbng.res.in (Adimurthi), sid@math.tifrbng.res.in (S. Mishra), gowda@math.tifrbng.res.in (G.D. Veerappa Gowda).

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1. Introduction

We are interested in the following Hamilton–Jacobi equation,

\[ \begin{align*}
vt + H(k(x), v_x) &= 0, \\
v(x, 0) &= v_0(x),
\end{align*} \tag{1} \]

where \( H \) is the Hamiltonian and \( k \) is a spatially varying and possibly discontinuous coefficient. A special case of (1) is the so called 2-Hamiltonian case given by

\[ \begin{align*}
vt + g(v_x) &= 0 \quad \text{if } x < 0, \ t > 0, \\
v(x, 0) &= v_0(x) \\
vt + f(v_x) &= 0 \quad \text{if } x > 0, \ t > 0, \\
v(x, 0) &= v_0(x) \quad \forall x \in \mathbb{R}, \\
v &\in \text{Lip}(\mathbb{R} \times \mathbb{R}_+). \tag{2}
\end{align*} \]

In the above case, the Hamiltonian \( H \) is discontinuous in the space variable with a single discontinuity at the interface \( x = 0 \). Equations of the type (1) can arise in several applications like the synthetic aperture radar shape from shading equations in image processing. See [25] for details.

The connections that exist between single conservation laws and Hamilton–Jacobi equations in one space dimension are well known. In particular, let \( v \) solve (1), then by taking \( u = v_x \), it can be shown that \( u \) is a solution of the following single conservation law,

\[ \begin{align*}
&u_t + \left( f(k(x), u) \right)_x = 0, \\
&u(x, 0) = u_0(x). \tag{3}
\end{align*} \]

This is an example of a single conservation law with a spatially varying and possibly discontinuous flux function. Similarly (2) is connected to the following conservation law,

\[ \begin{align*}
&u_t + \left( H(x)f(u) + (1 - H(x))g(u) \right)_x = 0, \\
&u(x, 0) = u_0(x). \tag{4}
\end{align*} \]

Equation (4) is a special case of (3) and is called the 2-flux case.

Conservation laws with discontinuous flux arise in a wide variety of applications in physics and engineering. To mention a few, they arise while considering two-phase flow in a heterogeneous porous medium that models petroleum reservoir simulation. They also arise while modeling the action of an ideal clarifier–thickener unit that is used in waste-water treatment plants. Details of the applications can be seen in [28].

Hamilton–Jacobi equations with discontinuous Hamiltonian (with a time dependent discontinuous coefficient) were studied by Ostrov in [26]. Under the assumptions that the Hamiltonian is convex (or concave), he used a vanishing viscosity approximation of (1) and passed to the limit in a control formulation to prove existence of viscosity solutions. In [11], Coclite and Risebro studied (1) (with time dependent coefficients and convex fluxes) and obtained existence of
viscosity solutions by a front tracking approximation. Uniqueness was obtained by a suitable modification of the “doubling of variables” argument.

In [1], Adimurthi and Veerappa Gowda studied (2) with the assumptions that the Hamiltonians \( f, g \) are in \( C^2 \), are convex with superlinear growth and obtained explicit Hopf–Lax type formulas for the viscosity solution. As a consequence of this, they proposed a new interface entropy [18] condition and showed that \( u = v_x \) is a unique entropy solution to the corresponding conservation law (4) thus obtaining a characterization of it in terms of the explicit Hopf–Lax formula.

The corresponding study of the conservation laws (3), (4) is in a mature stage of development. Entropy formulations for (3) (or different special cases of it) have been proposed by Gimse, Risebro in [16,17], Diehl in [12,13], Klingenberg, Risebro in [19], Karlsen, Risebro and Towers in [22] and Audusse, Perthame in [8]. Existence results have been obtained by vanishing viscosity approximations in [20], by front tracking in [17,19] and as limits of numerical schemes in [3,10,21,23,27,29,30] and references therein.

More recently, the authors have embarked on a systematic study of (3), (4) (based on the theory developed in [1]) in a series of papers [2] namely [4–7] and [28]. In these papers, a new entropy framework for (3) is developed. This framework is based on a two-step approach. In the first step, an interface connection is defined and is used to characterize infinite classes of entropy solutions. Each of these classes of solutions is shown to form a stable semi-group in \( L^1 \).

The existence of solutions is shown by designing Godunov-type finite volume schemes based on exact Riemann solvers and showing that they converge to the entropy solution. In the second step, an optimization problem is defined on the set of connections and the optimizer is defined as the optimal entropy solution.

It is now widely accepted that there are more than one valid concepts of entropy solutions for (3) depending on the physics of the problem. Some of these entropy solutions correspond to different semigroups that can be characterized by different connections. For example the optimal entropy solutions of [4] correspond to the physically meaningful solutions for two-phase flows in heterogeneous porous media whereas a different semigroup (see [10]) is valid for the clarifier–thickener unit. The physical relevance of other semigroups is not encountered so far. Thus, the above solution concept provides flexibility in terms of incorporating different semigroups of solutions for different physical models.

Given the close relationship between conservation laws and Hamilton–Jacobi equations in one dimension, it is natural to examine whether multiple classes of “stable” solutions also exist for (1), (2). Given the results of [1,26], where the solutions of the conservation law were characterized in terms of solutions of the Hamilton–Jacobi equation, it is possible to obtain the infinite class of stable solutions for (4) from some explicit formulas of the solutions of the Hamilton–Jacobi equation (2). This paper seeks to provide answers to the above questions. Given an interface connection \((A, B)\) (for definition, see Section 2), we will obtain explicit Hopf–Lax type formulas for the solutions \( v \) of (2) (but with boundary conditions at the interface) and show that \( u = v_x \) is the corresponding \( AB \)-entropy solution of (4). Thus, the infinite classes of entropy solutions of (4) can be characterized in terms of explicit Hopf–Lax type formulas for the corresponding Hamilton–Jacobi equation. This also allows us to introduce multiple “stable” solution concepts at the level of the Hamilton–Jacobi equations (2).

The way we do so is based on defining two Neumann initial boundary value problems for (2) (see (10), (11)) with specification of the boundary values at the interface \( (x = 0) \). The boundary values will allow us to introduce the interface connection in the analysis. The key trick is the construction of appropriate boundary values based on the connection and involving existing results for viscosity solutions of initial boundary value problems for Hamilton–Jacobi equations.
This method provides an alternative way to show existence of infinitely many stable classes of entropy solutions for (4).

We will work with the assumption that the Hamiltonians \( f \) and \( g \) will either be convex or concave. This leads to three possible cases. Namely,

**Case I.** Both \( f \) and \( g \) are convex or concave.

Within the framework of \([1,11,26]\), it was assumed that both the Hamiltonians are either convex or concave. We will deal with this case in considerable detail.

**Case II.** \( g \) is concave and \( f \) is convex.

This corresponds to a case of sign-changing coefficients in (1) and was treated in [4].

**Case III.** \( g \) is convex and \( f \) is concave.

This is also a case corresponding to sign-changing coefficients in (1) and was treated in [7] for the conservation law. We treat this case also. To our knowledge, this is the first time where the above mixed cases have been tackled in literature.

We have organized this paper as follows. In Section 2, we deal with the case where both the Hamiltonians are convex (concave) and obtain the Hopf–Lax formulas for each connection. We will also present most of the preliminary material in this section. In Section 3, we deal with the case where \( f \) is convex and \( g \) is concave and in Section 4, with the case where \( g \) is convex and \( f \) is concave. Both these mixed cases are very different from each other and require different treatment. In fact in the case where \( g \) is convex and \( f \) is concave, in general Rankine–Hugoniot condition may not hold across the interface. In view of this we have introduced the concept of generalised Rankine–Hugoniot condition (see Definition 4.1) and prove existence theorem. Some generalizations are mentioned in Section 5 and conclusions from this paper are drawn in Section 6.

## 2. Case I: Convex–convex case

In this section, we will deal with (2) with the assumptions that both the Hamiltonians are convex (or concave). We start with a few definitions below,

**Definition 2.1.** Define the following classes of functions,

\[
CV(\mathbb{R}) = \{ h \in C^2(\mathbb{R}) : h \text{ is strictly convex and is of superlinear growth} \}.
\]

\[
CC(\mathbb{R}) = \{ h \in C^2(\mathbb{R}) : h \text{ is strictly concave and is of superlinear growth} \}.
\]

We are in a position to state the hypothesis on the Hamiltonians that will be considered in this section and on the initial data. Let \(-\infty < s < S < +\infty\) and \( I = [s, S] \). In this section, we are going to consider Hamiltonians with the following hypotheses.

**Hypotheses on the Hamiltonians.**

\((H_1)\) \( f, g \in CV(\mathbb{R}) \).

\((H_2)\) \( f(s) = g(s), f(S) = g(S) \).

\((H_3)\) Let \( f(\theta_f) = \min f, g(\theta_g) = \min g \) and assume that \( \theta_f < S, \theta_g > s \).
It is to be remarked that if the hypothesis (H₃) is not satisfied, then there exists only one solution.

**Remark 2.1.** In the above, we have assumed that both the Hamiltonians are convex. The case where both are concave i.e \((f, g \in CC(\mathbb{R}))\) can be similarly treated and we will only work with the convexity assumptions for the rest of this paper while mentioning the changes that are required when both \(f, g\) are concave.

**Remark 2.2.** The shapes of the Hamiltonians are shown in Fig. 1. Note that we do not have any extra assumptions on how the Hamiltonians are going to intersect in the interior of the domain. For example, in the Hamiltonians shown in Fig. 1, the point of intersection is undercompressive; i.e. let \(\alpha\) be the point of intersection in the interior of \(I\), then \((f'(\alpha) > 0\) and \(g'(\alpha) < 0)\).

Next, we state the hypotheses on initial data in this section.

**Hypotheses on the initial data \(v₀\).**

\((I₁) \; v₀ \in C^1(\mathbb{R}) \cup \text{Lip}(\mathbb{R}).\)
\((I₂) \; v₀(0) = 0 \text{ and } v₀'(x) \in I \; \forall x \in \mathbb{R}.\)

For sake of completeness, we will review some results for the corresponding conservation law (4) with the fluxes satisfying the hypotheses (H₁), (H₂). The study of (4) with the above hypotheses was carried out in [4] and we will recall results from that paper. First, we recall the definition of weak solutions of (4) below.
Definition 2.2 (Weak solution). $u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$ is said to be a weak solution for (4) if for all $\varphi \in C^\infty_0(\mathbb{R} \times \mathbb{R}_+)$, the following holds:

$$
\int_\infty^{-\infty} \int_0^\infty \left( u \frac{\partial \varphi}{\partial t} + (H(x)f(u) + (1 - H(x))g(u)) \frac{\partial \varphi}{\partial x} \right) dx \, dt + \int_{-\infty}^\infty u_0(x)\varphi(x,0) \, dx = 0. \tag{5}
$$

It is easy to see that $u$ satisfies (5) if and only if in the weak sense $u$ satisfies

$$
\begin{align*}
&u_t + g(u)x = 0, \quad x < 0, \quad t > 0, \\
u_t + f(u)x = 0, \quad x > 0, \quad t > 0,
\end{align*} \tag{6}
$$

and at $x = 0$, $u$ satisfies Rankine–Hugoniot (RH) condition, namely for almost all $t$

$$
f(u^+(t)) = g(u^-(t)), \tag{7}
$$

where $u^+(t) = \lim_{x \to 0^+} u(x, t)$, $u^-(t) = \lim_{x \to 0^-} u(x, t)$.

