Applied Mathematics Letters 24 (2011) 777–783

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A geometric perspective on the generalized Cobb–Douglas production functions

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ARTICLE INFO

Article history:
Received 24 March 2010
Received in revised form 23 December 2010
Accepted 28 December 2010

Dedicated to the memory of Professor Stere Ianuş (1939–2010)

Keywords:
Production function
Return to scale
Cobb–Douglas
Gaussian curvature
Developable hypersurface

ABSTRACT

In this work we obtain an interesting link between some fundamental notions in the theory of production functions and the differential geometry of hypersurfaces in Euclidean spaces. In fact we establish that a generalized Cobb–Douglas production function has decreasing/increasing return to scale if and only if the corresponding hypersurface has positive/negative Gaussian curvature. Moreover, this production function has constant return to scale if and only if the corresponding hypersurface is developable.

1. Introduction and preliminaries on production functions

The production functions are a mathematical formalization of the relationship between the output of a firm/industry/economy and the inputs that have been used in obtaining it. Therefore, a production function is a map Q of class C^∞, $Q : \mathbb{R}^n_+ \to \mathbb{R}_+$, $Q = Q(x_1, x_2, \ldots, x_n)$, where Q is the quantity of output, n is the number of the inputs and $x_1, x_2, \ldots, x_n$ are the factor inputs (such as labor, capital, land, raw materials etc.). In order for these functions to also model economic reality, they are required to have certain properties (see e.g. [1,2]). We recall now some of them with appropriate economic interpretations:

1. Q vanishes in the absence of an input; this means that factor inputs are necessary.
2. $\frac{\partial Q}{\partial x_i} > 0$, for all $i \in \{1, \ldots, n\}$, which indicates that the production function is strictly increasing with respect to any factor of production.
3. $\frac{\partial^2 Q}{\partial^2 x_i} > 0$, for all $i \in \{1, \ldots, n\}$, which means that the production has decreasing efficiency with respect to any factor of production.
4. $Q(x + y) \geq Q(x) + Q(y)$, $\forall x, y \in \mathbb{R}^n_+$, which signifies that the production has non-decreasing global efficiency.
5. Q is a homogeneous function, i.e. there exists a real number p such that $Q(\lambda \cdot x) = \lambda^p Q(x)$ for all $x \in \mathbb{R}^n_+$ and $\lambda \in \mathbb{R}$, which means that if the inputs are multiplied by same factor, then the output is multiplied by some power of this factor.

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doi:10.1016/j.aml.2010.12.038

If $\lambda = 1$ then the function is said to have a constant return to scale, if $\lambda > 1$ then we have an increased return to scale and if $\lambda < 1$ then we say that the function has a decreased return to scale.

We can easily see that the production function Q can be identified with a hypersurface of the $(n+1)$-dimensional Euclidean space through the map $f : \mathbb{R}^n_+ \to \mathbb{R}^{n+1}_+, f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, Q(x_1, \ldots, x_n))$. Indeed, the Jacobian matrix

$$J_f = \begin{pmatrix} 1 & 0 & \ldots & 0 & \frac{\partial Q}{\partial x_1} \\ 0 & 1 & \ldots & 0 & \frac{\partial Q}{\partial x_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & \frac{\partial Q}{\partial x_n} \end{pmatrix}$$

has rank n and therefore f is an immersion and defines a hypersurface (see e.g. [3]). We remark that in the particular case of two inputs, we have a surface, and an analysis of the Cobb–Douglas, CES and Sato production functions from the point of view of space differential geometry has been performed in [4,5].

In the analysis of production functions, a fundamental role is played by isoquants. Roughly speaking, they represent the sets of all possible combinations of inputs $x \in \mathbb{R}^n_+$ that result in the production of a given level of output Q^* (see e.g. [6,7]). If we have only two inputs, the isoquants are curves on a surface and in the general case they are $(n-1)$-dimensional submanifolds of the corresponding hypersurface of the production function.

It is well known that the classical treatment of the production functions makes use of the projections of production functions on a plane. Unfortunately, this approach leads to limited conclusions and a differential geometric treatment is more useful. Next, we will study the generalized Cobb–Douglas production function with n inputs, using the powerful tool of the differential geometry of hypersurfaces in Euclidean spaces.

