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The eigenmatrix of the linear association scheme on $R(2,m)$

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Abstract

Let $R(r,m)$ be the rth order Reed-Muller code of length 2^m. For $-1\leqslant r\leqslant s\leqslant m$, the action of the general affine group $\mathrm{AGL}(m,2)$ on $R(s,m)/R(r,m)$ defines a linear association scheme on $R(s,m)/R(r,m)$. In this paper, we determine the eigenmatrix of the linear association scheme on $R(2,m)$ $(=R(2,m)/R(-1,m))$. Our approach relies on the Möbius inversion and detailed calculations with the general linear group and the symplectic group over GF(2). As a consequence, we obtain explicit formulas for the weight enumerators of all cosets of $R(m-3,m)$. Such explicit formulas were not available previously. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Reed-Muller codes provided a working stage for the constructions of the Delsarte–Goethals codes by Delsarte and Goethals [4] and their formal duals by Hergert [7]. (The well known Kerdock code and Praparata code discovered earlier are special cases of the Delsarte–Goethals codes and their formal duals [11,13].) The blueprint for these constructions is the association scheme of the symplectic forms, i.e., the linear association scheme on $R(2,m)/R(1,m)$, where $R(r,m)$ is the rth order Reed-Muller code of length 2^m. The linear association scheme on $R(2,m)/R(1,m)$ is well understood. In particular, its eigenmatrix has been determined [4]. Using the eigenmatrix of the linear association scheme on $R(2,m)/R(1,m)$, one quickly obtains the weight enumerator of any coset of $R(m-3,m)$ in $R(m-2,m)$ through the MacWilliams identity.

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In general, for any $-1 \leqslant r \leqslant s \leqslant m$, the action of the general affine group $\mathrm{AGL}(m,2)$ on $R(s,m)/R(r,m)$ defines a linear association scheme on $R(s,m)/R(r,m)$. Conceivably, this association scheme contains much information about the Reed-Muller codes. However, very little is known about this association scheme in the general case. In this paper, we concentrate on the linear association scheme on $R(2,m)$ $(=R(2,m)/R(-1,m))$. Using the Möbius inversion and through detailed calculations with the general linear group and the symplectic group over GF(2), we are able to determine the eigenmatrix of this association scheme. A direct consequence of this result is explicit formulas for the weight enumerators of all cosets of $R(m-3,m)$ in $R(m,m)$. We remark that such explicit formulas have not been available. In an earlier paper [8], we classified all cosets of $R(m-3,m)$ in $R(m,m)$ and gave an inductive formula for the weight distribution of each such coset. However, the inductive formula contains some function which can be evaluated at specific values but has not been determined in general.

The following basic notations will be used throughout the paper. Let

$$\begin{aligned} \mathscr{P}_m &= \text{the GF(2)-algebra of all functions from } \mathrm{GF}(2)^m \text{ to GF(2)} \\ &= \mathrm{GF}(2)[X_1,\ldots,X_m]/(X_1^2 - X_1,\ldots,X_m^2 - X_m). \end{aligned} \tag{1.1}$$

The Hamming weight of an $f \in \mathscr{P}_m$ is $|f| = |f^{-1}(1)|$. For each $r \leqslant m$,

$$R(r,m) = \{f \in \mathscr{P}_m\colon\ \deg f \leqslant r\} \tag{1.2}$$

is the rth order Reed-Muller code of length 2^m. Recall that $R(r,m)^{\perp} = R(m-r-1,m)$. For any subset $\{i_1,\ldots,i_r\} \subset \{1,\ldots,m\}$, define

$$(X_{i_1}\ldots X_{i_r})^c = \prod_{i \in \{1,\ldots,m\}\setminus\{i_1,\ldots,i_r\}} X_i \in \mathscr{P}_m. \tag{1.3}$$

Certainly, $(\ )^c$ can be extended to a linear automorphism of $\mathscr{P}_m$. For any invertible affine transform of $\mathrm{GF}(2)^m$

$$\alpha : x \mapsto xA + b, \quad x \in \mathrm{GF}(2)^m, \tag{1.4}$$

where $A \in \mathrm{GL}(m,2)$, $b \in \mathrm{GF}(2)^m$, its action on an $f \in \mathscr{P}_m$ is defined as

$$\alpha(f) = f((X_1,\ldots,X_m)A + b). \tag{1.5}$$

The affine transform in (1.4) is denoted by $\alpha = (A,b)$. The group of all invertible affine transforms on $\mathrm{GF}(2)^m$, i.e., the general affine group on $\mathrm{GF}(2)^m$, is denoted by $\mathrm{AGL}(m,2)$. The general affine group $\mathrm{AGL}(m,2)$ acts on $R(r,m)$ as an automorphism group for all r, hence it also acts on $R(s,m)/R(r,m)$ as an automorphism group for $-1 \leqslant r \leqslant s \leqslant m$. For $f \in R(s,m)/R(r,m)$, its $\mathrm{AGL}(m,2)$-orbit will be denoted by $[f]$; its stabilizer in $\mathrm{AGL}(m,2)$ will be denoted by $\mathrm{AGL}(m,2)_f$.

For $0 \leqslant i \leqslant \lfloor m/2 \rfloor$, put

$$K(m,i) = \left.\begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix}\right\}_{2i} \in M_{m\times m}(\mathrm{GF}(2)). \tag{1.6}$$

Then

$$S(K(m,i)) = \{A \in \mathrm{GL}(m,2)\colon\ \mathrm{A}K(m,i)A^{\mathrm{T}} = K(m,i)\} \tag{1.7}$$

is the generalized symplectic group over GF(2).

Finally, for any binary code C, its weight enumerator will be denoted by $\mathscr{W}_C(x,y)$, namely,

$$\mathscr{W}_C(x,y) = \sum_{i=0}^{n} w_i(C)x^{n-i}y^i, \tag{1.8}$$

where n is the length of C and

$$w_i(C) = |\{a \in C\colon |a| = i\}|. \tag{1.9}$$

2. Linear association schemes

We refer the reader to [1,3] for general background in association schemes and to [7] for more details about linear association schemes on quotient spaces of the Reed-Muller codes. For our purpose, we will restrict ourselves to linear association schemes over the binary field. Let V and V' be two finite dimensional vector spaces over GF(2) of the same dimension and $\langle\cdot,\cdot\rangle : V \times V' \to \mathrm{GF}(2)$ a nondegenerate bilinear form. Let G be an automorphism group of V. Then G acts on V' too: For each $g \in G$ and $\alpha \in V'$, $g(\alpha) \in V'$ is defined by

$$\langle a, g(\alpha)\rangle = \langle g^{-1}(a), \alpha\rangle, \quad a \in V. \tag{2.1}$$

In this way, G also becomes an automorphism group of V'. V and V' have the same number of G-orbits. (This claim follows from the Burnside Lemma since for any invertible matrix A, A and $(A^{-1})^{\mathrm{T}}$ fix the same number of vectors.) Let $C_0,\ldots,C_d$ be the G-orbits in V and $C'_0,\ldots,C'_d$ the G-orbits in V'. Put $R_i = \{(a,b) \in V \times V\colon a-b \in C_i\}$, $0 \leqslant i \leqslant d$. Then $\mathscr{R} = \{R_0,\ldots,R_d\}$ forms an association scheme on V. In [7], such an association scheme is called a linear association scheme. Of course, $\mathscr{R}' = \{R'_0,\ldots,R'_d\}$, where $R'_i = \{(\alpha,\beta) \in V' \times V'\colon \alpha - \beta \in R'_i\}$, $0 \leqslant i \leqslant d$, also forms a linear association scheme on V'; in [7], $\mathscr{R}'$ is called the dual of $\mathscr{R}$. The first eigenmatrix $P=(p_j(i))$ and the second eigenmatrix $Q=(q_j(i))$ of $\mathscr{R}$ are given by

$$p_j(i)=\sum_{a\in C_j}(-1)^{\langle a,\alpha_i\rangle}, \tag{2.2}$$

$$q_j(i)=\sum_{\alpha\in C_j'}(-1)^{\langle a_i,\alpha\rangle}, \tag{2.3}$$

where $\alpha_i\in C_i'$ and $a_i\in C_i$ are arbitrary. Of course, the first and the second eigenmatrice of $\mathscr{R}'$ are Q and P respectively. If the sizes of the orbits C_i, C_i' $(0\leqslant i\leqslant d)$ are known, the matrices P and Q give rise to each other easily via

$$|C_i|q_j(i)=|C_j'|p_i(j),\quad 0\leqslant i,j\leqslant d. \tag{2.4}$$

In this paper, the eigenmatrix of an association scheme refers to its first eigenmatrix.

Quotient spaces of the Reed-Muller codes provide an interesting but challenging family of linear association schemes. For $-1\leqslant r\leqslant s\leqslant m$, define

$$\langle\cdot,\cdot\rangle:(R(s,m)/R(r,m))\times(R(m-r-1,m)/R(m-s-1,r))\to \mathrm{GF}(2),$$
$$(f,g)\mapsto|f\cdot g| \tag{2.5}$$

where the Hamming weight $|f\cdot g|\in\mathbb{Z}$ is viewed as an element in GF(2). Then $\langle\cdot,\cdot\rangle$ is a well defined nondegenerate bilinear form on $(R(s,m)/R(r,m))\times(R(m-r-1,m)/R(m-s-1,m))$. Using $\mathrm{AGL}(m,2)$ as an automorphism group of $R(s,m)/R(r,m)$, we obtain a linear association scheme on $R(s,m)/R(r,m)$ and its dual scheme on $R(m-r-1,m)/R(m-s-1,m)$. Note that we have two actions of $\mathrm{AGL}(m,2)$ on $R(m-r-1,m)/R(m-s-1,m)$: Initially, $\mathrm{AGL}(m,2)$ already acts on $R(m-r-1,m)/R(m-s-1,m)$; on the other hand, the $\mathrm{AGL}(m,2)$ action on $R(s,m)/R(r,m)$ induces an $\mathrm{AGL}(m,2)$ action on $R(m-r-1,m)/R(m-s-1,m)$ via (2.1). But this does not cause any ambiguity since one can easily see that the two $\mathrm{AGL}(m,2)$ actions on $R(m-r-1,m)/R(m-s-1,m)$ coincide.