The weak solutions may not be unique and we have to impose additional criteria, the so called entropy conditions, to select an unique solution. As in [4], we need to define the so called interior entropy conditions. For that, we define

Definition 2.3 (Entropy–entropy flux pair). For $i = 1, 2$, $(\varphi_i, \psi_i)$ are said to be entropy pairs if $\varphi_i$ is a convex function on $[s, S]$ and $(\psi'_1(\theta), \psi'_2(\theta)) = (\varphi'_1(\theta)f'(\theta), \varphi'_2(\theta)g'(\theta))$ for $\theta \in [s, S]$.

This allows us to define the interior entropy condition as follows.

Definition 2.4 (Interior entropy condition). A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is said to satisfy the interior entropy condition if it satisfies in the sense of distributions,

$$
\begin{align*}
&\frac{\partial \varphi_1(u)}{\partial t} + \frac{\partial \psi_1(u)}{\partial x} \leq 0 \quad \text{in } x > 0, \quad t > 0, \\
&\frac{\partial \varphi_2(u)}{\partial t} + \frac{\partial \psi_2(u)}{\partial x} \leq 0 \quad \text{in } x < 0, \quad t > 0.
\end{align*} \tag{8}
$$

As in [4], the interior entropy condition (8) is not enough to guarantee uniqueness and we need to impose appropriate “jump” conditions at the interface. We start with the following definitions.

Definition 2.5 (Connection). In case the fluxes $f$ and $g$ satisfy the hypothesis $(H_1)$, $(H_2)$ with $\theta_f$ be the unique minimum of $f$ and $\theta_g$ being the unique minimum of $g$, then the pair $(A, B) \in I \times I$ is said to be a connection if it satisfies the following:

1. $g(A) = f(B)$.
2. $A \leq \theta_g$, and $\theta_f \leq B$.

It is said to be undercompressive if $A < \theta_g$ and $B > \theta_f$. 
Some examples of connections are given in Fig. 1 for the undercompressive case. Note that there are infinitely many connections. Next we define the following.

**Definition 2.6 (Interface entropy functional \( I_{AB} \)).** Let \( u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+) \) such that \( u^+(t) = u(0+, t) \), \( u^-(t) = u(0-, t) \) exist for a.e. \( t \). We define the following interface entropy functional relative to the connection \((A, B)\):

\[
I_{AB}(t) = \text{sign}(u^-(t) - A)(g(u^-(t)) - g(A)) - \text{sign}(u^+(t) - B)(f(u^+(t)) - f(B)).
\]

Now, we are in position to define the following.

**Definition 2.7 (Interface entropy condition).** Let \( u \in L^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+) \) such that \( u^+(t) = u(0+, t) \), \( u^-(t) = u(0-, t) \) exist for a.e. \( t \), then we say that \( u \) satisfies the interface entropy condition relative to a connection \((A, B)\) if the following holds:

\[
I_{AB}(u^-(t), u^+(t)) \geq 0 \quad \text{a.e. } t. \tag{9}
\]

Note that this condition is defined with respect to each connection \((A, B)\). Now for every choice of connection \((A, B)\), we are in position to define the \(AB\)-entropy solution as follows.

**Definition 2.8 (\(AB\)-entropy solution).** A function \( u \in L^\infty(\mathbb{R} \times \mathbb{R}_+) \) is defined as the entropy solution of (4) relative to the connection \((A, B)\) if the following holds:

1. \( u \) is a weak solution of (4), that is \( u \) satisfies (5).
2. \( u \) satisfies the interior entropy condition (8).
3. \( u \) satisfies the interface entropy condition (9) relative to the connection \((A, B)\).

We call the entropy solution relative to the connection \((A, B)\) as an \(AB\)-entropy solution. Note that for each choice of connection, we have a different class of entropy solutions. Note that we implicitly assume that the traces exist at the interface while imposing that the interface entropy condition is satisfied. In [4], the following existence and uniqueness result was proved.

**Theorem 2.1.** Let the fluxes satisfy the hypotheses (H1), (H2). Then for every choice of connection \((A, B)\), the corresponding \(AB\)-entropy solutions of (4) exist. Furthermore, let \( u, v \in L^\infty(\mathbb{R} \times \mathbb{R}_+) \) be two \(AB\)-entropy solutions for (4) with initial data \( u_0, v_0 \), respectively. Then for any \( \bar{M} \geq M = \max\{\text{Lip}(f), \text{Lip}(g)\}, a < 0, b > 0, b - a \geq 2\bar{M}t \) the function

\[
t \mapsto \int_{a+\bar{M}t}^{b-\bar{M}t} |u(x, t) - v(x, t)| \, dx
\]

is non-increasing, and if \( u_0 = v_0 \) a.e., then it follows that \( u = v \) a.e.

The above theorem showed the existence and uniqueness of infinitely many stable semigroups of solutions for (4). The uniqueness was proved by a “doubling of variables” argument along with the interface entropy condition (9). Existence was a consequence of the convergence of a Godunov type finite volume scheme. We refer the reader to [4] for details.
Our aim in this section is to extend this framework to the Hamilton–Jacobi equation (2) and to obtain an independent proof of existence of an $AB$-entropy solution for (4) from the Hopf–Lax type formula for the viscosity solution of the Hamilton–Jacobi equation.

As mentioned in the introduction, the strategy adopted by us relies on defining the following two Neumann initial boundary value problems for (2) given by

\begin{align*}
vt + g(vx) &= 0 \quad \text{if } x < 0, \ t > 0, \\
v(x, 0) &= v_0(x) \quad \text{if } x < 0, \\
\frac{\partial v}{\partial x} (0, t) &= \lambda_-(t) \quad \text{if } t > 0
\end{align*}

(10)

and

\begin{align*}
vt + f(vx) &= 0 \quad \text{if } x > 0, \ t > 0, \\
v(x, 0) &= v_0(x) \quad \text{if } x > 0, \\
\frac{\partial v}{\partial x} (0, t) &= \lambda_+(t) \quad \text{if } t > 0.
\end{align*}

(11)

We will obtain explicit Hopf–Lax type formulas for the viscosity solutions of the above two initial boundary value problems, patch up the two solutions and differentiate it to obtain a corresponding $AB$-entropy solution. The key trick is to introduce the connection $(A, B)$ into the above equation. This is done by a proper construction of the boundary functions $\lambda_\pm (t)$ which is one of the highlights of this paper. We remark that the initial boundary value problems for conservation laws was studied by Bardos, LeRoux, Nédélec [9] and LeFloch [24]. The corresponding initial boundary value problem for a Hamilton–Jacobi equation was studied by Joseph, Veerappa Gowda [15]. We begin with a few definitions below.

**Definition 2.9 (Legendre transformations).** The Legendre transformation $h^*$ of $h$ is defined as

\[ h^*(p) = \begin{cases} 
\sup_q \{qp - h(q)\} & \text{if } h \in CV(\mathbb{R}), \\
\inf_q \{qp - h(q)\} & \text{if } h \in CC(\mathbb{R}).
\end{cases} \]

We recall some simple properties of the Legendre transforms below:

(i) If $h \in CV(\mathbb{R})(h \in CC(\mathbb{R}))$, then $h^* \in CV(\mathbb{R})(h^* \in CC(\mathbb{R}))$.

(ii) $(h^*)^* = h$.

(iii) $h^{*'} = h^{-1}$.

(iv) $h^*(h'(p)) = ph'(p) - h(p)$, $h(h^{*'}(p)) = ph^{*'}(p) - h^*(p)$.

**Definition 2.10 (Control curves).** Let $t > 0$ and $\gamma : [0, t] \to \mathbb{R}$ be a continuous function. $\gamma$ is said to be a control curve if there exist $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ such that the following holds:

(i) $\gamma_t = \gamma |_{[t_i, t_i+1]}$ is linear,

(ii) $\gamma(t) \neq 0$ for $t \in (t_3, t_2) \cup (t_1, t_0),$

(iii) $\gamma(t) = 0$ for $t \in (t_2, t_1)$. 
Denote the following set:

\[ \Gamma(t) = \{ \gamma \in C^0[0, T]: \gamma \text{ is a control curve} \}. \]

**Definition 2.11.**  
(*Positive control curves*) Let \( x \geq 0 \), define the positive control curves \( \Gamma_+(x, t) \) by

\[ \Gamma_+(x, t) = \{ \gamma \in \Gamma(t): \gamma(0) = x, \gamma \geq 0 \}. \]

(*Negative control curves*) Let \( x \leq 0 \), define the negative control curves \( \Gamma_-(x, t) \) by

\[ \Gamma_-(x, t) = \{ \gamma \in \Gamma(t): \gamma(0) = x; \gamma \leq 0 \}. \]

The form of the possible control curves, positive and negative control curves is shown in Fig. 2. We also need the following definition.

**Definition 2.12 (Cost functional).** Let \( v_0 \in C^0(\mathbb{R}) \) and \( h \in CC(\mathbb{R}) \cup CV(\mathbb{R}) \) and \( \gamma \in \Gamma(t) \), define the following cost functionals:

\[
J(\gamma, v_0, h) = v_0(\gamma(0)) + \int_0^t h^*(\gamma'(\theta)) d\theta, \quad (12)
\]

\[
J_+(\gamma, v_0, h) = v_0(\gamma(0)) + \int_{\{t: \gamma(t) > 0\}} h^*(\gamma'(\theta)) d\theta, \quad (13)
\]

\[
J_-(\gamma, v_0, h) = v_0(\gamma(0)) + \int_{\{t: \gamma(t) < 0\}} h^*(\gamma'(\theta)) d\theta. \quad (14)
\]
The above definitions will be used in specifying the explicit Hopf–Lax type formula for (2). As mentioned earlier, we will construct suitable boundary functions and then use explicit formulas for the solutions of the Neumann boundary value problems (10), (11). For this, we need the following theorem which describes an explicit formula for the solutions of Neumann boundary value problems for the Hamilton–Jacobi equation (11). The theorem below is reproduced from the paper of Joseph, Veerappa Gowda [15].

**Theorem 2.2.** Let \( h \in CV(\mathbb{R}) \) and \( v_0 \in C^1(\mathbb{R}^+) \), \( b \in L^1_{\text{loc}}(\mathbb{R}^+) \). Let \( \theta_h \) be the unique minimum of \( h \) and define for \( x \geq 0 \).

\[
v(x,t) = \inf_{\gamma \in \Gamma_+(x,t)} \left\{ J_+(\gamma, v_0, h) - \int_{\gamma=0} h(\max(b(\theta), \theta_h)) \, d\theta \right\}.
\]

Then \( v \in \text{Lip}(\mathbb{R}^+ \times \mathbb{R}^+) \) is the unique viscosity solution of (11) with \( h = f \) and \( \lambda_+(t) = b(t) \). Furthermore \( u = v_x \) is the weak solution of

\[
\begin{align*}
  u_t + (h(u))_x &= 0, \quad x > 0, \ t > 0, \\
  u(x, 0) &= u_0(x) = v_{0x}(x), \quad x > 0.
\end{align*}
\]

satisfying the interior entropy condition (8) and the boundary condition

\[
u(0+, t) = b(t),
\]

in the sense of Bardos, LeRoux, Nédélec [9] i.e. if \( h'(u(0+, t)) \geq 0 \), then for a.e. \( t > 0 \).

\[
u(0+, t) = \max\{b(t), \theta_h\}.
\]

**Remark 2.3.** The same holds if \( h \in CC(\mathbb{R}) \) with appropriate modifications. That is replace \( \inf \) by \( \sup \) and \( \max(\max(b(\theta), \theta_h)) \) by \( \min(\max(b(\theta), \theta_h)) \). If we look for \( x \leq 0 \), then replace \( \Gamma_+ \) by \( \Gamma_- \), \( J_+ \) by \( J_- \) and \( h'(u(0+, t)) \geq 0 \) by \( h'(u(0-, t)) \leq 0 \).

