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The 24 symmetry pairings of self-dual maps on the sphere

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Abstract

Given a self-dual map on the sphere, the collection of its self-dual permutations generates a transformation group in which the map automorphism group appears as a subgroup of index two. A careful examination of this pairing yields direct constructions of self-dual maps and provides a classification of self-dual maps.

1. Introduction

The concept of duality is essential to the study of a variety of finite combinatorial objects, e.g. planar graphs, polyhedra, simplicial complexes and matroids. It is natural to consider those objects which are self-dual, that is, isomorphic to their duals. The existence of several classes of self-dual graphs, for instance wheels and hyperwheels, see Figs. 23 and 16, was known to Kirkman [9], and in [11] it was shown that large self-dual graphs could be constructed by patching together a planar graph and its dual. Recent interest in this topic was sparked by questions of Grünbaum and Shephard [6], the examination of which led to several methods for the construction of self-dual polyhedra [1], and, more generally, self-dual graphs [12]. All of these constructions involve a sequence of moves which simultaneously modify the object and the dual object so as to preserve an isomorphism between them. An examination of self-dual tilings in the plane is contained in [2].

Grünbaum and Shephard's question involved the *self-dual permutation*, which is the permutation on the components of the self-dual object defined by composing the isomorphism onto the dual with the dual correspondence. Since the self-dual permutation reverses the dimension of the elements, it does not define an automorphism,

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however, its square is an automorphism, and a common mistake has been to suppose that the square is in fact the identity. Grünbaum and Shephard defined the *rank* of a self-dual object to be the smallest order of any of its self-dual permutations, and asked if the rank of a self-dual polyhedron could be greater than 2. An affirmative answer was given by Jendroľ [7], who also identified which symmetries could be exhibited by self-dual polyhedra [8].

In this paper we study the self-dual permutations directly in the form of the *self-dual pairing*. As a result, not only can questions of rank and symmetry be unified, but a complete classification is possible, yielding direct, non-recursive constructions.

As in [12], the objects will be spherical maps, although the results apply to polyhedra as well. Let $\Gamma=(V,E)$ be a finite connected planar graph, and let ρ be a tame embedding of Γ into the sphere, S^2. Two such embeddings, ρ and ρ' are regarded as equivalent if $\rho'=f\rho$ for some homeomorphism f of S^2. If Γ is a simple 3-connected graph, then all embeddings of Γ are equivalent. $S^2-\rho(\Gamma)$ consists of a disjoint union of open cells whose closures in S^2 are the faces of a realization of S^2 as a finite CW-complex called a *map*. An isomorphism of maps will be understood to be an isomorphism of cell complexes and we note that the CW-complex arising from an embedded graph will not in general be regular. It is well known that a graph is 3-connected if and only if the complex can be realized as a polyhedron [3], however we make no connectivity assumptions. We will use the following lemma, which seems to be well known though nowhere explicitly stated.

Lemma 1 (Straightening lemma). *If* Aut(G) *is the group of map automorphisms of the map G, then the map can be redrawn so that* Aut(G) *acts as a group of isometries on* S^2.

Proof. If a map is subdivided by adding a new vertex in the interior of each edge and face, and adding radial arcs from each new face vertex to each vertex on the boundary of that face — analogous to the barycentric subdivision of simplicial complexes — then the same group acts on the subdivided complex, and we may thus assume without loss of generality that every fixed point of the action lies on an edge or vertex of the map.

Aut(G) is a finite group acting on the sphere, so, see [5, p. 273], there is a Cayley graph for Aut(G) cellularly embedded in S^2 so that the action is induced by the regular action of Aut(G) on its Cayley graph. We may assume, by subdivision, that this Cayley graph is a subcomplex of G. By [5, Theorems 6.3.1 and 6.3.2] we see in fact that it must be the Cayley graph of one of the finite spherical isometry groups with generating set chosen such that the Cayley graph is either 3-connected, or a single cycle, and for which there is an embedding so that the action on the Cayley graph is by isometries, see [4, Sections 4.2 and 4.3]. We thus straighten the original map so that the Cayley graph conforms to this embedding. Consider one of the cells defined by the Cayley graph. Its boundary is a regular n-gon and the action on this boundary determines the isotropy subgroup of this cell. The action on the boundary is by rotations and reflections, and hence the map within this cell may be straightened so that the isotropy group acts isometrically on this face, and straightening the translates of this cell compatibly, and doing the same with the other cells, we may straighten the whole map.

All the edges in the embedding given from Lemma 1 need not be geodesics, for example if there are loops or parallel edges.