The linear association scheme so defined on $R(2,m)/R(1,m)$ is the association scheme of symplectic forms. The eigenmatrix of this association scheme has been determined by Delsarte and Goethals [4]. Let $C_i=\{f\in R(2,m)/R(1,m)\colon\ \mathrm{rank}\,f=2i\}$, $0\leqslant i\leqslant\lfloor m/2\rfloor$, where $\mathrm{rank}\,f$ is the rank of the skew-symmetric matrix associated to f. Then C_i $(0\leqslant i\leqslant\lfloor m/2\rfloor)$ are the $\mathrm{AGL}(m,2)$ orbits of $R(2,m)/R(1,m)$ and $C_i'=\{f^c\colon f\in C_i\}$ $(0\leqslant i\leqslant\lfloor m/2\rfloor)$ are the $\mathrm{AGL}(m,2)$ orbits of $R(m-2,m)/R(m-3,m)$. It is established in [4] that for $0\leqslant i,j\leqslant\lfloor m/2\rfloor$ and $g_i\in C_i'$, $\sum_{f\in C_j}(-1)^{\langle f,g_i\rangle}$ satisfies an equation which is recursive in i, j and m. As the solution of the recursive equation,

$$\sum_{f\in C_j}(-1)^{\langle f,g_i\rangle}=\sum_{k=0}^{j}(-1)^{j-k}4^{\binom{j-k}{2}}\begin{bmatrix}\lfloor m/2\rfloor-k\\ \lfloor m/2\rfloor-j\end{bmatrix}_4 \begin{bmatrix}\lfloor m/2\rfloor-i\\ k\end{bmatrix}_4 2^{k(2\lceil m/2\rceil-1)}, \tag{2.6}$$

where $[\ \ ]_b$ is the b-ary Gauss coefficient.

As we move to determine the eigenmatrix of the linear association scheme on $R(2,m)$, the calculations that we face will be more complicated and our approach will be different.

3. The AGL$(m,2)$-orbits of $R(2,m)$ and $R(m,m)/R(m-3,m)$

The first step in determining the eigenmatrix of the linear association scheme on $R(2,m)$ is to enumerate a list of AGL$(m,2)$ orbit representatives for $R(2,m)$ and $R(m,m)/R(m-3,m)$ and determine the orbit size, or equivalently, the size of the stabilizer, of each representative.

Proposition 3.1. *Let*

$$f_i = X_1X_2 + X_3X_4 + \cdots + X_{2i-1}X_{2i}, \quad 0 \leqslant i \leqslant \lfloor m/2 \rfloor, \tag{3.1}$$

$$g_i = X_1X_2 + X_3X_4 + \cdots + X_{2i-1}X_{2i} + X_m, \quad 0 \leqslant i < m/2, \tag{3.2}$$

$$h_i = X_1X_2 + X_3X_4 + \cdots + X_{2i-1}X_{2i} + 1, \quad 0 \leqslant i \leqslant \lfloor m/2 \rfloor. \tag{3.3}$$

Then $\{f_i\colon 0 \leqslant i \leqslant \lfloor m/2 \rfloor\} \cup \{g_i\colon 0 \leqslant i < m/2\} \cup \{h_i\colon 0 \leqslant i \leqslant \lfloor m/2 \rfloor\}$ *forms a complete list of* AGL$(m,2)$ *orbit representatives in* $R(2,m)$. *The sizes of their stabilizers in* AGL$(m,2)$ *are as follows.*

$$|\mathrm{AGL}(m,2)_{f_i}| = 2^{-i^2-i+(1/2)(m^2+m)} \prod_{k=1}^{m-2i} (2^k - 1) \prod_{k=1}^{i} (2^{2k} - 1), \quad 0 \leqslant i \leqslant \lfloor m/2 \rfloor. \tag{3.4}$$

$$|\mathrm{AGL}(m,2)_{g_i}| = 2^{-i^2-i+(1/2)(m^2+m)-1} \prod_{k=1}^{m-2i-1} (2^k - 1) \prod^{i} (2^{2k} - 1), \quad 0 \leqslant i < m/2. \tag{3.5}$$

$$|\mathrm{AGL}(m,2)_{h_i}| = 2^{-i^2-i+(1/2)(m^2+m)} \prod_{k=1}^{m-2i} (2^k - 1) \prod_{k=1}^{i} (2^{2k} - 1), \quad 0 \leqslant i \leqslant \lfloor m/2 \rfloor. \tag{3.6}$$

Proof. It is well known that the elements in (3.1)–(3.3) form a complete list of AGL$(m,2)$ orbit representatives in $R(2,m)$. Observe that for $f \in R(2,m)$, $f \sim f_i$ if and only if $|f| = 2^{m-1} - 2^{m-1-i}$ and $f \sim h_i$ if and only if $|f| = 2^{m-1} + 2^{m-1-i}$, where $\sim$ means AGL$(m,2)$ equivalence. On the other hand, the weight enumerator of $R(2,m)$ is known (see [15], for example). In particular, for $0 \leqslant i \leqslant \lfloor m/2 \rfloor$,

$$w_{2^{m-1}-2^{m-1-i}}(R(2,m)) = w_{2^{m-1}+2^{m-1-i}}(R(2,m)) = 2^{i^2+i} \frac{\prod_{k=m-2i+1}^{m} (2^k - 1)}{\prod_{k=1}^{i} (2^{2k} - 1)}. \tag{3.7}$$

Thus

$$\begin{aligned}|\mathrm{AGL}(m,2)_{f_i}| &= \frac{|\mathrm{AGL}(m,2)|}{|[f_i]|}\\ &= \frac{|\mathrm{AGL}(m,2)|}{w_{2^{m-1}-2^{m-1-i}}(R(2,m))}\\ &= 2^{-i^2-i+(1/2)(m^2+m)}\prod_{k=1}^{m-2i}(2^k-1)\prod_{k=1}^{i}(2^{2k}-1),\end{aligned} \tag{3.8}$$

and (3.4) is proved. Eq. (3.6) follows in the same way.

To prove (3.5), let $\alpha=(A,b)\in \mathrm{AGL}(m,2)$, where $A=(a_{kl})\in \mathrm{GL}(m,2)$ and $b=(b_1,\dots,b_m)\in \mathrm{GF}(2)^m$. For $0\leqslant i<m/2$, $\alpha(g_i)=g_i$ if and only if the following three equations are satisfied:

$$AK(m,i)A^{\mathrm{T}}=K(m,i),\ \text{i.e.,}\ A\in S(K(m,i)), \tag{3.9}$$

$$\begin{aligned}&a_{k1}a_{k2}+a_{k3}a_{k4}+\cdots+a_{k,2i-1}a_{k,2i}+b_1a_{k2}+b_2a_{k1}+b_3a_{k4}\\ &\quad +b_4a_{k3}+\cdots+b_{2i-1}a_{k,2i}+b_{2i}a_{k,2i-1}+a_{km}=\begin{cases}0, & 1\leqslant k\leqslant m-1,\\ 1, & k=m,\end{cases}\end{aligned} \tag{3.10}$$

$$b_1b_2+b_3b_4+\cdots+b_{2i-1}b_{2i}+b_m=0. \tag{3.11}$$

It is known that

$$|S(K(m,i))| = 2^{-i^2+i+(1/2)(m^2-m)}\prod_{k=1}^{i}(2^{2k}-1)\prod_{k=1}^{m-2i}(2^k-1). \tag{3.12}$$

(Cf. [8].) Elements in $S(K(m,i))$ are of the form

$$A=\begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix},\quad A_{11}\in S(K(2i,i)),\quad A_{22}\in \mathrm{GL}(m-2i,2). \tag{3.13}$$

Given $A=(a_{kl})\in S(K(m,i))$, then Eq. (3.10) with $k=2i+1,\dots,m$ is equivalent to $(a_{2i+1,m},\dots,a_{mm})=(0,\dots,0,1)$; for $1\leqslant k\leqslant 2i$, Eq. (3.10) determines a unique $(b_1,\dots,b_{2i})$. Thus we have

$$\begin{aligned}|\mathrm{AGL}(m,2)_{g_i}| &= |S(K(2i,i))|2^{2i(m-2i)}(2^{m-2i}-2)\cdots(2^{m-2i}-2^{m-2i-1})2^{m-1-2i}\\ &= 2^{-i^2-i+(1/2)(m^2+m)-1}\prod_{k=1}^{i}(2^{2k}-1)\prod_{k=1}^{m-2i-1}(2^k-1),\end{aligned} \tag{3.14}$$

and (3.5) is proved. □

Proposition 3.2. *Let*

$$F_i=(X_1X_2)^c+(X_3X_4)^c+\cdots+(X_{2i-1}X_{2i})^c,\quad 0\leqslant i\leqslant \lfloor m/2\rfloor, \tag{3.15}$$

$$G_i=(X_1X_2)^c+(X_3X_4)^c+\cdots+(X_{2i-1}X_{2i})^c+X_m^c,\quad 0\leqslant i<m/2, \tag{3.16}$$

$$H_i=(X_1X_2)^c+(X_3X_4)^c+\cdots+(X_{2i-1}X_{2i})^c+1^c,\quad 0\leqslant i\leqslant \lfloor m/2\rfloor. \tag{3.17}$$