The above theorem provides an explicit formula for the solution of the Neumann boundary value problem for the Hamilton–Jacobi equation and will be used by us to obtain the Hopf–Lax type formula for (2) once the appropriate boundary functions \( \lambda_\pm(t) \) are constructed. Next, we are going to construct the boundary functions in the convex–convex case. We need to define some auxiliary functions below.

Let \( v_0 \in C^1(\mathbb{R} \setminus \{0\}) \cup \text{Lip}(\mathbb{R}) \) such that \( v_0(0) = 0 \). For \( h \in CV(\mathbb{R}) \cup CC(\mathbb{R}) \) define \( b_\pm \) as follows,

\[
  b_+(t, v_0, h) = \begin{cases} 
  \inf_{\gamma \in \Gamma_+(0, t)} J_+(\gamma, v_0, h) & \text{if } h \in CV(\mathbb{R}), \\
  \sup_{\gamma \in \Gamma_+(0, t)} J_+(\gamma, v_0, h) & \text{if } h \in CC(\mathbb{R}), 
  \end{cases}
\]

\[
  b_-(t, v_0, h) = \begin{cases} 
  \inf_{\gamma \in \Gamma_-(0, t)} J_-(\gamma, v_0, h) & \text{if } h \in CV(\mathbb{R}), \\
  \sup_{\gamma \in \Gamma_-(0, t)} J_-(\gamma, v_0, h) & \text{if } h \in CC(\mathbb{R}).
  \end{cases}
\]
**Definition 2.13** (Characteristic sets). The characteristic sets $ch_{\pm}$ are defined by

$$
ch_{\pm}(t, v_0, h) = \left\{ \gamma \in \Gamma_{\pm}(0, t); \ b_{\pm}(t, v_0, h) = J(\gamma, v_0, h) \right\},
$$

$$
y_{+}(t, v_0, h) = \inf\left\{ \gamma(0): \ \gamma \in ch_{+}(t, v_0, h) \right\},
$$

$$
y_{-}(t, v_0, h) = \sup\left\{ \gamma(0): \ \gamma \in ch_{-}(t, v_0, h) \right\}.
$$

Elements in the characteristic set are called characteristic curves. We say that two characteristic curves $\gamma_1, \gamma_2 \in ch_{\pm}(t, v_0, h)$ intersect properly if $\exists \theta \in (0, t)$ such that the following holds:

$$
0 < \gamma_1(\theta) = \gamma_2(\theta) \quad \text{if} \ \gamma_1, \gamma_2 \in ch_{+}(t, v_0, h),
$$

$$
0 > \gamma_1(\theta) = \gamma_2(\theta) \quad \text{if} \ \gamma_1, \gamma_2 \in ch_{-}(t, v_0, h).
$$

We have the following lemma.

**Lemma 2.1.** With the above notation we have:

(i) $t \rightarrow b_{\pm}(t, v_0, h)$ are Lipschitz continuous functions.

(ii) No two characteristics intersect properly.

(iii) $t \rightarrow y_{+}(t, v_0, h)$ is a non-decreasing function and satisfies

$$
v_0'(y_{+}(t, v_0, h)) = h'\left(\frac{-y_{+}(t, v_0, h)}{t}\right) \quad \text{if} \ y_{+}(t, v_0, h) > 0
$$

and at the points of differentiability of $y_{+}$ and $b_{+}$, it satisfies

$$
b'_{+}(t, v_0, h) = \begin{cases} 
-h(v_0'(y_{+}(t, v_0, h))) & \text{if} \ y_{+}(t, v_0, h) > 0, \\
-h(\theta_h) & \text{if} \ y_{+}(t, v_0, h) = 0,
\end{cases}
$$

where

$$
h(\theta_h) = \begin{cases} 
\min h & \text{if} \ h \in CV(\mathbb{R}), \\
\max h & \text{if} \ h \in CC(\mathbb{R}).
\end{cases}
$$

(iv) $t \rightarrow y_{-}(t, v_0, h)$ is a non-increasing function and satisfies

$$
v_0'(y_{-}(t, v_0, h)) = h'\left(\frac{-y_{-}(t, v_0, h)}{t}\right) \quad \text{if} \ y_{-}(t, v_0, h) < 0
$$

and at the points of differentiability of $y_{-}, b_{-}$, it satisfies

$$
b'_{-}(t, v_0, h) = \begin{cases} 
-h(v_0'(y_{-}(t, v_0, h))) & \text{if} \ y_{-}(t, v_0, h) < 0, \\
-h(\theta_h) & \text{if} \ y_{-}(t, v_0, h) = 0.
\end{cases}
$$

The proof of the above lemma is a consequence of simple convex arguments that were set forth in [1]. For the sake of completeness, we repeat the arguments in Appendix A. The next step is to define the boundary functions below.
Construction of boundary functions.

Let \( f, g \in CV(\mathbb{R}) \cup CC(\mathbb{R}) \). Denote \( f^{-1}_{+} \) the inverse of \( f \) restricted to the increasing part of \( f \) and \( f^{-1}_{-} \) is the inverse of \( f \) restrict to the decreasing part of \( f \). Similarly define \( g^{-1}_{+} \) and \( g^{-1}_{-} \) for \( g \). Let \( v_0 \in C^1(\mathbb{R} \setminus \{0\}) \cup \text{Lip}(\mathbb{R}) \) and let \( b_{+}(t) = b_{+}(t, v_0, f) \) and \( b_{-}(t) = b_{-}(t, v_0, g) \) as defined above. Let the Hamiltonians \( f \) and \( g \) satisfy the hypotheses (H1), (H2) and \( (A, B) \) be any given connection. Then from Lemma 2.1, we get that \( -b_{+}'(t) \in f(I), -b_{-}'(t) \in g(I) \). We define the boundary functions as

**Definition 2.14 (Boundary data).** With the above notation, define the boundary values \( \lambda_{\pm}(t) = \lambda_{\pm}(t, A, B, v_0, f, g) \) by

\[
\begin{align*}
\lambda_{+}(t) &= \begin{cases} 
 f^{-1}_{-}(-b_{+}'(t)) & \text{if } -b_{+}'(t) > \max(-b_{-}'(t), f(B)), \\
 f^{-1}_{+}(\max(-b_{-}'(t), f(B))) & \text{if } -b_{+}'(t) \leq \max(-b_{-}'(t), f(B)),
\end{cases} 
\lambda_{-}(t) &= \begin{cases} 
 g^{-1}_{-}(-b_{-}'(t)) & \text{if } -b_{-}'(t) > \max(-b_{+}'(t), g(A)), \\
 g^{-1}_{+}(\max(-b_{+}'(t), g(A))) & \text{if } -b_{-}'(t) \leq \max(-b_{+}'(t), g(A)).
\end{cases}
\end{align*}
\]

(28)

(29)

We have the following lemma giving the properties of the boundary functions.

**Lemma 2.2.** \( \lambda_{\pm} \) satisfies the following properties:

(i) \( s \leq \lambda_{\pm} \leq S \).

(ii) \( f(\lambda_{+}(t)) = g(\lambda_{-}(t)) \).

(iii) If \( f'(\lambda_{+}(t)) > 0, g'(\lambda_{-}(t)) < 0 \) implies that \( \lambda_{+}(t) = B, \lambda_{-}(t) = A \).

**Proof.** Since \( v_0(x) \in I \) for all \( x \in \mathbb{R} \) and hence \( s \leq \lambda_{\pm}(t) \leq S \). Let \( -b_{+}'(t) > \max(-b_{-}'(t), f(B)) \geq -b_{-}'(t) \) and since \( f(B) = g(A) \), hence

\[
f(\lambda_{+}(t)) = -b_{+}'(t) = \max(-b_{+}'(t), g(A)) = g(\lambda_{-}(t)).
\]

(30)

Let \( -b_{+}'(t) \leq \max(-b_{-}'(t), f(B)) \). Suppose \( -b_{-}'(t) < f(B) \), then \( f(\lambda_{+}(t)) = \max(-b_{-}'(t), f(B)) = f(B) = \max(-b_{+}'(t), g(A)) = g(\lambda_{-}(t)) \). Suppose \( -b_{-}'(t) \geq f(B) \), then \( -b_{+}'(t) = \max(-b_{+}'(t), g(A)) \) and hence \( f(\lambda_{+}(t)) = \max(-b_{+}'(t), f(B)) = -b_{+}'(t) = g(\lambda_{-}(t)) \). This proves (ii).

Let \( f'(\lambda_{+}(t)) > 0, g'(\lambda_{-}(t)) < 0 \). Then from definition, \( -b_{+}'(t) \leq \max(-b_{-}'(t), f(B)) \) and \( -b_{-}'(t) \leq \max(-b_{+}'(t), g(A)) \). Suppose \( -b_{+}'(t) \leq f(B), \) then \( -b_{+}'(t) \leq \max(-b_{+}'(t), f(B)) \leq g(A) \). Hence \( f(\lambda_{+}(t)) = \max(-b_{+}'(t), f(B)) = f(B) = \max(-b_{-}'(t), g(A)) = g(\lambda_{-}(t)) \). Similarly if \( -b_{+}'(t) > g(A) \), then \( f(\lambda_{+}(t)) = f(B) = g(\lambda_{-}(t)) \). Suppose \( -b_{+}'(t) f(B) \) and \( -b_{-}'(t) \leq g(A) \), then \( -b_{+}'(t) \leq \max(-b_{-}'(t), f(B)) = -b_{+}'(t) \leq \max(-b_{+}'(t), g(A)) \leq -b_{+}'(t) \) which is a contradiction. Hence in all the cases, \( \lambda_{+}(t) = B, \lambda_{-}(t) = A \) this proves (iii) and hence the lemma.

We need some further definitions below.

**Definition 2.15.** Define \( \Gamma_{\pm}(x, t) \in \Gamma_{\pm}(x, t) \) by

\[
\Gamma_{\pm}(x, t) = \{ \gamma \in \Gamma_{\pm}(x, t); \{ \theta; \gamma(\theta) \neq 0 \} = (t_1, t) \text{ for some } t_1 \leq t \}.
\]

(31)
For $\gamma \in \overline{T}_\pm(x,t)$ denote $t_1(\gamma) \in [0,t]$ such that $\gamma > 0$ in $(t_1(\gamma), t)$.