2. Some basic concepts in the differential geometry of hypersurfaces

Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^{n+1}$ be a hypersurface. We denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^{n+1}$ and let $\| \cdot \|$ be the induced norm. In the theory of hypersurfaces there are two fundamental forms which are essential in the study of the geometry of hypersurfaces. The first fundamental form is given by

$$g = \sum_{i=1}^{n} g_{ii} \mathrm{d}x_i^2 + 2 \sum_{1 \le i < j \le n} g_{ij} \mathrm{d}x_i \mathrm{d}x_j,$$

where

$$g_{ii} = \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_i} \right\rangle, \quad i \in \{1, \ldots, n\} \tag{1}$$

and

$$g_{ij} = \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle, \quad 1 \le i < j \le n. \tag{2}$$

The second fundamental form is given by

$$h = \sum_{i=1}^{n} h_{ii} \mathrm{d}x_i^2 + 2 \sum_{1 \le i < j \le n} h_{ij} \mathrm{d}x_i \mathrm{d}x_j,$$

where

$$h_{ii} = \left\langle N, \frac{\partial^2 f}{\partial x_i^2} \right\rangle, \quad i \in \{1, \ldots, n\}, \tag{3}$$

$$h_{ij} = \left\langle N, \frac{\partial^2 f}{\partial x_i \partial x_j} \right\rangle, \quad 1 \le i < j \le n \tag{4}$$

and

$$N = \frac{\frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n}}{\left\| \frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n} \right\|} \tag{5}$$

is the so-called Gauss map of the hypersurface. We remark that the symmetric matrix of the first fundamental form is usually denoted by the Roman numeral $I = (g_{ij})_{i,j}$, while the symmetric matrix of the second fundamental form is denoted by $II = (h_{ij})_{i,j}$.

The Gaussian curvature of a point x on a hypersurface is defined by

$$K(x) = \frac{\det[II](x)}{\det[I](x)}. \tag{6}$$

A hypersurface having zero Gaussian curvature is said to be developable. In this case the hypersurface can be flattened onto a hyperplane without distortion and one can realize a "good" analysis of isoquants by projections, without losing essential information about their geometry. We remark that cylinders and cones are examples of developable surfaces, but the spheres are not under any metric (see [8]).

3. Generalized Cobb–Douglas production functions and their corresponding hypersurfaces

The Cobb–Douglas production function with n inputs, also called the generalized Cobb–Douglas production function, is given by $Q : \mathbb{R}^n_+ \to \mathbb{R}_+$,

$$Q(x_1, \ldots, x_n) = A \prod_{i=1}^{n} x_i^{\alpha_i}, \tag{7}$$

where $A > 0$ and $\alpha_i > 0$, for all $i \in \{1, \ldots, n\}$. The importance and advantages of the Cobb–Douglas production function in its generalized form were very well outlined in [9].

This function being homogeneous of degree $\varepsilon = \sum_{i=1}^{n} \alpha_i$, it is clear that the elasticity of scale is ε. It is also obvious that the geometric representation of the generalized Cobb–Douglas production function is given by the hypersurface $f : \mathbb{R}^n_+ \to \mathbb{R}^{n+1}_+$,

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, Q(x_1, \ldots, x_n)), \tag{8}$$

which we call the Cobb–Douglas hypersurface.

Next, we obtain necessary and sufficient conditions for the Cobb–Douglas hypersurface to be developable or to have positive/negative Gaussian curvature, using the elasticity of scale for the generalized Cobb–Douglas production function.

Theorem 3.1. *If (CDPF) denotes the generalized Cobb–Douglas production function and (CDH) denotes the Cobb–Douglas hypersurface, then we have the following.*

i. *(CDPF) has constant return to scale if and only if (CDH) is developable.*
ii. *(CDPF) has decreasing return to scale if and only if (CDH) has positive Gaussian curvature.*
iii. *(CDPF) has increasing return to scale if and only if (CDH) has negative Gaussian curvature.*