Then $\{F_i\colon 0\leqslant i\leqslant \lfloor m/2\rfloor\}\cup\{G_i\colon 0\leqslant i<m/2\}\cup\{H_i\colon 0\leqslant i\leqslant\lfloor m/2\rfloor\}$ *forms a complete list of* AGL$(m,2)$ *orbit representatives of* $R(m,m)/R(m-3,m)$. *The* AGL$(m,2)$ *orbit sizes of the representatives are as follows.*

$$|[F_i]| = 2^{i^2-i}\,\frac{\prod_{k=m-2i+1}^{m}(2^k-1)}{\prod_{k=1}^{i}(2^{2k}-1)},\quad 0\leqslant i\leqslant\lfloor m/2\rfloor. \tag{3.18}$$

$$|[G_i]| = 2^{i^2-i+m-1}\,\frac{\prod_{k=m-2i}^{m}(2^k-1)}{\prod_{k=1}^{i}(2^{2k}-1)},\quad 0\leqslant i< m/2. \tag{3.19}$$

$$|[H_i]| = 2^{i^2-i+m}\,\frac{\prod_{k=m-2i+1}^{m}(2^k-1)}{\prod_{k=1}^{i}(2^{2k}-1)},\quad 0\leqslant i\leqslant\lfloor m/2\rfloor. \tag{3.20}$$

Proof. In [8], we determined that F_i $(0\leqslant i\leqslant\lfloor m/2\rfloor)$ together with

$$\alpha_i = i\cdot 1^c + X_1^c+\cdots+X_i^c,\quad 1\leqslant i\leqslant m \tag{3.21}$$

and

$$\beta=(m+1)1^c+X_1^c+\cdots+X_m^c \tag{3.22}$$

form a complete list of AGL$(m,2)$ orbit representatives of $R(m,m)/R(m-3,m)$. Using the method in [8], we can also show that when m is odd,

$$G_i\sim\begin{cases}\alpha_{2i+2}, & 0\leqslant i<(m-1)/2,\\ \beta, & i=(m-1)/2,\end{cases} \tag{3.23}$$

$$H_i\sim\alpha_{2i+1},\quad 0\leqslant i\leqslant(m-1)/2, \tag{3.24}$$

and when m is even,

$$G_i\sim\alpha_{2i+2},\quad 0\leqslant i<m/2, \tag{3.25}$$

$$H_i\sim\begin{cases}\alpha_{2i+1}, & 0\leqslant i<m/2,\\ \beta, & i=m/2,\end{cases} \tag{3.26}$$

where $\sim$ means AGL$(m,2)$-equivalence. Eqs. (3.18)–(3.20) are from Theorem 3.7 of [8]. □

4. Several counting lemmas

When we compute the eigenmatrix of the linear association scheme on $R(2,m)$ in the next section, we will need the sizes of the following sets:

$$\Theta(i,j)=\left\{A=(a_{kl})\in \mathrm{GL}(m,2)\colon \sum_{k=1}^{j}\sum_{l=1}^{i}\begin{vmatrix}a_{2k-1,2l-1} & a_{2k-1,2l}\\ a_{2k,2l-1} & a_{2k,2l}\end{vmatrix}=0\right\},$$
$$0\leqslant i,j\leqslant\lfloor m/2\rfloor. \tag{4.1}$$

$$\Theta_0(i,j)=\{A=(a_{kl})\in\Theta(i,j)\colon (a_{m1},\ldots,a_{m,2i})=0\},$$
$$0\leqslant i\leqslant\lfloor m/2\rfloor,\ 0\leqslant j<m/2. \tag{4.2}$$

$$\Theta_{0,0}(i,j)=\{A=(a_{kl})\in\Theta_0(i,j)\colon a_{mm}=0\},\quad 0\leqslant i\leqslant\lfloor m/2\rfloor,\ 0\leqslant j<m/2. \tag{4.3}$$

The goal of this section is to make the sizes of the above sets available. The relevance of $\Theta(i,j)$ to our problem is clear: For $0\leqslant i,j\leqslant\lfloor m/2\rfloor$ and $A\in\mathrm{GL}(m,2)$, $\langle A(f_i),F_j\rangle=0$ if and only if $A\in\Theta(i,j)$. The relevance of $\Theta_0(i,j)$ and $\Theta_{0,0}(i,j)$ will be clear in the next section.

Lemma 4.1. *For* $0\leqslant i,j\leqslant\lfloor m/2\rfloor$, *we have*

$$|\Theta(i,j)|$$
$$=2^{(1/2)(m^2-m)-1}\prod_{k=1}^{m}(2^k-1)+2^{-i^2+i+(1/2)(m^2-m)-1}\prod_{k=1}^{i}(2^{2k}-1)\prod_{k=1}^{m-2i}(2^k-1)$$
$$\times\sum_{k=0}^{i}(-1)^{i-k}\begin{bmatrix}\lfloor m/2\rfloor-k\\ \lfloor m/2\rfloor-i\end{bmatrix}_4\begin{bmatrix}\lfloor m/2\rfloor-j\\ k\end{bmatrix}_4 4^{\binom{i-k}{2}}2^{k(2\lceil m/2\rceil-1)}. \tag{4.4}$$

Proof. From (2.6), we have

$$\sum_{k=0}^{i}(-1)^{i-k}4^{\binom{i-k}{2}}\begin{bmatrix}\lfloor m/2\rfloor-k\\ \lfloor m/2\rfloor-i\end{bmatrix}_4\begin{bmatrix}\lfloor m/2\rfloor-j\\ k\end{bmatrix}_4 2^{k(2\lceil m/2\rceil-1)}$$

$$=\sum_{f\sim f_i\ \text{in}\ R(2,m)/R(1,m)}(-1)^{\langle f,F_j\rangle}$$

$$=\frac{1}{|S(K(m,i))|}\sum_{A\in\mathrm{GL}(m,2)}(-1)^{\langle A(f_i),F_j\rangle}$$

$$=\frac{1}{|S(K(m,i))|}(2|\Theta(i,j)|-|\mathrm{GL}(m,2)|). \tag{4.5}$$

(In (4.5), note that when $\mathrm{GL}(m,2)$ acts on $R(2,m)/R(1,m)$, the stabilizer of f_i is the generalized symplectic group $S(K(m,i))$.) Now (4.4) follows from (4.5) and (3.12). □

The sizes of $\Theta_0(i,j)$ and $\Theta_{0,0}(i,j)$ are more difficult to compute. For $0\leqslant i,j\leqslant\lfloor m/2\rfloor$, we let

$$\theta(i,j)=\left\{A=(a_{kl})\in M_{2j\times 2i}(\mathrm{GF}(2))\colon \sum_{k=1}^{j}\sum_{l=1}^{i}\begin{vmatrix}a_{2k-1,2l-1} & a_{2k-1,2l}\\ a_{2k,2l-1} & a_{2k,2l}\end{vmatrix}=0\right\}, \tag{4.6}$$

$$\theta(i,j;\rho)=\{A\in\theta(i,j)\colon\ \mathrm{rank}\,A=\rho\}. \tag{4.7}$$

Lemma 4.2. *For* $0 \leqslant i \leqslant \lfloor m/2 \rfloor$ *and* $0 \leqslant j < m/2$, *we have*

$$|\Theta_0(i,j)| = \sum_{\rho \leqslant \min\{2i,2j\}} 2^{(m-2i-2j)\rho + (1/2)(\rho^2-\rho+m^2-m)} \times \prod_{k=1}^{m-2i}(2^k-1) \prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)|\theta(i,j;\rho)|, \tag{4.8}$$

$$|\Theta_{0,0}(i,j)| = \begin{cases} 0 & \text{if } 2i = m, \\ \sum_{\rho \leqslant \min\{2i,2j\}} 2^{(m-2i-2j)\rho+(1/2)(\rho^2-\rho+m^2-m)}(2^{m-2i-1}-1) \\ \quad \times \prod_{k=1}^{m-2i-1}(2^k-1) \prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)|\theta(i,j;\rho)| & \text{if } 2i < m. \end{cases} \tag{4.9}$$

Proof. Given $A \in \theta(i,j;\rho)$, suppose that we choose matrices $A_1 \in M_{(m-2j-1)\times 2i}(\mathrm{GF}(2))$ and $A_2 \in M_{m\times(m-2i)}(\mathrm{GF}(2))$ successively such that

$$\begin{bmatrix} A & \\ A_1 & A_2 \\ 0 & \end{bmatrix} \in \Theta_0(i,j). \tag{4.10}$$

Then the number of choices for A_1 is

$$2^{(m-2j-1)\rho}(2^{m-2j-1}-1)\cdots(2^{m-2j-1}-2^{2i-\rho-1}) = 2^{(m-2j-1)\rho+(1/2)(2i-\rho)(2i-\rho-1)} \prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1) \tag{4.11}$$

and the number of choices for A_2 is

$$(2^m-2^{2i})\cdots(2^m-2^{m-1}) = 2^{(1/2)(m-2i)(m+2i-1)}\prod_{k=1}^{m-2i}(2^k-1). \tag{4.12}$$