We now state the main theorem of this section giving the explicit Hopf–Lax type formula for (2) as

**Theorem 2.3.** Let $f, g$ satisfy the hypotheses (H1), (H2) of this section and the initial data $v_0$ satisfy the hypotheses (I1), (I2), then the following hold:

(i) The boundary functions $\lambda_\pm(t)$ (as defined in (28)) satisfy for a.e. $t > 0$,

$$I_{AB}(\lambda_-(t), \lambda_+(t)) \geq 0.$$  (32)

(ii) Defining the value functions $v_\pm$ by

$$v_+(x,t) = \inf_{\gamma \in I_+^{\pm}(x,t)} \left\{ J_+(\gamma, v_0, f) - \int_{\{\gamma = 0\}} f(\lambda_+(\theta)) d\theta \right\} \text{ if } x \geq 0,$$  (33)

$$v_-(x,t) = \inf_{\gamma \in I_-^{\pm}(x,t)} \left\{ J_-(-\gamma, v_0, g) - \int_{\{\gamma = 0\}} g(\lambda_-(\theta)) d\theta \right\} \text{ if } x \leq 0,$$  (34)

then $v_\pm$ are Lipschitz continuous functions such that $v_+(0,t) = v_-(0,t)$ $\forall t > 0$ and $v_-, v_+$ are respectively the viscosity solutions of (10) and (11).

(iii) From (33), (34), define the Lipschitz continuous function

$$v(x,t) = \begin{cases} v_+(x,t) & \text{ if } x \geq 0, t > 0, \\ v_-(x,t) & \text{ if } x \leq 0, t > 0. \end{cases}$$  (35)

Then $u = v_x$ is a weak solution of (4) with $u_0 = v_{0x}$, satisfies the interior entropy condition (8) and interface entropy condition (9).

**Proof.** Let $\lambda_\pm(t)$ be defined by (28) and $v_\pm(x,t)$ be the associated value functions defined in (33), (34). Then we have the following.

**Claim 1.** $v_\pm$ is given by

$$v_+(x,t) = \inf_{\gamma \in I_+^{\pm}(x,t)} \left\{ J_+(\gamma, v_0, f) - \int_{0}^{t_1(\gamma)} f(\lambda_+(\theta)) d\theta \right\},$$  (36)

$$v_-(x,t) = \inf_{\gamma \in I_-^{\pm}(x,t)} \left\{ J_-(-\gamma, v_0, g) - \int_{0}^{t_1(\gamma)} g(\lambda_-(\theta)) d\theta \right\}. $$  (37)

By the superlinearity of the Hamiltonians $f$ and $g$, minimizers exist. Let $\gamma \in I_+(x,t)$ be a minimizer for $v_+(x,t)$. Let $0 \leq t_1 \leq t_2 \leq t$ be such that $\gamma > 0$ in $(0, t_1) \cup (t_2, t)$ and $\gamma = 0$ on $(t_1, t_2)$. Hence from definition of $b_+(t)$ we have
\[ v_+(x, t) = v_0(\gamma(0)) + \int_0^{t_1} f^*(\gamma'(\theta)) d\theta + \int_{t_2}^t f^*(\gamma'(\theta)) d\theta - \int_{t_1}^{t_2} f(\lambda_+(\theta)) d\theta \]

\[ = b_+(t_1) + \int_{t_2}^t f^*(\gamma'(\theta)) d\theta - \int_{t_1}^{t_2} f(\lambda_+(\theta)) d\theta. \]

From Lemma 2.2(i), we have \( b'_+(\theta) \geq -f(\lambda_+(\theta)) \) and hence by integrating we get that

\[ b_+(t_1) \geq -\int_0^{t_1} f(\lambda_+(\theta)) d\theta. \] (38)

This implies that

\[ v_+(x, t) \geq \int_{t_2}^t f^*(\gamma'(\theta)) d\theta - \int_{t_1}^{t_2} f(\lambda_+(\theta)) d\theta \]

\[ \geq \inf_{\eta \in \Gamma_+(x, t)} \left\{ J_+(\eta, v_0, f) - \int_{\{\eta = 0\}} f(\lambda_+(\theta)) d\theta \right\} \]

\[ \geq v_+(x, t). \]

This proves the first part of (36). Similarly we can prove the second part of (36) and this proves the claim. As a consequence of this and the fact that \( f(\lambda_+(\theta)) = g(\lambda_-(\theta)) \) we obtain that

\[ v_+(0, t) = -\int_0^t f(\lambda_+(\theta)) d\theta = -\int_0^t g(\lambda_-(\theta)) d\theta = v_-(0, t). \] (39)

From the above identity, we can now define the Lipschitz continuous function \( v \) given by (35) and make the following claim about it.

**Claim 2.** \( u = v_x \) is a weak solution of (4).
\[ + \int_{-\infty}^{\infty} v_{0x}(x)\varphi(x, 0) \, dx + \int_{0}^{\infty} \left( v(0+, t) - v(0-, t) \right) \, dt \]
\[
= \int_{-\infty}^{\infty} \int_{0}^{\infty} \left[ u\varphi_{t} + (H(x)f(u)) + \left(1 - H(x)\right)g(u)\varphi_{x} \right] \, dx \, dt + \int_{-\infty}^{\infty} u_{0}(x)\varphi(x, 0) \, dx.
\]

Hence \( u \) satisfies (5) and is a weak solution of (4).

**Claim 3.** For almost every \( t > 0 \), \( u(x \pm, t) \) exist and \( u(0+, t) = \lambda_{+}(t), u(0-, t) = \lambda_{-}(t) \).

In order to prove this, as in Lax, Evans, Oleinik [14], Adimurthi, Veerappa Gowda [15] and Joseph, Veerappa Gowda [15], we give explicit formulas for \( u \). We will do this for \( x \geq 0 \) and a similar formula holds for \( x \leq 0 \).

Let \( x \geq 0 \), define the characteristics \( ch_{+}(x, t), \bar{F}_{+}(x, t) \) as the minimizers of (35). By super-linearity and strict convexity of \( f, ch_{+}(x, t) \neq \phi \) and no two characteristics intersect properly, i.e. if \( \gamma, \eta \in ch_{+}(x, t) \), then \( \{ \theta; \gamma(\theta) = \eta(\theta) > 0 \} = \phi \) (see [1, Appendix]). In view of this define for \( t > 0 \),

\[
y_{+}(x, t) = \min \{ y(0); \gamma \in ch_{+}(x, t) \},
\]
\[
t_{+}(x, t) = \max \{ t_{1}(\gamma); \gamma \in ch_{+}(x, t) \},
\]
\[
R(t) = \min \{ x; t_{+}(x, t) = 0 \}.
\]

Using the fact that characteristics do not intersect properly, it follows that \( x \to y_{+}(x, t), x \to t_{+}(x, t) \) are non-decreasing and non-increasing functions, respectively (see [1]). Furthermore if \( x > R(t), t_{+}(x, t) > 0 \) and \( x < R(t), y_{+}(x, t) = 0, t_{+}(x, t) > 0 \). Let \( \eta \in ch_{+}(x, t) \) be such that \( t_{1}(\eta) = t_{+}(x, t), y_{+}(x, t) = \eta(0) \). If \( y_{+}(x, t) > 0 \), then \( \frac{\partial}{\partial y(0)} J_{+}(\gamma, v_{0}, f) \) at \( \gamma(0) = \eta(0) \), and hence,

\[
v'_{0}(y_{+}(x, t)) = f^{*}(\frac{x - y_{+}(x, t)}{t}). \tag{40}
\]

Furthermore if \( x \) is a point of differentiability of \( y_{+}(x, t) \), then

\[
v_{x}(x, t) = v_{0}'(y_{+}(x, t)) + f^{*}(\frac{x - y_{+}(x, t)}{t})(t - y_{+}(x, t)) = f^{*}(\frac{x - y_{+}(x, t)}{t}). \tag{41}
\]

If \( y_{+}(x, t) = 0 \), then \( 0 < t_{+}(x, t) \leq t \). Hence if \( x > 0 \), then \( t_{+}(x, t) < t \) and \( \frac{\partial}{\partial y(0)} J_{+}(\gamma, v_{0}, t) = 0 \) at \( t_{1}(\gamma) = t_{+}(x, t) \). Therefore if \( t_{+}(x, t) \) is the Lebesgue point of \( f(\lambda_{+}(\theta)) \), then

\[
-f(\lambda_{+}(t_{1})) - f^{*}(\frac{x}{t - t_{1}}) + \frac{x}{t - t_{1}} f^{*}(\frac{x}{t - t_{1}}) \bigg|_{t_{1}=t_{+}(x, t)} = 0. \tag{42}
\]
Hence at the points of differentiability of $t_1(x, t)$ we have that

$$v_x(x, t) = t'_+(x, t) \left[-f(\lambda_+(t_1)) - f^*(\frac{x}{t-t_1}) - \frac{x}{t-t_1} f^*(\frac{x}{t-t_1})\right]_{t=t_+(x, t)}$$

$$+ f^*(\frac{x}{t-t_+(x, t)})$$

$$= f^*(\frac{x}{t-t_+(x, t)})$$

and from (41), we have

$$f(\lambda_+(t_+(x, t))) = f\left(f^*(\frac{x}{t-t_+(x, t)})\right).$$

(43)

Hence $\lambda_+(t_+(x, t)) = f^*(\frac{x}{t-t_+(x, t)})$ and this gives the Lax–Oleinik type explicit formula for the solution given by

$$u(x, t) = v_x(x, t) = \left\{ \begin{array}{ll}
  f^*(\frac{x-y_+(x, t)}{t-t_+(x, t)}) & \text{if } y_+(x, t) > 0, \\
  \lambda_+(t_+(x, t)) & \text{if } 0 < t_+(x, t) < t.
\end{array} \right. $$

(44)

This implies that $u(x \pm t)$ exist for all $x > 0$ and a.e. $t > 0$. Suppose for some $t > 0$, $y_+(x, t) > 0$ for all $x > 0$. Then from Claim 1 and (41) we obtain that

$$-\int_0^t f(\lambda_+(\theta)) d\theta = v_0(0, t) = b_+(t).$$

(45)

Hence for a.e. $t \in [\theta, b_+ + t]$ we have that $-b'_+(t) = f(\lambda_+(t))$, hence from the above, $\lambda_+(t) = f^{-1}(-b'_+(t))$. Now observe that $y(0+, t) = y_+(t)$ and hence Step 6 in Appendix A and (41) gives that

$$\lambda_+(t) = f^{-1}(-b'_+(t)) = f^{-1}\left(f\left(v_0'(y_+(t))\right)\right)$$

$$= v_0'(y_+(t)) = f^\ast\left(\frac{-y_+(t)}{t}\right)$$

$$= v_x(0+, t) = u(0+, t).$$

(46)

This together with the second part of (41) gives $u(0+, t) = \lambda_+(t)$ a.e. $t > 0$. This proves the claim.

**Claim 4.** Let $f, g \in CV(\mathbb{R})$. Then for a.e. $t > 0$, $I_{AB}(t) = I_{AB}(u^-(t), u^+(t)) \geq 0$.

From claim (3), $u^+(t) = \lambda_+(t)$, $u^-(t) = \lambda_-(t)$ and from (28), $f(u^+(t)) = f(\lambda_+(t)) \geq f(B)$ and $g(u^-(t)) = g(\lambda_-(t)) \geq g(A)$. Also from Lemma 2.2(ii) $f(u^+(t)) = g(u^-(t))$. Hence

$$I_{AB}(t) = (g(u^-(t)) - g(A))(\text{sign}(u^-(t) - A) - \text{sign}(u^+(t) - B)).$$

(47)
If \( u^-(t) > A \), then \( I_{AB}(t) \geq 0 \). If \( u^-(t) < A \) and \( u^+(t) > B \), then \( f'(u^+(t)) > 0 \), \( g'(u^-(t)) < 0 \) and hence from Lemma 2.2(iii), \( u^-(t) = A \), then \( u^+(t) = B \) which is a contradiction. This proves the claim.