Proof. From (7) and (8) we have for all $i, j \in \{1, \ldots, n\}$, $i \neq j$,

$$\frac{\partial f}{\partial x_i} = \left(0, \ldots, 0, \underbrace{1}_{\substack{\uparrow \\ i}}, 0, \ldots, 0, \frac{\alpha_i}{x_i} Q(x_1, \ldots, x_n) \right), \tag{9}$$

$$\frac{\partial^2 f}{\partial x_i^2} = \left(0, \ldots, 0, \frac{\alpha_i(\alpha_i - 1)}{x_i^2} Q(x_1, \ldots, x_n) \right) \tag{10}$$

and

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \left(0, \ldots, 0, \frac{\alpha_i \alpha_j}{x_i x_j} Q(x_1, \ldots, x_n) \right). \tag{11}$$

Using (9) in (1) and (2) we obtain

$$g_{ii} = 1 + \frac{\alpha_i^2}{x_i^2} Q^2(x_1, \ldots, x_n) \tag{12}$$

and

$$g_{ij} = \frac{\alpha_i \alpha_j}{x_i x_j} Q^2(x_1, \ldots, x_n) \tag{13}$$

for all $i, j \in \{1, \ldots, n\}$, $i \neq j$.

On the other hand, from (5) and (9), after some long but straightforward computation, we obtain that the Gauss map of the hypersurface is

$$N = \frac{(-1)^{n+1}}{\left[1 + Q^2(x_1, \ldots, x_n) \sum_{i=1}^{n} \left(\frac{\alpha_i}{x_i}\right)^2\right]^{\frac{1}{2}}} \left(\frac{\alpha_1 Q(x_1, \ldots, x_n)}{x_1}, \ldots, \frac{\alpha_n Q(x_1, \ldots, x_n)}{x_n}, -1\right).$$

Substituting now the above expression and (10) and (11) into (3) and (4) we obtain

$$h_{ii} = \frac{(-1)^{n+2}\alpha_i(\alpha_i - 1)Q(x_1, \ldots, x_n)}{x_i^2 \left[1 + Q^2(x_1, \ldots, x_n) \sum_{i=1}^{n} \left(\frac{\alpha_i}{x_i}\right)^2\right]^{\frac{1}{2}}} \tag{14}$$

and

$$h_{ij} = \frac{(-1)^{n+2}\alpha_i\alpha_j Q(x_1, \ldots, x_n)}{x_i x_j \left[1 + Q^2(x_1, \ldots, x_n) \sum_{i=1}^{n} \left(\frac{\alpha_i}{x_i}\right)^2\right]^{\frac{1}{2}}}, \tag{15}$$

for all $i, j \in \{1, \ldots, n\}$, $i \neq j$.

From (12) and (13) we deduce

$$\det[I] = \det[(g_{ij})_{i,j}] = 1 + Q^2(x_1, \ldots, x_n) \sum_{i=1}^{n} \left(\frac{\alpha_i}{x_i}\right)^2 \tag{16}$$

and from (14) and (15) we obtain

$$\det[II] = \det[(h_{ij})_{i,j}] = \frac{\left(1 - \sum_{i=1}^{n} \alpha_i\right) Q^n(x_1, \ldots, x_n) \prod_{i=1}^{n} \alpha_i}{\left[1 + Q^2(x_1, \ldots, x_n) \sum_{i=1}^{n} \left(\frac{\alpha_i}{x_i}\right)^2\right]^{\frac{n}{2}} \prod_{i=1}^{n} x_i^2}. \tag{17}$$

Using now (16) and (17) in (6) we derive

$$K(x) = \frac{\left(1 - \sum_{i=1}^{n} \alpha_i\right) Q^n(x_1, \ldots, x_n) \prod_{i=1}^{n} \alpha_i}{\left[1 + Q^2(x_1, \ldots, x_n) \sum_{i=1}^{n} \left(\frac{\alpha_i}{x_i}\right)^2\right]^{\frac{n}{2}+1} \prod_{i=1}^{n} x_i^2}. \tag{18}$$

Therefore, we have that (*CDPF*) has constant return to scale if and only if $\varepsilon = \sum_{i=1}^{n} \alpha_i = 1$ and from (18) we deduce the equivalence with the vanishing of Gauss curvature, i.e. (*CDH*) is developable. Similarly we conclude from (18) that (*CDPF*) has decreasing/increasing return to scale if and only if (*CDH*) has positive/negative Gaussian curvature. □