Therefore we have

$$\begin{aligned} |\Theta_0(i,j)| &= \sum_\rho 2^{(m-2j-1)\rho+(1/2)(2i-\rho)(2i-\rho-1)+(1/2)(m-2i)(m+2i-1)} \\ &\quad \times \prod_{k=1}^{m-2i}(2^k-1)\prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)|\theta(i,j;\rho)| \\ &= \sum_\rho 2^{(m-2i-2j)\rho+(1/2)(\rho^2-\rho+m^2-m)}\prod_{k=1}^{m-2i}(2^k-1) \\ &\quad \times \prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)|\theta(i,j;\rho)|. \end{aligned} \tag{4.13}$$

Obviously, $|\Theta_{0,0}(i,j)|=0$ when $2i=m$. So, assume that $2i<m$. Given $A\in\theta(i,j;\rho)$, suppose that we choose $A_1\in M_{(m-2j-1)\times 2i}(\mathrm{GF}(2))$, $A_2\in M_{(m-1)\times 1}(\mathrm{GF}(2))$ and $A_3\in M_{m\times(m-2i-1)}(\mathrm{GF}(2))$ successively such that

$$\begin{bmatrix} A & & \\ & & A_2 \\ A_1 & A_3 & \\ 0 & & 0 \end{bmatrix} \in \Theta_{0,0}(i,j). \tag{4.14}$$

Then the number of choices for A_1 is again given by (4.11); the number of choices for A_2 is $2^{m-1}-2^{2i}=2^{2i}(2^{m-2i-1}-1)$; the number of choices for A_3 is

$$(2^m-2^{2i+1})\cdots(2^m-2^{m-1})=2^{(1/2)(m+2i)(m-2i-1)}\prod_{k=1}^{m-2i-1}(2^k-1). \tag{4.15}$$

Thus

$$\begin{aligned}|\Theta_{0,0}(i,j)| &= \sum_{\rho} 2^{(m-2j-1)\rho+(1/2)(2i-\rho)(2i-\rho-1)+2i+(1/2)(m+2i)(m-2i-1)}(2^{m-2i-1}-1)\\ &\quad\times\prod_{k=1}^{m-2i-1}(2^k-1)\prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)|\theta(i,j;\rho)|\\ &= \sum_{\rho} 2^{(m-2i-2j)\rho+(1/2)(\rho^2-\rho+m^2-m)}(2^{m-2i-1}-1)\\ &\quad\times\prod_{k=1}^{m-2i-1}(2^k-1)\prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)|\theta(i,j;\rho)|,\end{aligned} \tag{4.16}$$

and (4.9) is proved. □

Remark. From (4.8) and (4.9), we see that

$$|\Theta_{0,0}(i,j)|=\frac{2^{m-2i-1}-1}{2^{m-2i}-1}|\Theta_0(i,j)|,\quad 0\leqslant i,j<m/2. \tag{4.17}$$

Because of Lemma 4.2, in order to determine $|\Theta_0(i,j)|$ and $|\Theta_{0,0}(i,j)|$, it suffices to determine $|\theta(i,j;\rho)|$.

For each subspace $H\subset\mathrm{GF}(2)^{2i}$, define

$$\theta_{=}(i,j;H)=\{A\in\theta(i,j)\colon \text{the row space of } A \text{ is } H\}, \tag{4.18}$$

$$\theta_{\leqslant}(i,j;H)=\{A\in\theta(i,j)\colon \text{the row space of } A\subset H\}. \tag{4.19}$$

Then clearly,

$$|\theta(i,j;\rho)|=\sum_{H\subset\mathrm{GF}(2)^{2i},\dim H=\rho}|\theta_{=}(i,j;H)| \tag{4.20}$$

and

$$|\theta_{\leqslant}(i,j;H)|=\sum_{L\subset H}|\theta_{=}(i,j;L)|. \tag{4.21}$$

Using the Möbius inversion, we have

$$|\theta_{=}(i,j;H)| = \sum_{L \subset H} \mu(L,H)|\theta_{\leqslant}(i,j;L)|, \tag{4.22}$$

where μ is the Möbius function of the partially ordered set of subspaces of $\mathrm{GF}(2)^{2i}$. The function μ has been determined by Rota [14] (also see [2]) as follows.

$$\mu(L,H) = (-1)^{\dim H - \dim L} 2^{\binom{\dim H - \dim L}{2}}. \tag{4.23}$$

Therefore, by (4.20), (4.22) and (4.23), we have

$$\begin{aligned} |\theta(i,j;\rho)| &= \sum_{\substack{H \subset \mathrm{GF}(2)^{2i} \\ \dim H = \rho}} \sum_{L \subset H} (-1)^{\rho - \dim L} 2^{\binom{\rho - \dim L}{2}} |\theta_{\leqslant}(i,j;L)| \\ &= \sum_{\sigma \leqslant \rho} \sum_{\substack{L \subset \mathrm{GF}(2)^{2i} \\ \dim L = \sigma}} \sum_{\substack{H \supset L \\ \dim H = \rho}} (-1)^{\rho-\sigma} 2^{\binom{\rho-\sigma}{2}} |\theta_{\leqslant}(i,j;L)| \\ &= \sum_{\sigma \leqslant \rho} \sum_{\substack{L \subset \mathrm{GF}(2)^{2i} \\ \dim L = \sigma}} (-1)^{\rho-\sigma} 2^{\binom{\rho-\sigma}{2}} \begin{bmatrix} 2i - \sigma \\ \rho - \sigma \end{bmatrix}_2 |\theta_{\leqslant}(i,j;L)|. \end{aligned} \tag{4.24}$$

Let $L \subset \mathrm{GF}(2)^{2i}$ be any σ-dimensional subspace and let $x_1, \ldots, x_\sigma$ be a basis of L. Since

$$\begin{bmatrix} x_1 \\ \vdots \\ x_\sigma \end{bmatrix} K(2i,i) \begin{bmatrix} x_1 \\ \vdots \\ x_\sigma \end{bmatrix}^{\mathrm{T}} \in M_{\sigma \times \sigma}(\mathrm{GF}(2)) \tag{4.25}$$

is a skew-symmetric matrix with all diagonal entries equal to 0, there exists a $P \in \mathrm{GL}(\sigma, 2)$ such that

$$P \begin{bmatrix} x_1 \\ \vdots \\ x_\sigma \end{bmatrix} K(2i,i) \begin{bmatrix} x_1 \\ \vdots \\ x_\sigma \end{bmatrix}^{\mathrm{T}} P^{\mathrm{T}} = K(\sigma, t), \quad t \leqslant \lfloor \sigma/2 \rfloor, \tag{4.26}$$

where $K(2i,i)$ and $K(\sigma,t)$ are defined in (1.6). Clearly, the number t in (4.26) is independent of the choice of the basis of L. We shall call the subspace L of type (σ,t) if (4.26) holds, or equivalently, if L has a basis $x_1, \ldots, x_\sigma$ such that

$$\begin{bmatrix} x_1 \\ \vdots \\ x_\sigma \end{bmatrix} K(2i,i) \begin{bmatrix} x_1 \\ \vdots \\ x_\sigma \end{bmatrix}^{\mathrm{T}} = K(\sigma, t). \tag{4.27}$$

Let $L \subset \mathrm{GF}(2)^{2i}$ be a subspace of type (σ,t) with a basis $x_1, \ldots, x_\sigma$ satisfying (4.27). Let $B = (b_{kl}) \in M_{2j \times \sigma}(\mathrm{GF}(2))$ and put

$$(a_{kl}) = A = B \begin{bmatrix} x_1 \\ \vdots \\ x_\sigma \end{bmatrix} \in M_{2j \times 2i}(\mathrm{GF}(2)). \tag{4.28}$$

Then we have

$$AK(2i,i)A^{\mathrm{T}} = BK(\sigma,t)B^{\mathrm{T}}. \tag{4.29}$$

Summing up the entries of both sides of (4.29) at positions $(1,2),(3,4),\dots,(2j-1,2j)$, we have

$$\sum_{k=1}^{j}\sum_{l=1}^{i}\begin{vmatrix} a_{2k-1,2l-1} & a_{2k-1,2l} \\ a_{2k,2l-1} & a_{2k,2l}\end{vmatrix} = \sum_{k=1}^{j}\sum_{l=1}^{t}\begin{vmatrix} b_{2k-1,2l-1} & b_{2k-1,2l} \\ b_{2k,2l-1} & b_{2k,2l}\end{vmatrix}. \tag{4.30}$$

Therefore,

$$\begin{aligned}
&|\theta_{\leqslant}(i,j;L)| \\
&= \left|\left\{(a_{kl}) = B\begin{bmatrix} x_1 \\ \vdots \\ x_\sigma\end{bmatrix} : B \in M_{2j\times\sigma}(\mathrm{GF}(2)), \sum_{k=1}^{j}\sum_{l=1}^{i}\begin{vmatrix} a_{2k-1,2l-1} & a_{2k-1,2l} \\ a_{2k,2l-1} & a_{2k,2l}\end{vmatrix} = 0\right\}\right| \\
&= \left|\left\{B = (b_{kl}) \in M_{2j\times\sigma}(\mathrm{GF}(2))\colon \sum_{k=1}^{j}\sum_{l=1}^{t}\begin{vmatrix} b_{2k-1,2l-1} & b_{2k-1,2l} \\ b_{2k,2l-1} & b_{2k,2l}\end{vmatrix} = 0\right\}\right| \\
&= 2^{2j\sigma} - (\text{the weight of } X_1X_2 + X_3X_4 + \cdots + X_{4jt-1}X_{4jt} \text{ in } R(2,2j\sigma)) \\
&= 2^{2j\sigma-1} + 2^{2j\sigma-2jt-1}.
\end{aligned} \tag{4.31}$$