Thus we have proved all the parts of the Theorem 2.3. \( \square \)

The above theorem provides the explicit Hopf–Lax type formula for the Hamilton–Jacobi equations (2) and for the \( AB \)-entropy solutions of (4) and hence provides an alternative characterization of the \( AB \)-entropy solutions in terms of the Hamilton–Jacobi equations.

3. Case II: Concave–convex case

In the last section, we considered the case where both the Hamiltonians were either convex or concave. This case is the most interesting since it arises in many applications. The mixed case where one of the Hamiltonians is convex and the other concave can also arise in some situations. This is also a special case of considering a sign-changing coefficient \( k \) in (1). This case has received less attention in literature and we will deal with it in the subsequent sections. The case of (3) and (4) with sign-coefficients has been covered in some papers like [4,7], etc. and the lesson learned there is that the results are not symmetric with respect to whether \( f \) is convex and \( g \) is concave or \( f \) is concave and \( g \) is convex. So, we will deal with both the cases separately. In this section, we deal with the case where \( f \) is convex and \( g \) is concave. We start with hypotheses on the Hamiltonians.

Hypotheses on the Hamiltonians.

(\( \bar{H}_1 \)) Let \( \bar{I} = [s, \infty) \), \( f \in CV(I) \) with the unique minimum being denoted as \( \theta_f \) and \( g \in CC(I) \) with the unique maximum being denoted by \( \theta_g \).

(\( \bar{H}_2 \)) \( f(s) = g(s) \).

We also have the following hypothesis on the initial data:

(\( I_2 \)) \( v_0(0) = 0 \) and \( v'_0(x) \geq s, x \in \mathbb{R} \).

We assume that the initial data \( v_0 \) in this case satisfies (\( I_1 \)) and (\( I_2 \)).

Remark 3.1. The shape of the above Hamiltonians is shown in Fig. 3. (4) was considered with the fluxes satisfying the above hypothesis in [4] and the entire theory of \( AB \)-entropy solutions for (4) was developed and existence and stability shown.

In this section, we will obtain explicit Hopf–Lax formulas for the equations. To start with, we need some definitions.

Definition 3.1 (Connection). In case the Hamiltonians \( f \) and \( g \) satisfy the hypothesis (\( \bar{H}_1 \)), (\( \bar{H}_2 \)) with \( \theta_f \) be the unique minimum of \( f \) and \( \theta_g \) being the unique maximum of \( g \) (the concave–convex case), then \( (A, B) \in \bar{I} \times \bar{I} \) is said to be a connection if \( \theta_g \leq A, \theta_f \leq B \) and \( f(B) = g(A) \). It is said to be undercompressive if \( A > \theta_g \) and \( B > \theta_f \).

Some examples of connections are also shown in Fig. 3. We also impose the interface entropy condition (9) with respect to these connections. The definitions of weak solutions of (4) and interior entropy conditions remain the same. An \( AB \)-entropy solution with respect to a given
connection \((A, B)\) is a weak solution that satisfies the interior entropy conditions and the interface entropy conditions with respect to the connection \((A, B)\). An existence and uniqueness result for \(AB\)-entropy solutions similar to Theorem 2.1 was proved in [4]. Here, we are going to obtain an explicit Hopf–Lax type formula for the \(AB\)-entropy solution in terms of explicit Hopf–Lax type formula for the Neumann boundary value problems (10), (11) as in Section 2. The key issue as in Section 2 is a proper construction of the boundary functions \(\lambda_{\pm}\) and use of these boundary functions for obtaining the explicit Hopf–Lax type formulas. We will use the definitions of control curves, characteristics and their intersections as in Section 2. We start with a specification of the boundary functions.

**Definition 3.2 (Boundary data).** In this case, the boundary functions are given by:

**Case 1.** \(g(s) \leq g(A)\):

\[
\begin{align*}
\lambda_+(t) &= f_+^{-1}\left(\min\left(-b'_- (t), f (B)\right)\right), \\
\lambda_-(t) &= \begin{cases} 
g_+^{-1}\left(-b'_- (t)\right) & \text{if } -b'_- (t) < g(A), \\
A & \text{if } -b'_- (t) \geq g(A).
\end{cases}
\end{align*}
\]

**Case 2.** \(g(s) \geq g(A)\):

\[
\begin{align*}
\lambda_+(t) &= \begin{cases} 
f^{-1}_-\left(-b'_+ (t)\right) & \text{if } -b'_+ (t) > f(B), \\
B & \text{if } -b'_+ (t) \leq f(B).
\end{cases} \\
\lambda_-(t) &= g_+^{-1}\left(\max\left(-b'_+ (t), g(A)\right)\right).
\end{align*}
\]
The properties of the boundary functions are given in the following lemma.

**Lemma 3.1.** The boundary functions \( \lambda_{\pm} \) have the following properties:

(i) \( \lambda_{\pm}(t) \in I \).

(ii) \( f(\lambda_{+}(t)) = g(\lambda_{-}(t)) \).

(iii) \( I(t) = (g(\lambda_{-}(t)) - g(A)) \text{sign}(\lambda_{-}(t) - A) - (f(\lambda_{+}(t)) - f(B)) \text{sign}(\lambda_{+}(t) - B) \geq 0 \).

**Proof.** (i) and (ii) follow from the definition. Let \( g(s) \leq g(A) \), then \( g(\lambda_{-}(t)) \leq g(A) \) and \( \lambda_{-}(t) \leq A \). Hence from (i) we have,

\[
I(t) = (g(A) - g(\lambda_{-}(t)))(\text{sign}(A - \lambda_{-}(t)) - \text{sign}(B - \lambda_{+}(t))) \geq 0.
\]

Suppose \( g(s) \geq g(A) \), then \( f(\lambda_{+}(t)) \geq f(B) \) and \( \lambda_{+}(t) \leq B \). Hence,

\[
I(t) = (f(\lambda_{+}(t)) - f(B))(\text{sign}(\lambda_{-}(t) - A) + \text{sign}(B - \lambda_{+}(t))) \geq 0.
\]

This proves the lemma.

Next, we have the following theorem giving the explicit Hopf–Lax type formula in this case.

**Theorem 3.1.** Let the Hamiltonians \( f \) and \( g \) satisfy the hypothesis \((\mathcal{H}_1), (\mathcal{H}_2)\) and the initial data satisfies \((I_1), (I_2)\), then:

(i) Defining the value functions \( v_{\pm} \) by

\[
v_{+}(x,t) = \inf_{\gamma \in \Gamma_{+}(x,t)} \left\{ J_{+}(\gamma, v_0, f) - \int_{\{\gamma = 0\}} f(\lambda_{+}(\theta)) \, d\theta \right\} \quad \text{if } x \geq 0, \quad (52)
\]

\[
v_{-}(x,t) = \sup_{\gamma \in \Gamma_{-}(x,t)} \left\{ J_{-}(\gamma, v_0, g) - \int_{\{\gamma = 0\}} g(\lambda_{-}(\theta)) \, d\theta \right\} \quad \text{if } x \leq 0 \quad (53)
\]

we have \( v_{\pm} \) are Lipschitz continuous functions such that \( v_{+}(0,t) = v_{-}(0,t) \) \( \forall t > 0 \) and \( v_{-}, v_{+} \) are respectively the viscosity solutions of (10) and (11).

(ii) From (52), (53), define the Lipschitz continuous function

\[
v(x,t) = \begin{cases} v_{+}(x,t) & \text{if } x \geq 0, \ t > 0, \\ v_{-}(x,t) & \text{if } x \leq 0, \ t > 0. \end{cases} \quad (54)
\]

Then \( u = v_x \) is a weak solution of (4) with \( u_0 = v_{0x} \), satisfies the interior entropy condition (8) and interface entropy condition (9).

The proof of the above theorem is exactly the same as that of Theorem 2.3 and we omit the details. Thus, we have obtained explicit formulas for the \( AB \)-entropy solution for (4) with the case where \( g \) is concave and \( f \) is convex. This handles the case of mixed type geometries which are not usually covered in literature.

4. Case III: Convex–concave case

In the previous section, we considered the case where \( f \) is convex and \( g \) is concave and obtain explicit Hopf–Lax type formulas for the Hamilton–Jacobi equation (2) and for the \( AB \)-entropy solutions of the conservation law (4). In this section, we consider the situation where \( f \)
Fig. 4. Flux shapes in Case III (the convex–concave case).

is concave and \( g \) is convex. This case is very different from the previous case as was shown for the conservation law (4) in [7]. We start with the hypotheses on the Hamiltonians,

\begin{itemize}
  
  \item[(H1)] Let \( \bar{I} = [s, \infty) \), \( g \in CV(I) \) with the unique minimum being denoted as \( \theta_g \) and \( f \in CC(I) \) with the unique maximum being denoted by \( \theta_h \).

  \item[(H2)] \( f(s) = g(s) \).

\end{itemize}

Furthermore, we assume that the initial data satisfies the hypothesis \( (I_1), (I_2) \). \( f \) and \( g \) satisfying the above hypotheses are shown in Fig. 4. We would like to mention that (4) with fluxes satisfying the above hypotheses has been analyzed in [7]. In the above quoted reference, it was shown that the main difficulty in this case is not the uniqueness of entropy solutions but their existence as the Rankine–Hugoniot conditions (7) are not satisfied and we have to replace them with a weaker notion of solutions. For that we need to recall following definition.

**Definition 4.1** (Generalized Rankine–Hugoniot condition). \( u \in L^\infty(\mathbb{R} \times \mathbb{R}_+) \) such that \( u^+(t) = u(0^+, t) \) and \( u^-(t) = u(0^-, t) \) exist. Then \( u \) is said to satisfy the generalized Rankine–Hugoniot solution if the following holds:

\[
\begin{aligned}
  \text{if } u^+(t) < \theta_f \text{ then } & f(u^+(t)) = g(u^-(t)), \\
  \text{if } u^-(t) < \theta_g \text{ then } & f(u^+(t)) = g(u^-(t)), \\
  \text{if } u^+(t) \geq \theta_f \text{ or } u^-(t) \geq \theta_g \text{ then } & f(u^+(t)) \leq g(u^-(t)).
\end{aligned}
\]

Note that by demanding that the generalized Rankine–Hugoniot condition holds at the interface, we are asking for a criteria that is weaker than the Rankine–Hugoniot condition (7). This also serves as a replacement of the interface entropy condition. As in [7], the notion of solutions
that we are going to use in this case is the so-called generalized entropy solutions which are defined below.

**Definition 4.2** (Generalized entropy solution). \( u \in L^\infty(\mathbb{R} \times \mathbb{R}_+) \) is defined to be a generalized entropy solution of (4) if the following hold:

1. \( u \) satisfies (6) in sense of distributions i.e. \( u \) is a weak solution of (4) away from the interface \( x = 0 \).
2. \( u \) satisfies the interior entropy solution (8).
3. \( u(0+, t), u(0-, t) \) exist and satisfy (55).