4. Cobb–Douglas and Tzitzéica hypersurfaces

In 1907 the Romanian geometer Gheorghe Ţiţeica, who has published most of his papers under the name Georges Tzitzéica (see [10]), discovered a class of surfaces of the Euclidean 3-space, nowadays called Tzitzéica surfaces, having the property that the ratio between the Gaussian curvature of the surface at a point P and the fourth power of the distance from a fixed point O to the tangent plane at P is constant [11]. This class of surfaces of great interest, having important applications both in mathematics and in physics (see e.g. [12–17]), has been generalized to higher dimensions as follows. A Tzitzéica hypersurface is a hypersurface satisfying

$$K(x) = k \cdot d^{n+2}(x), \tag{19}$$

where $K(x)$ denotes the Gaussian curvature at a point x, $d(x)$ is the distance between the origin and the hyperplane tangent to the hypersurface at x and k is a real constant. It is known that the simplest Tzitzéica hypersurface is that described by the equation $x_1 \cdot \ldots \cdot x_{n+1} = 1$. Next, we investigate whether the Cobb–Douglas hypersurfaces can lead to examples of Tzitzéica hypersurfaces.

Since the hyperplane tangent to a hypersurface defined through a map $f : U \to \mathbb{R}^{n+1}$, $f = f(x_1, \ldots, x_n)$, U being an open subset of $\mathbb{R}^n$, contains the vectors $\frac{\partial f}{\partial x_i}$, for all $i \in \{1, \ldots, n\}$, it is clear from (9) that the hyperplane tangent to a Cobb–Douglas hypersurface at a point $x = (x_1, \dots, x_n)$ has the equation

$$(H_{tg}) : \det \begin{bmatrix} X^1 - x_1 & X^2 - x_2 & \dots & X^n - x_n & X^{n+1} - Q(x_1, \dots, x_n) \\ 1 & 0 & \dots & 0 & \frac{\alpha_1}{x_1} Q(x_1, \dots, x_n) \\ 0 & 1 & \dots & 0 & \frac{\alpha_2}{x_2} Q(x_1, \dots, x_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \frac{\alpha_n}{x_n} Q(x_1, \dots, x_n) \end{bmatrix} = 0$$

from which we derive

$$(X^1 - x_1)\frac{\alpha_1 Q(x_1, \dots, x_n)}{x_1} + \dots + (X^n - x_n)\frac{\alpha_n Q(x_1, \dots, x_n)}{x_n} - [X^{n+1} - Q(x_1, \dots, x_n)] = 0.$$

Therefore, applying a classical formula in $(n+1)$-dimensional Euclidean geometry, we deduce that the distance between the origin $O = (0, \dots, 0)$ and the hyperplane tangent to the hypersurface at $x = (x_1, \dots, x_n)$ is

$$d(x) = \frac{Q(x_1, \dots, x_n)\left|1 - \sum_{i=1}^{n} \alpha_i\right|}{\left[1 + Q^2(x_1, \dots, x_n)\sum_{i=1}^{n}\left(\frac{\alpha_i}{x_i}\right)^2\right]^{\frac{1}{2}}}. \quad (20)$$

We are able now to prove the following result.

Lemma 4.1. *The equation $x_{n+1} = A\prod_{i=1}^{n} x_i^{\alpha_i}$, where $A > 0$, defines a Tzitzéica hypersurface if and only if $\alpha_1 = \dots = \alpha_n = -1$ or $\sum_{i=1}^{n} \alpha_i = 1$.*

Proof. We consider the hypersurface $(\mathcal{H})$ defined by $x_{n+1} = A\prod_{i=1}^{n} x_i^{\alpha_i}$, where $A > 0$. One can distinguish two cases.

Case I. If $\sum_{i=1}^{n} \alpha_i = 1$, then we have from (18) that the Gaussian curvature of the hypersurface at a point x is $K(x) = 0$ and from (20) we deduce the distance between the origin O and the hyperplane tangent to the hypersurface at x is $d(x) = 0$. Then the conclusion is obvious from (19).