On the other hand, for $0 \leqslant 2t \leqslant \sigma \leqslant 2i$, let $\mathcal{N}^{(2i)}_{(\sigma,t)}$ be the set of all subspaces of type (σ,t) in $\mathrm{GF}(2)^{2i}$. To determine $|\mathcal{N}^{(2i)}_{(\sigma,t)}|$, suppose that we choose $x_1,\dots,x_\sigma \in \mathrm{GF}(2)^{2i}$ consecutively such that they are linearly independent and

$$\begin{bmatrix} x_1 \\ \vdots \\ x_\sigma\end{bmatrix} K(2i,i) \begin{bmatrix} x_1 \\ \vdots \\ x_\sigma\end{bmatrix}^{\mathrm{T}} = K(\sigma,t). \tag{4.32}$$

Then the number of choices for $x_1,x_2;\ x_3,x_4;\dots;x_{2t-1},x_{2t}$ are $2^{2i}-1$, $2^{2i-1};2^{2i-2}-1,\ 2^{2i-3};\dots;2^{2i-2t+2}-1,\ 2^{2i-2t+1}$. The numbers of choices for x_{2t+1}, $x_{2t+2},\dots,x_\sigma$ are $2^{2i-2t}-1,\ 2^{2i-2t-1}-2,\dots,2^{2i-\sigma+1}-2^{\sigma-2t-1}$. Thus

$$\begin{aligned}
|\mathcal{N}^{(2i)}_{(\sigma,t)}| &= \frac{1}{|S(K(\sigma,t))|}2^{(2i-1)+(2i-3)+\cdots+(2i-2t+1)} \\
&\quad \prod_{k=i-t+1}^{i}(2^{2k}-1)\prod_{k=2i-\sigma+1}^{2i-2t}(2^k-2^{2i-2t-k}) \\
&= \frac{2^{-t^2+2it+(1/2)(\sigma-2t)(\sigma-2t-1)}\prod_{k=i-\sigma+t+1}^{i}(2^{2k}-1)}{2^{-t^2+t+(1/2)(\sigma^2-\sigma)}\prod_{k=1}^{t}(2^{2k}-1)\prod_{k=1}^{\sigma-2t}(2^k-1)} \\
&= 2^{2t^2+2it-2\sigma t}\frac{\prod_{k=i-\sigma+t+1}^{i}(2^{2k}-1)}{\prod_{k=1}^{t}(2^{2k}-1)\prod_{k=1}^{\sigma-2t}(2^k-1)}.
\end{aligned} \tag{4.33}$$

Combining (4.24), (4.31) and (4.33), we now have

$$
\begin{aligned}
|\theta(i,j;\rho)| &= \sum_{\sigma\leqslant\rho}\sum_{2t\leqslant\sigma}\sum_{L\in\mathcal{N}^{(2i)}_{(\sigma,t)}}(-1)^{\rho-\sigma}2^{\binom{\rho-\sigma}{2}}\begin{bmatrix}2i-\sigma\\ \rho-\sigma\end{bmatrix}_2 |\theta_{\leqslant}(i,j;L)| \\
&= \sum_{\sigma\leqslant\rho}\sum_{2t\leqslant\sigma}|\mathcal{N}^{(2i)}_{(\sigma,t)}|(-1)^{\rho-\sigma}2^{\binom{\rho-\sigma}{2}}\begin{bmatrix}2i-\sigma\\ \rho-\sigma\end{bmatrix}_2 (2^{2j\sigma-1}+2^{2j\sigma-2jt-1}) \\
&= \sum_{\sigma\leqslant\rho}\sum_{2t\leqslant\sigma}(-1)^{\rho-\sigma}2^{\binom{\rho-\sigma}{2}+2t^2+2it-2\sigma t+2j\sigma-1}(1+2^{-2jt})\begin{bmatrix}2i-\sigma\\ \rho-\sigma\end{bmatrix}_2 \\
&\quad\times\frac{\prod_{k=i-\sigma+t+1}^{i}(2^{2k}-1)}{\prod_{k=1}^{t}(2^{2k}-1)\prod_{k=1}^{\sigma-2t}(2^k-1)}.
\end{aligned} \tag{4.34}
$$

Now we are ready to determine $|\Theta_0(i,j)|$ and $|\Theta_{0,0}(i,j)|$.

Lemma 4.3. *For* $0\leqslant i\leqslant\lfloor m/2\rfloor$ *and* $0\leqslant j<m/2$, *we have*

$$
\begin{aligned}
&|\Theta_0(i,j)| \\
&\quad= \sum_{\substack{(t,\sigma,\rho)\\ 2t\leqslant\sigma\leqslant\rho\leqslant\min\{2i,2j\}}}(-1)^{\rho-\sigma}2^{\binom{\rho-\sigma}{2}+2t^2+2it-2\sigma t+2j\sigma-1+(m-2i-2j)\rho+(1/2)(\rho^2-\rho+m^2-m)} \\
&\qquad(1+2^{-2jt})\begin{bmatrix}2i-\sigma\\ \rho-\sigma\end{bmatrix}_2 \\
&\qquad\times\frac{\prod_{k=i-\sigma+t+1}^{i}(2^{2k}-1)\prod_{k=1}^{m-2i}(2^k-1)\prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)}{\prod_{k=1}^{t}(2^{2k}-1)\prod_{k=1}^{\sigma-2t}(2^k-1)},
\end{aligned} \tag{4.35}
$$

$$
|\Theta_{0,0}(i,j)| = \begin{cases}0 & \text{if } 2i=m,\\ \dfrac{2^{m-2i-1}-1}{2^{m-2i}-1}|\Theta_0(i,j)| & \text{if } 2i<m.\end{cases} \tag{4.36}
$$

Proof. Eq. (4.36) has already been proved. (Cf. the remark after the proof of Lemma 4.2.) Eq. (4.35) follows from (4.8) and (4.34). □

5. The eigenmatrix of the linear association scheme on *R*(2, *m*)

Let $f_i, g_i, h_i, F_i, G_i, H_i$ be the AGL$(m,2)$ orbit representatives of $R(2,m)$ and $R(m,m)/R(m-3,m)$ given in (3.1)–(3.3) and (3.15)–(3.17). The eigenmatrix of the linear association scheme on $R(2,m)$ is a matrix with rows labeled by

$$\mathcal{R}=\{F_i\colon 0\leqslant i\leqslant\lfloor m/2\rfloor\}\cup\{G_i\colon 0\leqslant i<m/2\}\cup\{H_i\colon 0\leqslant i\leqslant\lfloor m/2\rfloor\}, \tag{5.1}$$

and columns labeled by

$$\mathcal{C}=\{f_i\colon 0\leqslant i\leqslant\lfloor m/2\rfloor\}\cup\{g_i\colon 0\leqslant i<m/2\}\cup\{h_i\colon 0\leqslant i\leqslant\lfloor m/2\rfloor\}. \tag{5.2}$$

For each $F \in \mathscr{R}$ and $f \in \mathscr{C}$, the (F, f) entry of the eigenmatrix is

$$\sum_{u \in [f]} (-1)^{\langle u,F \rangle} = 2|\mathscr{O}(f,F)| - |[f]|, \tag{5.3}$$

where

$$\mathscr{O}(f,F) = \{u \in [\,f\,]\colon \langle u,F \rangle = 0\}. \tag{5.4}$$

(We remind the reader that $[\,f\,]$ is the $\mathrm{AGL}(m,2)$ orbit of f in $R(2,m)$.)

Lemma 5.1. *We have*

$$|\mathscr{O}(f_i,F_j)| = \frac{1}{|\mathrm{AGL}(m,2)_{f_i}|} 2^m |\Theta(i,j)|, \quad 0 \leqslant i,j \leqslant \lfloor m/2 \rfloor, \tag{5.5}$$

$$|\mathscr{O}(f_i,G_j)| = \frac{1}{|\mathrm{AGL}(m,2)_{f_i}|} \left[2^{-2i+(1/2)(m^2+m)} (2^{2i}-1) \prod_{k=1}^{m-1} (2^k-1) + 2^m |\Theta_0(i,j)| \right], \\ 0 \leqslant i \leqslant \lfloor m/2 \rfloor,\ 0 \leqslant j < m/2, \tag{5.6}$$

$$|\mathscr{O}(f_i,H_j)| = \frac{1}{|\mathrm{AGL}(m,2)_{f_i}|} [2^{m-i-1}(2^i-1)|\mathrm{GL}(m,2)| + 2^{m-i}|\Theta(i,j)|], \\ 0 \leqslant i,j \leqslant \lfloor m/2 \rfloor, \tag{5.7}$$

$$|\mathscr{O}(g_i,F_j)| = \frac{1}{|\mathrm{AGL}(m,2)_{g_i}|} 2^m |\Theta(i,j)|, \quad 0 \leqslant i < m/2,\ 0 \leqslant j \leqslant \lfloor m/2 \rfloor, \tag{5.8}$$

$$|\mathscr{O}(g_i,G_j)| = \frac{1}{|\mathrm{AGL}(m,2)_{g_i}|} [2^{-2i+(1/2)(m^2+m)} (2^{2i}-1) \prod_{k=1}^{m-1} (2^k-1) + 2^m |\Theta_{0,0}(i,j)|], \\ 0 \leqslant i,j < m/2, \tag{5.9}$$

$$|\mathscr{O}(g_i,H_j)| = \frac{1}{|\mathrm{AGL}(m,2)_{g_i}|} 2^{m-1} |\mathrm{GL}(m,2)|, \quad 0 \leqslant i < m/2,\ 0 \leqslant j \leqslant \lfloor m/2 \rfloor, \tag{5.10}$$

$$|\mathscr{O}(h_i,F_j)| = \frac{1}{|\mathrm{AGL}(m,2)_{h_i}|} 2^m |\Theta(i,j)|, \quad 0 \leqslant i,j \leqslant \lfloor m/2 \rfloor, \tag{5.11}$$

$$|\mathscr{O}(h_i,G_j)| = \frac{1}{|\mathrm{AGL}(m,2)_{h_i}|} \left[2^{-2i+(1/2)(m^2+m)} (2^{2i}-1) \prod_{k=1}^{m-1} (2^k-1) + 2^m |\Theta_0(i,j)| \right], \\ 0 \leqslant i \leqslant \lfloor m/2 \rfloor,\ 0 \leqslant j < m/2, \tag{5.12}$$

$$|\mathscr{O}(h_i,H_j)| = \frac{1}{|\mathrm{AGL}(m,2)_{h_i}|} [2^{m-i-1}(2^i+1)|\mathrm{GL}(m,2)| - 2^{m-i}|\Theta(i,j)|], \\ 0 \leqslant i,j \leqslant \lfloor m/2 \rfloor. \tag{5.13}$$