In [7], we were able to show that the generalized entropy solutions exist and are unique. Note that in this case, we do not have infinite semigroups of stable solutions but a unique weak solution characterized by the generalized Rankine–Hugoniot condition (55). Our aim in this section is to carry out an analysis of (2) with the above hypothesis and obtain an alternative representation given by the explicit Hopf–Lax type formulas for the generalized entropy solutions of (4). The approach is fairly standard and we follow the same method as in the previous sections. The first step is to obtain a representation of the boundary functions \( \lambda_{\pm} \) which is given below.

**Definition 4.3.** If \( f \) and \( g \) obey \((H_1), (H_2)\), then define \( \lambda_{\pm} \) as follows,

\[
\lambda_+(t) = \begin{cases} 
    f_+^{-1}(\max(-b'_+(t), f(s))) & \text{if } -b'_+(t) > \max(-b'_-(t), f(s)), \\
    f_+^{-1}(-b'_+(t)) & \text{if } -b'_+(t) \leq \max(-b'_-(t), f(s)),
\end{cases}
\]

\[
\lambda_-(t) = \begin{cases} 
    g_+^{-1}(\min(-b'_+(t), g(s))) & \text{if } -b'_-(t) < \min(-b'_+(t), g(s)), \\
    g_+^{-1}(-b'_-(t)) & \text{if } -b'_-(t) \geq \min(-b'_+(t), g(s)).
\end{cases}
\]

The properties of the boundary functions are given in the following lemma.

**Lemma 4.1.** \( \lambda_{\pm} \) defined above satisfy \( f(\lambda_+(t)) \leq g(\lambda_-(t)) \). Furthermore if \( \lambda_+(t) < \theta_f \) or \( \lambda_-(t) < \theta_g \), then \( f(\lambda_+(t)) = g(\lambda_-(t)) \).

**Proof.** Let \(-b'_+(t) > \max(-b'_-(t), f(s))\) then \(-b'_+(t) > f(s) = g(s)\). If \(-b'_-(t) < f(s) = \min(-b'_+(t), g(s))\), then \( f(\lambda_+(t)) = s = g(\lambda_-(t)) \). If \(-b'_+(t) \geq f(s) = \min(-b'_+(t), g(s))\) and hence \( f(\lambda_+(t)) = -b'_+(t) = g(\lambda_-(t)) \). Furthermore if \( \lambda_+(t) < \theta_f \) then by definition \(-b'_+(t) > \max(-b'_-(t), f(s)) \) and hence \( f(\lambda_+(t)) = g(\lambda_-(t)) \).

Let \(-b'_+(t) \leq \max(-b'_-(t), f(s))\). Suppose \(-b'_-(t) < f(s)\) then \(-b'_+(t) \leq f(s)\). If \(-b'_-(t) < -b'_+(t) = \min(-b'_+(t), f(s))\) then \( f(\lambda_+(t)) = -b'_+(t) = g(\min(-b'_-(t), g(s))) \). If \(-b'_+(t) \geq -b'_+(t) = \min(-b'_+(t), g(s))\), then \( f(\lambda_+(t)) = -b'_+(t) \leq -b'_-(t) = g(\lambda_-(t)) \). Suppose \(-b'_+(t) \geq f(s)\), then \(-b'_+(t) \leq -b'_-(t) \) and hence \(-b'_+(t) \geq \min(-b'_+(t), g(s)) \).

This implies that \( f(\lambda_+(t)) = -b'_+(t) \leq -b'_-(t) = g(\lambda_-(t)) \). Furthermore if \( \lambda_-(t) < \theta_g \), then \(-b'_-(t) < \min(-b'_+(t), g(s))\) and \( g(\lambda_-(t)) = \min(-b'_+(t), g(s)) \). If \(-b'_+(t) \leq \max(-b'_-(t), f(s)) \leq f(s)\), then \( f(\lambda_+(t)) = -b'_-(t) = \min(-b'_+(t), g(s)) = g(\lambda_-(t)) \). If \(-b'_+(t) > \max(-b'_-(t), f(s)) = f(s)\), then \( f(\lambda_+(t)) = \max(-b'_-(t), f(s)) = f(s) = \min(-b'_+(t), g(s)) = g(\lambda_-(t)) \). This proves the lemma. \( \square \)
Next we use the above boundary functions to provide for explicit Hopf–Lax type formulas for (10), (11) in the theorem below.

**Theorem 4.1.** Let the Hamiltonians $f$ and $g$ satisfy the hypotheses $(H_1)$, $(H_2)$ and the initial data satisfy $(I_1)$ and $(I_2)$. Then we have:

(i) Define the associated value functions by

$$v_+(x, t) = \sup_{\gamma \in \Gamma_+(x, t)} \left\{ J_+(\gamma, v_0, f) - \int_{\{\gamma = 0\}} f(\lambda_+(\theta)) \, d\theta \right\} \quad \text{if } x \geq 0,$$

$$v_-(x, t) = \inf_{\gamma \in \Gamma_-(x, t)} \left\{ J_-(\gamma, v_0, g) - \int_{\{\gamma = 0\}} g(\lambda_-(\theta)) \, d\theta \right\} \quad \text{if } x \leq 0.$$  

Then $v_+, v_-$ are Lipschitz continuous and viscosity solutions of (11) and (10), respectively.

(ii) Furthermore define

$$u(x, t) = \left\{ \begin{array}{ll}
\frac{\partial}{\partial x} v_+(x, t) & \text{if } x > 0, \ t > 0, \\
\frac{\partial}{\partial x} v_-(x, t) & \text{if } x < 0, \ t > 0.
\end{array} \right.$$  

Then $u$ is a generalized entropy solution of (4).

**Proof.** The proof of this theorem is exactly as the proof of Theorem 2.3. From Claim 3 in the proof of Theorem 2.3, it follows that for a.e. $t > 0$,

$$u^+(t) = u(0^+, t) = \lambda_+(t), \quad u^-(t) = u(0^-, t) = \lambda_-(t).$$

Also from Lemma 4.1, $f(u^+(t)) = f(\lambda_+(t)) \leq g(\lambda_-(t)) = g(u^-(t))$ and if $u^+(t) < \theta_f$ or $u^-(t) < \theta_g$ then $f(u^+(t)) = g(u^-(t))$. This together with Theorem 2.2 gives that $u$ is a solution of (4) satisfying the generalized Rankine–Hugoniot condition (55). This proves the theorem.

Thus we obtain an alternative representation for the generalized entropy solutions of (4) in this case. The main difference between the previous cases and this case is the fact that the solution $v$ is no longer continuous at the interface.

5. Generalizations

In the above sections, we have dealt either with a finite interval on a one-sided infinite domain. We can also deal with the case where the domain is the entire real line. We do so in this section. Since, the method has already been demonstrated in the previous sections, we will just construct the boundary functions and state the theorem without any proof. Some results for the conservation law (4) were mentioned in [7] and were a combination of the results of Sections 2 and 3.

If $f, g \in CV(\mathbb{R})$, then the result is a straightforward extension of Theorem 2.3 by taking $s = -\infty$ and $S = \infty$. So we restrict ourselves to the following two sets of hypotheses:
Case IV. In this case, the Hamiltonians $f$ and $g$ are such that $f \in CV(\mathbb{R})$ and $g \in CC(\mathbb{R})$.

Case V. In this case, the Hamiltonians $f$ and $g$ are such that $g \in CV(\mathbb{R})$ and $h \in CC(\mathbb{R})$.

Define,

**Definition 5.1.** Let

$$A(f, g) = \{ \theta; f(\theta) = g(\theta), f'(\theta) \leq 0, g'(\theta) \geq 0 \}.$$  \hfill (62)

Then if $f, g$ are either in Case IV or V, it follows that either $A(f, g) = \emptyset$ or there exists a unique point $s$ such that $A(f, g) = \{ s \}$. Now define the intersection point $s_0$ by $s_0 = s$ if $A(f, g) = \{ s \}$.

If $A(f, g) = \emptyset$, then in Case IV

$$s_0 = \begin{cases} \theta_f & \text{if } \theta_R \geq \theta_f, \\ \theta_g & \text{if } \theta_R \leq \theta_f \end{cases}$$ \hfill (63)

and then in Case V

$$s_0 = \begin{cases} \theta_g & \text{if } \theta_R \geq \theta_f, \\ \theta_f & \text{if } \theta_R \leq \theta_f \end{cases}$$ \hfill (64)

Let $v_0 \in C^1(\mathbb{R} \setminus \{0\}) \cup \text{Lip}(\mathbb{R})$, $b_+(t) = b_+(t, v_0, f)$, $b_-(t) = b_-(t, v_0, g)$ be defined as in (19). Let $(A, B)$ be a connection with respect to $f$ and $g$ (the same definition as in Section 3). Then define the boundary values $\lambda_{\pm}(t)$ in different cases as follows.

Case IV. We have to consider the following subcases.

Case IV.1: $g(s_0) \leq g(A)$. Then define $\lambda_{\pm}$ as

$$\lambda_+(t) = \begin{cases} f^{-1}_-(\min(-b_-(t), g(A))) & \text{if } -b_+(t) < \min(-b'_-(t), g(A)), \\ f^{-1}_-(b'_+(t)) & \text{if } -b'_+(t) \geq \min(-b'_-(t), g(A)) \end{cases}$$ \hfill (65)

$$\lambda_-(t) = \begin{cases} g_+(b'_-(t)) & \text{if } -b'_-(t) < g(A), \\ A & \text{if } -b'_-(t) \geq g(A) \end{cases}$$ \hfill (66)

Case IV.2: $g(s_0) \geq g(A)$. Then define $\lambda_{\pm}$ as

$$\lambda_-(t) = \begin{cases} g^{-1}_+(b'_-(t)) & \text{if } -b'_-(t) < g(A), \\ A & \text{if } -b'_-(t) \geq g(A) \end{cases}$$ \hfill (67)

$$\lambda_-(t) = \begin{cases} g^{-1}_-((-b'_+(t), f(B))) & \text{if } -b'_-(t) > \min(-b'_+(t), f(B)), \\ g^{-1}_-(b'_-(t)) & \text{if } -b'_-(t) \leq \min(-b'_+(t), f(B)) \end{cases}$$ \hfill (68)

Case IV.3: Range of $(f) \cap \text{Range of } (g) = \emptyset$. In this case connection $(A, B)$ does not exist. Define,
\[ \lambda_+(t) = f_+^{-1}(-b'_+(t)), \quad (69) \]
\[ \lambda_-(t) = g_+^{-1}(-b'_-(t)). \quad (70) \]

**Case V.** We have to consider the following subcases.

**Case V.1:** \( g(s_0) \leq g(A) \). Then define \( \lambda_{\pm} \) by

\[ \lambda_+(t) = \begin{cases} 
  f_+^{-1}(-b'_+(t)) & \text{if } -b'_+(t) < f(B), \\
  B & \text{if } -b'_+(t) \geq f(B),
\end{cases} \quad (71) \]
\[ \lambda_-(t) = \begin{cases} 
  g_+^{-1}(\max(-b'_+(t), f(B))) & \text{if } -b'_-(t) < \max(-b'_+(t), f(B)), \\
  g_+^{-1}(-b'_-(t)) & \text{if } -b'_-(t) \geq \max(-b'_+(t), f(B)).
\end{cases} \quad (72) \]

**Case V.2:** \( g(s_0) \geq g(A) \). Then define \( \lambda_{\pm} \) by

\[ \lambda_+(t) = \begin{cases} 
  f_+^{-1}(\min(-b'_-(t), g(A))) & \text{if } -b'_+(t) > \min(-b'_-(t), g(A)), \\
  f_+^{-1}(-b'_+(t)) & \text{if } -b'_+(t) \leq \min(-b'_-(t), g(A)),
\end{cases} \quad (73) \]
\[ \lambda_-(t) = \begin{cases} 
  g_+^{-1}(-b'_-(t)) & \text{if } -b'_-(t) > g(A), \\
  A & \text{if } -b'_-(t) \leq g(A).
\end{cases} \quad (74) \]

**Case V.3:** Range of \( f \cap \) Range of \( g \) = \( \emptyset \). In this case connection \( (A, B) \) does not exist. Define

\[ \lambda_+(t) = f_+^{-1}(-b'_+(t)), \quad (75) \]
\[ \lambda_-(t) = g_+^{-1}(-b'_-(t)). \quad (76) \]

We need some more definitions given below.