Case II. If $\sum_{i=1}^{n} \alpha_i \neq 1$, then we denote by ϵ the signum of $(1 - \sum_{i=1}^{n} \alpha_i)$ and from (18) and (20) we obtain

$$\frac{K(x)}{d^{n+2}(x)} = \frac{\prod_{i=1}^{n} \alpha_i}{Q^2(x_1, \dots, x_n)\epsilon^{n+2}\left(1 - \sum_{i=1}^{n} \alpha_i\right)^{n+1}\prod_{i=1}^{n} x_i^2}. \quad (21)$$

Using now (7) in (21) we derive

$$\frac{K(x)}{d^{n+2}(x)} = \frac{\prod_{i=1}^{n} \alpha_i}{A^2\epsilon^{n+2}\left(1 - \sum_{i=1}^{n} \alpha_i\right)^{n+1}\prod_{i=1}^{n} x_i^{2\alpha_i+2}}. \quad (22)$$

From (22) we deduce that the Gaussian curvature $K(x)$ and $d^{n+2}(x)$ are proportional for any x if and only if $2\alpha_i + 2 = 0$, for all $i \in \{1, 2, \dots, n\}$. Therefore we conclude that in this case $(\mathcal{H})$ is a Tzitzéica hypersurface if and only if $\alpha_1 = \dots = \alpha_n = -1$.

The proof is now complete. □

Remark 4.2. In the above Lemma we can identify two examples of Tzitzéica hypersurfaces. Firstly, we have obtained a new class of Tzitzéica hypersurfaces, given by $x_{n+1} = A\prod_{i=1}^{n} x_i^{\alpha_i}$, where $\sum_{i=1}^{n} \alpha_i = 1$, and secondly, we can identify the hypersurface given by $x_{n+1} = \frac{A}{x_1 \cdot \dots \cdot x_n}$, which was originally discovered by Tzitzéica himself for $n = 2$ and later generalized by Calabi [18].

Remark 4.3. We have that for a generalized Cobb–Douglas production function the values of $\alpha_1, \dots, \alpha_n$ are positive constants (see e.g. [19]); in fact they are the output elasticities of the factor inputs determined by available technology (see also [20]). Hence, while there is a mathematical possibility of $\alpha_1 = \alpha_2 = \dots = \alpha_n = -1$ in the above lemma, in reality such a production function does not exist. Therefore, we can state the following result.

Theorem 4.4. *The generalized Cobb–Douglas production function has constant return to scale if and only if the corresponding Cobb–Douglas hypersurface is a Tzitzéica hypersurface.*

Remark 4.5. Next we give some diagrammatic representations for the Cobb–Douglas production functions with two inputs and one output in a three-dimensional space, for all the three cases: decreasing, increasing and constant return to scale. With the MATLAB command

$$\gg ezsurf('1.2 * x1^\wedge(0.4) * x2^\wedge(0.2)', [0, 10], [0, 10])$$

we obtain the picture shown in Fig. 1 of the Cobb–Douglas production function $Q(x_1, x_2) = 1.2x_1^{0.4}x_2^{0.2}$ having decreasing return to scale ($\lambda = 0.6 < 1$). Similarly we obtain the pictures for the Cobb–Douglas production function $Q(x_1, x_2) = 1.1x_1^{0.3}x_2^{0.9}$ having increasing return to scale (Fig. 2) and for the Cobb–Douglas production function $Q(x_1, x_2) = 1.4x_1^{0.5}x_2^{0.5}$ having constant return to scale (Fig. 3).