Proof. Let $\alpha=(A,b)\in \mathrm{AGL}(m,2)$, where $A=(a_{kl})\in \mathrm{GL}(m,2)$ and $b=(b_1,\dots,b_m)\in \mathrm{GF}(2)^m$. Then

$$\begin{aligned}\alpha(f_i)=&\left(\sum_{k=1}^{m}X_k a_{k1}+b_1\right)\left(\sum_{k=1}^{m}X_k a_{k2}+b_2\right)+\cdots\\&+\left(\sum_{k=1}^{m}X_k a_{k,2i-1}+b_{2i-1}\right)\left(\sum_{k=1}^{m}X_k a_{k,2i}+b_{2i}\right)\\=&\sum_{0\leqslant k<p\leqslant m}\left(\sum_{l=1}^{i}\begin{vmatrix}a_{k,2l-1}&a_{k,2l}\\a_{p,2l-1}&a_{p,2l}\end{vmatrix}\right)X_kX_p\\&+\sum_{k=1}^{m}\left[\sum_{l=1}^{i}a_{k,2l-1}a_{k,2l}+b_1a_{k2}+b_2a_{k1}+\cdots+b_{2i-1}a_{k,2i}+b_{2i}a_{k,2i-1}\right]X_k\\&+b_1b_2+\cdots+b_{2i-1}b_{2i}.\end{aligned}\tag{5.14}$$

Thus $\langle\alpha(f_i),F_j\rangle=0$ if and only if

$$\sum_{k=1}^{j}\sum_{l=1}^{i}\begin{vmatrix}a_{2k-1,2l-1}&a_{2k-1,2l}\\a_{2k,2l-1}&a_{2k,2l}\end{vmatrix}=0,\tag{5.15}$$

i.e., $A\in\Theta(i,j)$. Therefore

$$|\mathcal{O}(f_i,F_j)|=\frac{1}{|\mathrm{AGL}(m,2)_{f_i}|}2^m|\Theta(i,j)|.\tag{5.16}$$

Also from (5.14), we see that $\langle\alpha(f_i),G_j\rangle=0$ if and only if

$$\begin{aligned}&\sum_{k=1}^{j}\sum_{l=1}^{i}\begin{vmatrix}a_{2k-1,2l-1}&a_{2k-1,2l}\\a_{2k,2l-1}&a_{2k,2l}\end{vmatrix}\\&+\sum_{k=1}^{m-1}\sum_{l=1}^{i}\begin{vmatrix}a_{k,2l-1}&a_{k,2l}\\a_{m,2l-1}&a_{m,2l}\end{vmatrix}+\sum_{l=1}^{i}a_{m,2l-1}a_{m,2l}\\&+b_1a_{m2}+b_2a_{m1}+\cdots+b_{2i-1}a_{m,2i}+b_{2i}a_{m,2i-1}\\&=0.\end{aligned}\tag{5.17}$$

In Eq. (5.17), for each A with $(a_{m1},\dots,a_{m,2i})\neq 0$, the number of choices for b is 2^{m-1}; for each A with $(a_{m1},\dots,a_{m,2i})=0$, (5.17) is satisfied if and only if $A\in\Theta_0(i,j)$. Thus we have

$$\begin{aligned}&|\mathcal{O}(f_i,G_j)|\\&=\frac{1}{|\mathrm{AGL}(m,2)_{f_i}|}[(2^m-2^{m-2i})(2^m-2)\cdots(2^m-2^{m-1})\cdot 2^{m-1}+|\Theta_0(i,j)|\cdot 2^m]\\&=\frac{1}{|\mathrm{AGL}(m,2)_{f_i}|}\left[2^{-2i+(1/2)(m^2+m)}(2^{2i}-1)\prod_{k=1}^{m-1}(2^k-1)+2^m|\Theta_0(i,j)|\right].\end{aligned}\tag{5.18}$$

Again from (5.14), $\langle \alpha(f_i), H_j \rangle = 0$ if and only if

$$\sum_{k=1}^{j}\sum_{l=1}^{i}\begin{vmatrix} a_{2k-1,2l-1} & a_{2k-1,2l} \\ a_{2k,2l-1} & a_{2k,2l}\end{vmatrix} + \left(\sum_{k=1}^{m} a_{k1} + b_1\right)\left(\sum_{k=1}^{m} a_{k2} + b_2\right) + \cdots + \left(\sum_{k=1}^{m} a_{k,2i-1} + b_{2i-1}\right)\left(\sum_{k=1}^{m} a_{k,2i} + b_{2i}\right) = 0. \tag{5.19}$$

In Eq. (5.19), if $A \in \Theta(i,j)$, the number of choices of $(b_1,\ldots,b_{2i})$ is $2^{2i-1}+2^{i-1}$; if $A \notin \Theta(i,j)$, the number of choices of $(b_1,\ldots,b_{2i})$ is $2^{2i-1}-2^{i-1}$. Thus

$$\begin{aligned}|\mathcal{O}(f_i,H_j)| &= \frac{1}{|\mathrm{AGL}(m,2)_{f_i}|}[|\Theta(i,j)|(2^{2i-1}+2^{i-1})2^{m-2i} \\ &\quad + (|\mathrm{GL}(m,2)| - |\Theta(i,j)|)(2^{2i-1}-2^{i-1})2^{m-2i}] \\ &= \frac{1}{|\mathrm{AGL}(m,2)_{f_i}|}[2^{m-i-1}(2^i-1)|\mathrm{GL}(m,2)| + 2^{m-i}|\Theta(i,j)|]. \end{aligned} \tag{5.20}$$

To prove (5.8)–(5.10), observe that for $\alpha = (A,b) \in \mathrm{AGL}(m,2)$, where $A = (a_{kl}) \in \mathrm{GL}(m,2)$, $b = (b_1,\ldots,b_m) \in \mathrm{GF}(2)^m$,

$$\begin{aligned}\alpha(g_i) &= \left(\sum_{k=1}^{m} X_k a_{k1} + b_1\right)\left(\sum_{k=1}^{m} X_k a_{k2} + b_2\right) + \cdots \\ &\quad + \left(\sum_{k=1}^{m} X_k a_{k,2i-1} + b_{2i-1}\right)\left(\sum_{k=1}^{m} X_k a_{k,2i} + b_{2i}\right) + \sum_{k=1}^{m} X_k a_{km} + b_m \\ &= \sum_{0\leqslant k<p\leqslant m}\left(\sum_{l=1}^{i}\begin{vmatrix} a_{k,2l-1} & a_{k,2l} \\ a_{p,2l-1} & a_{p,2l}\end{vmatrix}\right) X_k X_p \\ &\quad + \sum_{k=1}^{m}\left[\sum_{l=1}^{i} a_{k,2l-1}a_{k,2l} + b_1 a_{k2} + b_2 a_{k1} + \cdots + b_{2i-1}a_{k,2i} \right. \\ &\quad \left. + b_{2i}a_{k,2i-1} + a_{km}\right] X_k + b_1 b_2 + \cdots + b_{2i-1}b_{2i} + b_m. \end{aligned} \tag{5.21}$$

Therefore, $\langle \alpha(g_i), F_j \rangle = 0$ if and only if $A \in \Theta(i,j)$ and (5.8) follows. Eq. (5.21) also shows that $\langle \alpha(g_i), G_j \rangle = 0$ if and only if

$$\begin{aligned}&\sum_{k=1}^{j}\sum_{l=1}^{i}\begin{vmatrix} a_{2k-1,2l-1} & a_{2k-1,2l} \\ a_{2k,2l-1} & a_{2k,2l}\end{vmatrix} + \sum_{k=1}^{m-1}\sum_{l=1}^{i}\begin{vmatrix} a_{k,2l-1} & a_{k,2l} \\ a_{m,2l-1} & a_{m,2l}\end{vmatrix} + \sum_{l=1}^{i} a_{m,2l-1}a_{m,2l} \\ &\quad + b_1 a_{m2} + b_2 a_{m1} + \cdots + b_{2i-1}a_{m,2i} + b_{2i}a_{m,2i-1} + a_{mm} \\ &\quad = 0. \end{aligned} \tag{5.22}$$