**Definition 5.2** (*Generalized Rankine–Hugoniot condition*). Let \( u \) be a solution of (6) such that \( u_{\pm}(t) \) exist a.e. \( t > 0 \). Then \( u \) is said to satisfy the generalized Rankine–Hugoniot condition if for a.e. \( t > 0 \),

**Case IV:** In this case, \( f(u^+(t)) \geq g(u^-(t)) \) and whenever \( u^+(t) > \theta_f \) or \( u^-(t) > \theta_g \), then \( f(u^+(t)) = g(u^-(t)) \).

**Case V:** In this case, \( f(u^+(t)) \leq g(u^-(t)) \) and whenever \( u^+(t) < \theta_f \) or \( u^-(t) < \theta_g \), then \( f(u^+(t)) = g(u^-(t)) \).

**Definition 5.3** (*Interface entropy condition*). Let \( (A, B) \) be a connection and let \( I_{AB} \) be as defined in Section 2. Let \( u \in L^1_{\text{loc}}(R \times R_+) \) be such that \( u_{\pm}(t) \) exist a.e. \( t > 0 \). Then \( u \) is said to satisfy the interface entropy condition with respect to \( (A, B) \) if:

**Case IV:** We have to distinguish the following subcases:

**Case IV(i):** Let \( g(s_0) \leq g(A) \).
If \( u^{-}(t) \geq s_{0} \) and for every \( l \leq s_{0} \), \( f(u) \geq \min(g(u^{-}(t)), g(A)) \), then \( u^{-}(t) \) satisfies

\[
I_{AB}(u^{-}(t), l) \geq 0. 
\] (77)

**Case IV(ii):** Let \( g(s_{0}) \geq g(A) \).

If \( u^{+}(t) \leq s_{0} \) and for every \( l \leq s_{0} \) with \( g(l) \geq \min(f(u^{+}(t)), f(B)) \), then \( u^{+}(t) \) satisfies

\[
I_{AB}(l, u^{+}(t)) \geq 0. 
\] (78)

**Case V:** We have to consider the following subcases:

**Case V(i):** Let \( g(s_{0}) \leq g(A) \).

If \( u^{+}(t) \leq s_{0} \) and for every \( l \leq s_{0} \), \( g(l) \geq \min(f(u^{+}(t)), f(B)) \), then \( u^{+}(t) \) satisfies

\[
I_{AB}(l, u^{+}(t)) \geq 0. 
\] (79)

**Case V(ii):** Let \( g(s_{0}) \geq g(A) \).

If \( u^{-}(t) \leq s_{0} \) and for every \( l \geq s_{0} \), \( f(l) \geq \max(g(u^{-}(t)), g(A)) \), then \( u^{-}(t) \) satisfies

\[
I_{A,B}(u^{-}(t), l) \geq 0. 
\] (80)

Then under the above hypothesis on \( f, g, v_{0}, A, B \) and boundary values \( \lambda_{\pm}(t) \) we have the following theorem.

**Theorem 5.1.** Let \( v_{\pm}(x, t) \) be the value functions defined by

\[
v_{+}(x, t) = \begin{cases} 
\inf_{\gamma \in \Gamma_{+}(x, t)} \left( J_{+}(\gamma, v_{0}, f) - \int_{\{\gamma = 0\}} f(\lambda_{+}(\theta)) \, d\theta \right) & \text{if } f \in CV(\mathbb{R}), \\
\sup_{\gamma \in \Gamma_{+}(x, t)} \left( J_{+}(\gamma, v_{0}, f) - \int_{\{\gamma = 0\}} f(\lambda_{+}(\theta)) \, d\theta \right) & \text{if } f \in CC(\mathbb{R}),
\end{cases}
\] (81)

\[
v_{-}(x, t) = \begin{cases} 
\inf_{\gamma \in \Gamma_{-}(x, t)} \left( J_{-}(\gamma, v_{0}, g) - \int_{\{\gamma = 0\}} g(\lambda_{-}(\theta)) \, d\theta \right) & \text{if } g \in CV(\mathbb{R}), \\
\sup_{\gamma \in \Gamma_{-}(x, t)} \left( J_{-}(\gamma, v_{0}, g) - \int_{\{\gamma = 0\}} g(\lambda_{-}(\theta)) \, d\theta \right) & \text{if } g \in CC(\mathbb{R}),
\end{cases}
\] (82)

and \( u \) by

\[
u_{+}(x, t) = \begin{cases} 
\frac{\partial v_{+}}{\partial x} & \text{if } x > 0, \\
\frac{\partial v_{-}}{\partial x} & \text{if } x < 0.
\end{cases}
\] (83)

Then \( v_{\pm} \) are Lipschitz continuous functions and are viscosity solutions of (11) and (10). Furthermore \( u^{\pm}(t) \) exist for a.e. \( t > 0 \) and are the solutions of (4) in both cases IV and V satisfying the corresponding generalized Rankine–Hugoniot conditions and interface entropy conditions defined above.
6. Conclusion

In this paper, we have considered the Hamilton–Jacobi equation (2) and the related single conservation law (4). A well-established theory of $AB$-entropy solutions of (4) exists and the aim of this paper was to obtain explicit Hopf–Lax type formulas for the solutions of (2) that corresponds to $AB$-entropy solutions of (4). This is done by using two Neumann boundary value problems at the interface $x = 0$ for the Hamilton–Jacobi equation (2). The main issue is the proper specification of boundary functions which are in turn based on the corresponding connection. This enables us to derive Hopf–Lax type formulas and obtain an alternative characterization of the $AB$-entropy solutions of (4). We have considered the case where both the Hamiltonians are convex (concave). Furthermore, we also deal with the case where one of them is concave and the other convex. Depending of which of the Hamiltonians is convex and which is concave, we have established Hopf–Lax type formulas for the $AB$-entropy solutions or the generalized entropy solutions depending on which case we are dealing with. The main advantage of the Hopf–Lax type formulas is their explicitness. Also as in the continuous case, we are able to obtain close connections between the viscosity solutions of the Hamilton–Jacobi equations and the entropy solutions of the conservation laws even in the case where the Hamiltonian is discontinuous in the space variable.

Appendix A

In this section, we will provide a proof for Lemma 2.1. The proof was already presented in [1]. We repeat it here for the sake of completeness.

Proof of Lemma 2.1. It is enough to prove the lemma for $h \in CV(\mathbb{R})$ and $b_+(t, v_0, h)$. The other cases follow in similar manner. For notational convenience, denote $b_+(t) = b_+(t, v_0, h)$, $c_+(t) = c_+(t, v_0, h)$, $y_+(t) = y_+(t, v_0, h)$. Proof will follow in several steps.

Step 1. Let $c_+(t) \neq \emptyset$ and there exists $c > 0$ such that $\frac{v(0)}{t_1} \leq c \forall \gamma \in c_+(t)$ where $\gamma > 0$ in $(0, t_1)$ and $\gamma = 0$ on $[t_1, t]$.

Let $\gamma_0(\theta) = 0$, then $\gamma_0 \in \Gamma_+(0, t)$ and $b_+(t) \leq th^*(0)$. If $b_+(t) = th^*(0)$, then $\gamma_0 \in c_+(t)$. Otherwise there exists a sequence $\gamma_k \in \Gamma_+(0, t)$ such that

\[ b_+(t) = \lim_{k \to \infty} J(\gamma_k, y_0, h), \quad (A.1) \]

\[ J(\gamma_k, y_0, h) < th^*(0) \quad \text{for all } k. \quad (A.2) \]

Let $y_k = \gamma_k(0)$ and $0 \leq t_k \leq t$ be such that $\gamma_k(\theta) = 0$ in $(t_k, t)$ and $\gamma_k(\theta) > 0$ in $(0, t_k)$. Then from (A.2) we have that

\[ v_0(y_k) + t_k h^*\left(-\frac{y_k}{t_k}\right) \leq t_k h^*(0). \quad (A.3) \]

Since $v_0(0) = 0$ and $v_0 \in \text{Lip}(\mathbb{R})$, hence there exists $M > 0$ such that $|v_0(y)| \leq M|y|, \forall y \in \mathbb{R}$. Hence from (A.3) we have for $z_k = y_k/t_k$,

\[ z_k \left\{-M + \frac{h^*(-z_k)}{z_k}\right\} \leq \frac{v_0(y_k)}{t_k} + h^*\left(-\frac{y_k}{t_k}\right) \leq h^*(0). \quad (A.4) \]
Since $h^*(-\theta) \to \infty$ as $|\theta| \to \infty$, hence the above inequality implies that $\{z_k\}$ is bounded. Hence we can extract a subsequence still denoted by $\{y_k\}$ and $\{t_k\}$ converging to $y_0$ and $t_0$. Suppose $t_0 = 0$, then $y_0 = 0$ and hence $b_+(t) = v_0(0) + t h^*(0)$ which contradicts our assumption. Therefore $t_0 \neq 0$ and hence $b_+(t) = v_0(y_0) + t_0 h^*(0) + (t - t_0) h^*(-y_0/t_0) = J_+ (\hat{\gamma}, v_0, h)$ where $\hat{\gamma} = 0$ for $\theta \in (t_0, t)$ and $\hat{\gamma}(\theta) = \frac{y_0(\theta - \theta)}{\theta}$ for $\theta \in [0, t_0]$. Hence $\hat{\gamma} \in ch_+(t)$.

From (A.3) and (A.4) and by superlinearity of $h^*$ it follows that there exists $c > 0$ such that $\frac{|y^0(\theta)|}{\theta} \leq c$ for all $\gamma \in ch_+(t)$. This proves Step 1.

**Step 2.** Let $\gamma \in ch_+(t)$ and let $0 \leq t_1 \leq t$ be such that $\gamma(\theta) \in (0, t_1)$ and $\gamma(\theta) = 0$ for $\theta \in (t_1, t)$. Let $\tau \in [t_1, t]$ and $\gamma_{\tau} = \gamma[0, \tau]$. Then $\gamma_{\tau} \in ch_+(\tau)$ and $b_+(t) = (t - \tau) h^*(0) + b_+(\tau)$.