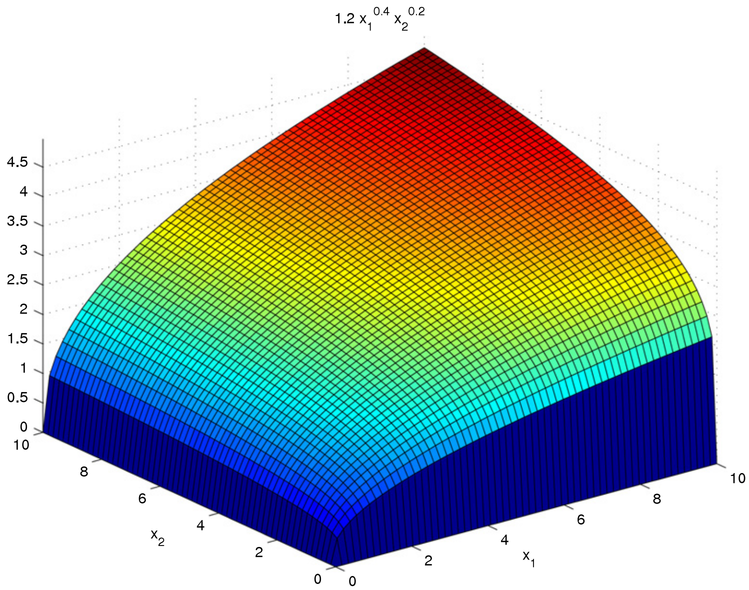

Fig. 1. Decreasing return to scale: $x_3 = 1.2x_1^{0.4}x_2^{0.2}$.

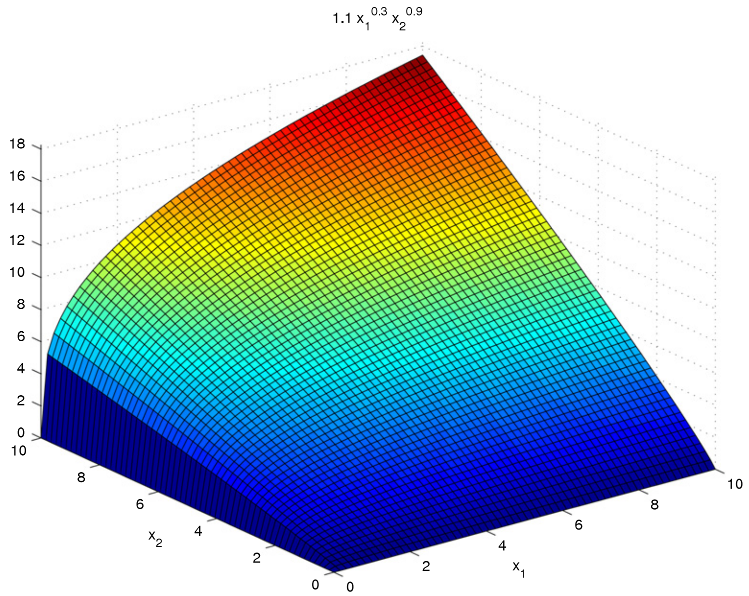

Fig. 2. Increasing return to scale: $x_3 = 1.1x_1^{0.3}x_2^{0.9}$.

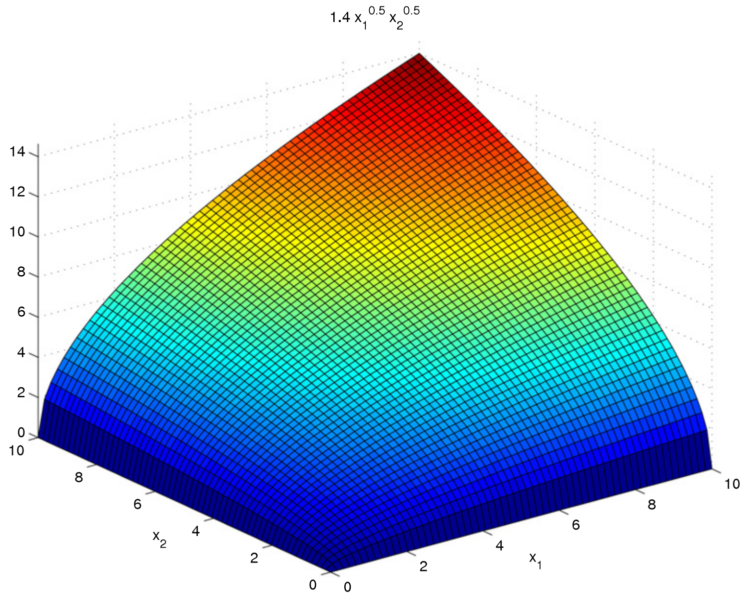

Fig. 3. Constant return to scale: $x_3 = 1.4x_1^{0.5}x_2^{0.5}$.

Acknowledgements

The author would like to thank the referees for helpful comments and suggestions. This work was partially supported by CNCSIS-UEFISCSU, project PNII-IDEI code 8/2008, contract no. 525/2009.

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