In Eq. (5.22), for each A with $(a_{m1},\dots,a_{m,2i})\neq 0$, the number of choices for b is 2^{m-1}; for each A with $(a_{m1},\dots,a_{m,2i})=0$, (5.22) is satisfied if and only if $A\in\Theta_{0,0}(i,j)$. Thus we have

$$
\begin{aligned}
&|\mathcal{O}(g_i,G_j)|\\
&=\frac{1}{|\mathrm{AGL}(m,2)_{g_i}|}[(2^m-2^{m-2i})(2^m-2)\cdots(2^m-2^{m-1})2^{m-1}+|\Theta_{0,0}(i,j)|2^m]\\
&=\frac{1}{|\mathrm{AGL}(m,2)_{g_i}|}\left[2^{-2i+(1/2)(m^2+m)}(2^{2i}-1)\prod_{k=1}^{m-1}(2^k-1)+2^m|\Theta_{0,0}(i,j)|\right].
\end{aligned}
\tag{5.23}
$$

Again by (5.21), $\langle\alpha(g_i),H_j\rangle=0$ if and only if

$$
\begin{aligned}
&\sum_{k=1}^{j}\sum_{l=1}^{i}\begin{vmatrix}a_{2k-1,2l-1} & a_{2k-1,2l}\\ a_{2k,2l-1} & a_{2k,2l}\end{vmatrix}+\left(\sum_{k=1}^{m}a_{k1}+b_1\right)\left(\sum_{k=1}^{m}a_{k2}+b_2\right)\\
&\quad+\cdots+\left(\sum_{k=1}^{m}a_{k,2i-1}+b_{2i-1}\right)\left(\sum_{k=1}^{m}a_{k,2i}+b_{2i}\right)+\sum_{k=1}^{m}a_{km}+b_m\\
&\quad=0.
\end{aligned}
\tag{5.24}
$$

Note that $2i<m$. Thus for every A and $(b_1,\dots,b_{m-1})$, Eq. (5.24) determines a unique b_m. Therefore

$$|\mathcal{O}(g_i,G_j)|=\frac{1}{|\mathrm{AGL}(m,2)_{g_i}|}2^{m-1}|\mathrm{GL}(m,2)|. \tag{5.25}$$

Eqs. (5.11)–(5.13) are proved in the same manner; we omit the details. □

Using Eqs. (5.5)–(5.13), (4.4), (4.35), (4.36) and (3.4)–(3.6) in (5.3), we can obtain explicit formulas for the entries of the eigenmatrix of the linear association scheme on $R(m,2)$. We omit the tedious computations and record the results in the following theorem.

Theorem 5.2. *We have*

$$
\begin{aligned}
&\sum_{u\in[f_i]}(-1)^{\langle u,F_j\rangle}\\
&=2^{2i}\sum_{k=0}^{i}(-1)^{i-k}\begin{bmatrix}\lfloor m/2\rfloor-k\\ \lfloor m/2\rfloor-i\end{bmatrix}_4\begin{bmatrix}\lfloor m/2\rfloor-j\\ k\end{bmatrix}_4 4^{\binom{i-k}{2}}2^{k(2\lceil m/2\rceil-1)},\\
&0\leqslant i,j\leqslant\lfloor m/2\rfloor,
\end{aligned}
\tag{5.26}
$$

$$\sum_{u\in[f_i]}(-1)^{\langle u,G_j\rangle}$$

$$=\frac{2^{i^2+i}(3-2^{-2i+1}-2^m)\prod_{k=m-2i+1}^{m}(2^k-1)}{(2^m-1)\prod_{k=1}^{i}(2^{2k}-1)}$$

$$+\,2^{i^2+i}\sum_{\substack{(t,\sigma,\rho)\\ 2t\leqslant\sigma\leqslant\rho\leqslant\min\{2i,2j\}}}(-1)^{\rho-\sigma}2^{\binom{\rho-\sigma}{2}+2t^2+2it-2\sigma t+2j\sigma+(m-2i-2j)\rho+(1/2)(\rho^2-\rho)}$$

$$\times(1+2^{-2jt})\begin{bmatrix}2i-\sigma\\ \rho-\sigma\end{bmatrix}_2\frac{\prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)}{\prod_{k=1}^{i-\sigma+t}(2^{2k}-1)\prod_{k=1}^{t}(2^{2k}-1)\prod_{k=1}^{\sigma-2t}(2^k-1)},$$

$$0\leqslant i\leqslant\lfloor m/2\rfloor,\ \ 0\leqslant j<m/2. \tag{5.27}$$

$$\sum_{u\in[f_i]}(-1)^{\langle u,H_j\rangle}=2^i\sum_{k=0}^{i}(-1)^{i-k}\begin{bmatrix}\lfloor m/2\rfloor-k\\ \lfloor m/2\rfloor-i\end{bmatrix}_4\begin{bmatrix}\lfloor m/2\rfloor-j\\ k\end{bmatrix}_4 4^{\binom{i-k}{2}}2^{k(2\lceil m/2\rceil-1)},$$

$$0\leqslant i,j\leqslant\lfloor m/2\rfloor, \tag{5.28}$$

$$\sum_{u\in[g_i]}(-1)^{\langle u,F_j\rangle}$$

$$=2^{2i+1}(2^{m-2i}-1)\sum_{k=0}^{i}(-1)^{i-k}\begin{bmatrix}\lfloor m/2\rfloor-k\\ \lfloor m/2\rfloor-i\end{bmatrix}_4\begin{bmatrix}\lfloor m/2\rfloor-j\\ k\end{bmatrix}_4 4^{\binom{i-k}{2}}2^{k(2\lceil m/2\rceil-1)},$$

$$0\leqslant i<m/2,\ \ 0\leqslant j\leqslant\lfloor m/2\rfloor, \tag{5.29}$$

$$\sum_{u\in[g_i]}(-1)^{\langle u,G_j\rangle}$$

$$=2^{i^2+i+1}(3-2^{-2i+1}-2^m)\frac{\prod_{k=m-2i}^{m-1}(2^k-1)}{\prod_{k=1}^{i}(2^{2k}-1)}$$

$$+\,2^{i^2+i+1}(2^{m-2i-1}-1)$$

$$\times\sum_{\substack{(t,\sigma,\rho)\\ 2t\leqslant\sigma\leqslant\rho\leqslant\min\{2i,2j\}}}(-1)^{\rho-\sigma}2^{\binom{\rho-\sigma}{2}+2t^2+2it-2\sigma t+2j\sigma+(m-2i-2j)\rho+(1/2)(\rho^2-\rho)}$$

$$\times(1+2^{-2jt})\begin{bmatrix}2i-\sigma\\ \rho-\sigma\end{bmatrix}_2\frac{\prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)}{\prod_{k=1}^{i-\sigma+t}(2^{2k}-1)\prod_{k=1}^{t}(2^{2k}-1)\prod_{k=1}^{\sigma-2t}(2^k-1)},$$

$$0\leqslant i,j<m/2, \tag{5.30}$$

$$\sum_{u\in[g_i]}(-1)^{\langle u,H_j\rangle}=0,\quad 0\leqslant i<m/2,\ 0\leqslant j\leqslant\lfloor m/2\rfloor, \tag{5.31}$$

$$\sum_{u\in[h_i]}(-1)^{\langle u,F_j\rangle}=2^{2i}\sum_{k=0}^{i}(-1)^{i-k}\begin{bmatrix}\lfloor m/2\rfloor-k\\ \lfloor m/2\rfloor-i\end{bmatrix}_4\begin{bmatrix}\lfloor m/2\rfloor-j\\ k\end{bmatrix}_4 4^{\binom{i-k}{2}}2^{k(2\lceil m/2\rceil-1)},$$
$$0\leqslant i,j\leqslant\lfloor m/2\rfloor, \tag{5.32}$$

$$\sum_{u\in[h_i]}(-1)^{\langle u,G_j\rangle}$$
$$=\frac{2^{i^2+i}(3-2^{-2i+1}-2^m)\prod_{k=m-2i+1}^{m}(2^k-1)}{(2^m-1)\prod_{k=1}^{i}(2^{2k}-1)}$$
$$+2^{i^2+i}\sum_{\substack{(t,\sigma,\rho)\\ 2t\leqslant\sigma\leqslant\rho\leqslant\min\{2i,2j\}}}(-1)^{\rho-\sigma}2^{\binom{\rho-\sigma}{2}+2t^2+2it-2\sigma t+2j\sigma+(m-2i-2j)\rho+(1/2)(\rho^2-\rho)}$$
$$\times(1+2^{-2jt})\begin{bmatrix}2i-\sigma\\ \rho-\sigma\end{bmatrix}_2\frac{\prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)}{\prod_{k=1}^{i-\sigma+t}(2^{2k}-1)\prod_{k=1}^{t}(2^{2k}-1)\prod_{k=1}^{\sigma-2t}(2^k-1)},$$
$$0\leqslant i\leqslant\lfloor m/2\rfloor,\ 0\leqslant j<m/2, \tag{5.33}$$

$$\sum_{u\in[h_i]}(-1)^{\langle u,H_j\rangle}=-2^{i}\sum_{k=0}^{i}(-1)^{i-k}\begin{bmatrix}\lfloor m/2\rfloor-k\\ \lfloor m/2\rfloor-i\end{bmatrix}_4\begin{bmatrix}\lfloor m/2\rfloor-j\\ k\end{bmatrix}_4 4^{\binom{i-k}{2}}2^{k(2\lceil m/2\rceil-1)},$$
$$0\leqslant i,j\leqslant\lfloor m/2\rfloor. \tag{5.34}$$