Let $\delta \in ch_+(\tau)$. Then define $\tilde{\delta}$ by $\tilde{\delta}(\theta) = \delta(\theta)$ for $\theta \in (0, \tau)$ and $\tilde{\delta}(\theta) = 0$ for $\theta \in [\tau, t]$. Then

$$v_0(\gamma(0)) + \int_0^\tau h^*(\gamma'(\theta)) \, d\theta + (t - \tau) h^*(0) = b_+(t)$$

$$\leq v_0(\tilde{\delta}(0)) + \int_0^\tau h^*(\tilde{\delta}'(\theta)) \, d\theta + (t - \tau) h^*(0),$$

and hence

$$v_0(\gamma_{\tau}(0)) + \int_0^\tau h^*(\gamma_{\tau}'(\theta)) \, d\theta \leq v_0(\tilde{\delta}(0)) + \int_0^\tau h^*(\tilde{\delta}'(\theta)) \, d\theta. \quad (A.5)$$

This implies that $b_+(\tau) = J(\gamma_{\tau}, v_0, h)$ and $b_+(t) = (t - \tau) h^*(0) + b_+(\tau)$. This proves Step 2.

**Step 3.** Let $\gamma \in ch_+(t)$, $\delta \in ch_+(\tau)$ then $\gamma$ and $\delta$ do not intersect properly.

Suppose $\gamma$ and $\delta$ intersect properly. Then from Step 2 we can assume that $\gamma > 0$, $\delta > 0$ in $(0, t)$, $(0, \tau)$, respectively, and there exists $0 < \theta_0 < \min(t, \tau)$ such that $0 < \gamma(\theta_0) = \delta(\theta_0)$. Without loss of generality we can assume that $t \leq \tau$ and let $\gamma(\theta) = -\frac{\theta_0}{t} (\theta - t)$ and $\delta(\theta) = -\frac{\tau_0}{\tau} (\theta - \tau)$. At $\theta_0$ we have $y_2(1 - \frac{\theta_0}{t}) = y_1(1 - \frac{\theta_0}{t})$ and hence $\frac{\gamma_2}{\tau} = (1 - \frac{\theta_0}{t}) \frac{\gamma_1}{t} + \frac{\theta_0 \gamma_2}{t \tau}$. Since $h^*$ is strictly convex we have

$$th^*\left(-\frac{y_2}{t}\right) < (t - \theta_0) h^*\left(-\frac{y_1}{t}\right) + \theta_0 h^*\left(-\frac{y_2}{\tau}\right). \quad (A.6)$$

Now,

$$v_0(y_1) + th^*\left(-\frac{y_1}{t}\right) = b_+(t) \leq v_0(y_1) + th^*\left(-\frac{y_2}{t}\right)$$

$$< v_0(y_1) + (t - \theta_0) h^*\left(-\frac{y_1}{t}\right) + \theta_0 h^*\left(-\frac{y_2}{\tau}\right). \quad (A.7)$$
This implies that

\[ h^*(\frac{-y_1}{\tau}) < h^*(\frac{-y_2}{\tau}). \]  \hspace{1cm} (A.8)

Using the above estimate and \( y_1/\tau = (1 - \theta_0/\tau)y_2/\tau + (\theta_0/\tau)(y_1/\tau) \) we obtain

\[ v_0(y_2) + \tau h^*(\frac{-y_2}{\tau}) = b_+(\tau) \]
\[ \leq v_0(y_2) + \tau h^*(\frac{-y_1}{\tau}) \]
\[ < v_0(y_2) + (\tau - \theta_0)h^*(\frac{-y_2}{\tau}) + \theta_0 h^*(\frac{-y_1}{\tau}) \]
\[ < v_0(y_2) + \tau h^*(\frac{-y_2}{\tau}) \]  \hspace{1cm} (A.9)

which is a contradiction. This proves Step 3.

**Step 4.** For \( \gamma \in ch_+(t) \), let \( 0 \leq t_1(\gamma) \leq t \) be such that \( \gamma = 0 \) on \([t_1(\gamma), t]\) and \( \gamma > 0 \) in \((0, t_1(\gamma))\). Define

\[ \alpha(t) = \inf\{t_1(\gamma); \gamma \in ch_+(t)\}, \]
\[ y_+(t) = \inf\{\gamma(0); \gamma \in ch_+(t)\}. \]

Then \( t \mapsto \alpha(t), t \mapsto y_+(t) \) are non-decreasing functions.

Let \( t < \tau \) and \( \gamma \in ch_+(\tau) \) such that \( \alpha(t) = t_1(\gamma) \). If \( t \leq \alpha(t) \) then clearly \( \alpha(t) \leq \alpha(\tau) \). Hence let \( \alpha(t) < t < \tau \) and \( \delta(\theta) = \gamma(\theta) \) for \( \theta \in [0, t] \). Then from Step 2, \( \delta \in ch_+(t) \) and hence \( \alpha(t) \leq t_1(\gamma) = \alpha(\tau) \). Next suppose \( y_+(t) = 0 \), then \( \alpha(t) \leq \alpha(\tau) \leq y_+(\tau) = 0 \) and hence \( y_+(\tau) = 0 \). Let \( y_+(\tau) > 0 \) and \( \tilde{\gamma} \in ch_+(t) \) such that \( y_+(t) = \tilde{\gamma}(0) \). Suppose \( y_+(t) > y_+(\tau) \) then \( \gamma \) and \( \tilde{\gamma} \) intersect properly which is a contradiction from Step 3. Hence \( y_+(t) \leq y_+(\tau) \) and this proves Step 4.

**Step 5.** \( b^+ \in \text{Lip}(\mathbb{R}^+) \). From Step 4, for each \( t \) we can choose a unique \( \gamma_t \in ch_+(t) \) such that \( \gamma_t(0) = y_+(t), \gamma_t = 0 \) in \([\alpha(t), t]\). Let \( t < \tau \) and \( \gamma_t \) and \( \gamma_\tau \) be the corresponding characteristics. Suppose \( \alpha(\tau) \leq t < \tau \), then from Step 2 we have \( y_+(t) = y_+(\tau), \alpha(t) = \alpha(\tau) \) and hence

\[ b_+(t) - b_+(\tau) = v_0(y_+(\tau)) + (t - \alpha(\tau))h^*(0) + \alpha(\tau)h^*(\frac{-y_+(\tau)}{\alpha(\tau)}) - v_0(y_+(\tau)) \]
\[ - (\tau - \alpha(\tau))h^*(0) - \alpha(\tau)h^*(\frac{-y_+(\tau)}{\alpha(\tau)}) - (t - \tau)h^*(0). \]  \hspace{1cm} (A.10)
Suppose \( t < \alpha(\tau) \). Let \( \delta(\theta) = -\frac{y_+^{\tau}(\theta)}{t}(\theta - t) \) for \( \theta \in [0, t] \). Then from Step 1 we have

\[
b_+(t) - b_+(\tau) \leq v_0(\delta(0)) + th^*\left(\frac{-y_+(\tau)}{t}\right) - v_0(y_+(\tau)) - (\tau - \alpha(\tau))h^*(0) - \alpha(\tau)h^*\left(\frac{-y_+(\tau)}{\alpha(\tau)}\right) - t\left(h^*\left(\frac{-y_+(\tau)}{\alpha(\tau)}\right) - th^*\left(\frac{-y_+(\tau)}{t}\right)\right) \leq (\tau - t)\left|h^*(0)\right| + (t - \alpha(\tau))h^*\left(\frac{-y_+(\tau)}{\alpha(\tau)}\right) - t\left(h^*\left(\frac{-y_+(\tau)}{\alpha(\tau)}\right) - th^*\left(\frac{-y_+(\tau)}{t}\right)\right) \leq (\tau - t)\left|h^*(0)\right| + \sup_{\theta \in [0, c]} \left|h^*(\theta)\right| + C \sup_{\theta \in [0, c]} \left|h^*(\theta)\right| \right). \tag{A.11}
\]

Also we have

\[
b_+(t) - b_+(\tau) \geq v_0(y_+(t)) + (t - \alpha(\tau))h^*(0) + \alpha(\tau)h^*\left(\frac{-y_+(t)}{\alpha(\tau)}\right) - v_0(y_+(\tau)) - (\tau - \alpha(\tau))h^*(0) - \alpha(\tau)h^*\left(\frac{-y_+(\tau)}{\alpha(\tau)}\right) = -(\tau - t)h^*(0). \tag{A.12}
\]

Hence combining the above inequalities to obtain \( b_+ \in \text{Lip}(\mathbb{R}^+) \) and this proves Step 5.

**Step 6.** We have the following.

1. \( v'_0(y_+(t)) = h^*\left(\frac{-y_+(t)}{t}\right) \) if \( y_+(t) > 0 \).
2. At the points of differentiability of \( b_+(t), y_+(t) \), we have

\[
b'_+(t) = \begin{cases} -h(v'_0(y_+(t))) & \text{if } y_+(t) > 0, \\
-\theta h & \text{if } y_+(t) = 0. \end{cases} \tag{A.13}
\]

For \( \gamma \in \Gamma_+(0, t) \) let \( 0 \leq t_1 = t_1(\gamma) \leq t \) be such that \( \gamma = 0 \) on \([t_1(\gamma), t]\) and \( \gamma > 0 \) in \((0, t_1(\gamma))\).

Let \( \gamma(0) = y = y(\gamma) \). Then

\[
J(\gamma, v_0, h) = v_0(y) + (t - t_1)h^*(0) + t_1h^*\left(\frac{-y}{t_1}\right). \tag{A.14}
\]

Let \( y_+(t) > 0 \). Then either \( \alpha(t) \in (0, t) \) or \( \alpha(t) = t \). If \( \alpha(t) < t \), then we have at \( y = y_+(t), t = \alpha(t) \)

\[
\frac{\partial}{\partial y} J(\gamma, v_0, h) = \frac{\partial}{\partial t_1} J(\gamma, v_0, h) = 0. \tag{A.15}
\]
Hence
\[ v'_0(y_+(t)) - h^*(\frac{-y_+(t)}{\alpha(t)}) = 0 \quad (A.16) \]
and
\[
0 = -h^*(0) + h^*(\frac{-y_+(t)}{\alpha(t)}) + \frac{y_+(t)}{\alpha(t)} h^*(\frac{-y_+(t)}{t}) \\
= h(\theta_h) - h\left(h^*(\frac{-y_+(t)}{\alpha(t)})\right) \\
= h(\theta_h) - h\left(v'_0(y_+(t))\right). \quad (A.17)
\]
Therefore \( v'_0(y_+(t)) = \theta_h \). Combining this with the first equation we obtain
\[
0 = h'(\theta_h) = h'\left(v'_0(y_+(t))\right) = \frac{-y_+(t)}{\alpha(t)} < 0,
\]
which is a contradiction. Hence \( \alpha(t) = t \) and then \( v'_0(y_+(t)) = h^*(\frac{-y_+(t)}{t}) \). This proves (1).

Let \( t \) be a point of differentiability of \( b_+ \) and \( y_+ \). Let \( y_+(t) > 0 \). Then from previous analysis \( \alpha(t) = t \) and
\[
b_+(t) = v_0(y_+(t)) + th^*(\frac{-y_+(t)}{t}). \quad (A.18)
\]
Hence,
\[
b'_+(t) = v'_0\left(y_+(t) - h^*(\frac{-y_+(t)}{t})\right)y'_+(t) + h^*(\frac{-y_+(t)}{t}) + \frac{y_+(t)}{t} h^*(\frac{-y_+(t)}{t}) \\
= -h\left(h^*(\frac{-y_+(t)}{t})\right) \\
= -h\left(v'_0(y_+(t))\right). \quad (A.19)
\]
If \( y_+(t) = 0 \), then \( b_+(t) = v_0(0) + th^*(0) \) and hence \( b'_+(t) = h^*(0) = -h(\theta_h) \). This proves Step 6 and hence the lemma.

References