6. The weight enumerators of the cosets of $R(m-3,m)$

Using Lemma 5.1, we can determine the weight enumerators of cosets of $R(m-3,m)$. In fact for each

$$F\in\mathscr{C}=\{F_i\colon 0\leqslant i\leqslant\lfloor m/2\rfloor\}\cup\{G_i\colon 0\leqslant i<m/2\}\cup\{H_i\colon 0\leqslant i\leqslant\lfloor m/2\rfloor\}, \tag{6.1}$$

by the MacWilliams identity, we have

$$\mathscr{W}_{F+R(m-3,m)}(x,y)+\mathscr{W}_{R(m-3,m)}(x,y)=\frac{2}{|R(2,m)|}\mathscr{W}_{\langle F,R(m-3,m)\rangle^\perp}(x+y,x-y), \tag{6.2}$$

or, equivalently,

$$\mathscr{W}_{F+R(m-3,m)}(x,y)=-\frac{1}{|R(2,m)|}\mathscr{W}_{R(2,m)}(x+y,x-y)$$
$$+\frac{2}{|R(2,m)|}\mathscr{W}_{\langle F,R(m-3,m)\rangle^\perp}(x+y,x-y). \tag{6.3}$$

In Eq. (6.3),

$$\mathscr{W}_{R(2,m)}(x,y)=\sum_{i=0}^{\lfloor m/2\rfloor}2^{i(i+1)}\frac{\prod_{k=m-2i+1}^{m}(2^k-1)}{\prod_{k=1}^{i}(2^{2k}-1)}(xy)^{2^{m-1}-2^{m-i-1}}(x^{2^{m-i}}+y^{2^{m-i}})$$
$$+\left[2^{\binom{m}{2}+m+1}-2\sum_{i=0}^{\lfloor m/2\rfloor}2^{i(i+1)}\frac{\prod_{k=m-2i+1}^{m}(2^k-1)}{\prod_{k=1}^{i}(2^{2k}-1)}\right](xy)^{2^{m-1}}. \tag{6.4}$$

(See [12] or [15].) As for the other term in the right-hand side of (6.3), we have

$$\begin{aligned}\mathscr{W}_{\langle F,R(m-3,m)\rangle^{\perp}}(x,y)&=\sum_{i\leqslant\lfloor m/2\rfloor}|\mathscr{O}(f_i,F)|x^{2^m-|f_i|}y^{|f_i|}\\&\quad+\sum_{i<m/2}|\mathscr{O}(g_i,F)|x^{2^m-|g_i|}y^{|g_i|}\\&\quad+\sum_{i\leqslant\lfloor m/2\rfloor}|\mathscr{O}(h_i,F)|x^{2^m-|h_i|}y^{|h_i|}\\&=\sum_{i\leqslant\lfloor m/2\rfloor}|\mathscr{O}(f_i,F)|x^{2^{m-1}+2^{m-1-i}}y^{2^{m-1}-2^{m-1-i}}\\&\quad+\sum_{i\leqslant\lfloor m/2\rfloor}|\mathscr{O}(h_i,F)|x^{2^{m-1}-2^{m-1-i}}y^{2^{m-1}+2^{m-1-i}}\\&\quad+\left(2^{m+\binom{m}{2}}-\sum_{i\leqslant\lfloor m/2\rfloor}(|\mathscr{O}(f_i,F)|+|\mathscr{O}(h_i,F)|)\right)(xy)^{2^{m-1}},\end{aligned} \tag{6.5}$$

where $|\mathscr{O}(f_i,F)|$ and $|\mathscr{O}(h_i,F)|$ have been determined in Lemma 5.1. Putting (6.4) and (6.5) together in (6.3) and using Lemma 5.1, we then have an explicit formula for $\mathscr{W}_{F+R(m-3,m)}(x,y)$. We record the results below but omit the details of the computations.

Theorem 6.1. *We have*

$$\mathscr{W}_{F_j+R(m-3,m)}(x,y)=2^{-\binom{m}{2}-m-1}\sum_{i\leqslant\lfloor m/2\rfloor}$$
$$\times\left(\sum_{k=0}^{i}(-1)^{i-k}4^{\binom{i-k}{2}}2^{k(2\lceil m/2\rceil-1)+2i}\begin{bmatrix}\lfloor m/2\rfloor-k\\ \lfloor m/2\rfloor-i\end{bmatrix}_4\begin{bmatrix}\lfloor m/2\rfloor-j\\ k\end{bmatrix}_4\right)$$
$$\times(x^2-y^2)^{2^{m-1}-2^{m-1-i}}\left[(x+y)^{2^{m-i}}+(x-y)^{2^{m-i}}\right]$$

$$- 2^{-\binom{m}{2}-m}\left(\sum_{i\leqslant\lfloor m/2\rfloor-j}(-1)^{\lfloor m/2\rfloor-i}2^{i(2\lceil m/2\rceil+1)}\begin{bmatrix}\lfloor m/2\rfloor-j\\ i\end{bmatrix}_4\prod_{k=1}^{\lfloor m/2\rfloor-i}(2^{2k}-1)\right)$$
$$\times (x^2-y^2)^{2^{m-1}},$$
$$0\leqslant j\leqslant\lfloor m/2\rfloor, \tag{6.6}$$

$$\mathscr{W}_{G_j+R(m-3,m)}(x,y)$$
$$=2^{-\binom{m}{2}-m-1}\sum_{i\leqslant\lfloor m/2\rfloor}\left[\frac{2^{i^2+i}(3-2^{-2i+1}-2^m)\prod_{k=m-2i+1}^{m}(2^k-1)}{(2^m-1)\prod_{k=1}^{i}(2^{2k}-1)}\right.$$
$$+2^{i^2+i}\sum_{\substack{(t,\sigma,\rho)\\ 2t\leqslant\sigma\leqslant\rho\leqslant\min\{2i,2j\}}}(-1)^{\rho-\sigma}2^{\binom{\rho-\sigma}{2}+2t^2+2it-2\sigma t+2j\sigma+(m-2i-2j)\rho+(1/2)(\rho^2-\rho)}$$
$$\left.\times(1+2^{-2jt})\begin{bmatrix}2i-\sigma\\ \rho-\sigma\end{bmatrix}_2\frac{\prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)}{\prod_{k=1}^{i-\sigma+t}(2^{2k}-1)\prod_{k=1}^{t}(2^{2k}-1)\prod_{k=1}^{\sigma-2t}(2^k-1)}\right]$$
$$\times(x^2-y^2)^{2^{m-1}-2^{m-1-i}}[(x+y)^{2^{m-i}}+(x-y)^{2^{m-i}}]$$
$$+\left\{\frac{2^{-m+1}}{2^m-1}-2^{-\binom{m}{2}-m}\sum_{i\leqslant\lfloor m/2\rfloor}\left[\frac{2^{i^2+i}(3-2^m)\prod_{k=m-2i+1}^{m}(2^k-1)}{(2^m-1)\prod_{k=1}^{i}(2^{2k}-1)}\right.\right.$$
$$+2^{i^2+i}\sum_{\substack{(t,\sigma,\rho)\\ 2t\leqslant\sigma\leqslant\rho\leqslant\min\{2i,2j\}}}(-1)^{\rho-\sigma}2^{\binom{\rho-\sigma}{2}+2t^2+2it-2\sigma t+2j\sigma+(m-2i-2j)\rho+(1/2)(\rho^2-\rho)}$$
$$\left.\left.\times(1+2^{-2jt})\begin{bmatrix}2i-\sigma\\ \rho-\sigma\end{bmatrix}_2\frac{\prod_{k=m+\rho-2i-2j}^{m-2j-1}(2^k-1)}{\prod_{k=1}^{i-\sigma+t}(2^{2k}-1)\prod_{k=1}^{t}(2^{2k}-1)\prod_{k=1}^{\sigma-2t}(2^k-1)}\right]\right\}$$
$$\times(x^2-y^2)^{2^{m-1}},$$
$$0\leqslant j<m/2, \tag{6.7}$$

$$\mathscr{W}_{H_j+R(m-3,m)}(x,y)$$
$$=2^{-\binom{m}{2}-m-1}\sum_{i\leqslant\lfloor m/2\rfloor}\left(\sum_{k=0}^{i}(-1)^{i-k}4^{\binom{i-k}{2}}2^{k(2\lceil m/2\rceil-1)+i}\right.$$
$$\left.\begin{bmatrix}\lfloor m/2\rfloor-k\\ \lfloor m/2\rfloor-i\end{bmatrix}_4\begin{bmatrix}\lfloor m/2\rfloor-j\\ k\end{bmatrix}_4\right)$$
$$\times(x^2-y^2)^{2^{m-1}-2^{m-1-i}}[(x+y)^{2^{m-i}}-(x-y)^{2^{m-i}}],\quad 0\leqslant j\leqslant\lfloor m/2\rfloor. \tag{6.8}$$

For readers who would like to verify the formulas in Theorem 6.1, we remark that when deriving Eq. (6.6), we used the formula

$$\sum_{k=0}^{n}(-1)^{n-k}b^{\binom{n-k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_b x^k=(x-1)(x-b)\cdots(x-b^{n-1}) \tag{6.9}$$

(see [6]) to simplify part of the equation. When deriving Eq. (6.7), we used the formula

$$\sum_{i\leqslant\lfloor m/2\rfloor} 2^{i^2-i}\frac{\prod_{k=m-2i+1}^{m}(2^k-1)}{\prod_{k=1}^{i}(2^{2k}-1)}=\sum_{i\leqslant\lfloor m/2\rfloor}\frac{|\mathrm{GL}(m,2)|}{|S(K(m,i))|}=2^{\binom{m}{2}} \tag{6.10}$$

also for the purpose of simplification. Formula (6.10) is the class equation of $R(2,m)/R(1,m)$ under the action of $\mathrm{GL}(m,2)$.

7. Uncited references

[5,9,10]

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