2. Self-dual pairings

Any map G determines a *dual* map, G^*, obtained by placing a vertex f^* in the interior of each face f and, if two faces f and f' meet along an edge e, then an edge e^* is drawn conncting f^* and f'^* such that e^* intersects G only once transversely in the interior of the edge e. A map G is said to be *self-dual* if G and G^* are map isomorphic. It follows that if a map G is self-dual, then its graph is self-dual, that is, the two skeletons of G and G^* are isomorphic, however not all self-dual graphs arise in this manner [13].

Given a map $G=(V,E,F)$, we can perform the dual construction and regard the superposition of the dual map with the original map as a single map, G_2, whose vertex set consists of $V \cup F^* \cup (E \cap E^*)$, so the edges of G_2 are the "half-edges" of G and G^*, and every face of G_2 is a quadrilateral. We will color the half-edges in G_2 originating from G and G^* differently, say red and blue respectively. The following is clear.

Proposition 1. *Every map isomorphism δ from G to G^* induces a unique color reversing automorphism δ_2 of G_2 and conversely.*

Given a self-dual map G, we define $\mathrm{Dual}(G)$ to be the group of all color preserving and color reversing map automorphisms of G_2. If G is a self-dual map, then the subgroup $\mathrm{Aut}(G)$ of all color preserving map automorphisms of G_2, which is equivalent to the group of map automorphisms of G, has index 2 in $\mathrm{Dual}(G)$. The other coset of $\mathrm{Aut}(G)$ in $\mathrm{Dual}(G)$ comprises the set of *self-dualities of* G and we call the inclusion $\mathrm{Dual}(G) \rhd \mathrm{Aut}(G)$ the *self-dual pairing of* G. By Lemma 1, both $\mathrm{Dual}(G)$ and $\mathrm{Aut}(G)$ belong to the collection of finite groups of isometries of S^2. These are listed in Fig. 1 by their symbol, see [4], together with a schematic for their fundamental regions, in which each rotation is indicated by a small circle at a pole, a rotatory reflection is indicated by an arrow on the edge opposite to the small circle, and all unmarked edges indicate reflections. The value at each vertex indicates the number of regions which meet there in the spherical tessellation. Jendrol̆ [8] enumerated the possible symmetries of self-dual polyhedra, that is, identified all possibilities for $\mathrm{Aut}(G)$. We will see, however, that there are examples of self-dual polyhedra which have isomorphic symmetry groups, but which have different self-dual pairings, e.g., the groups $\mathrm{Dual}(G)$ are not isomorphic. By the same token, there are self-dual polyhedra for which

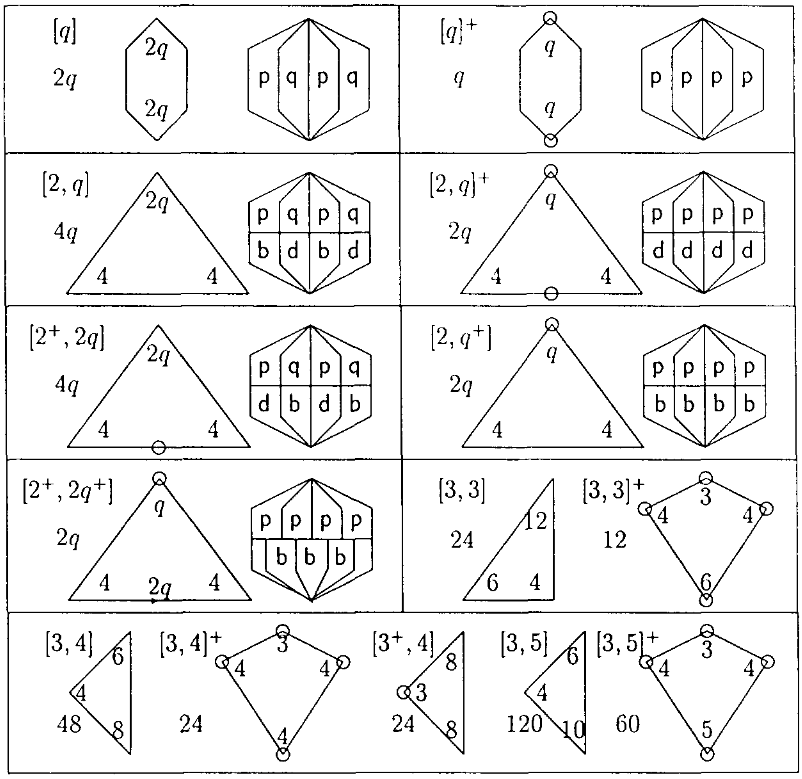


Fig. 1. The finite groups of the sphere. The number below the group symbol is the order of the group, and the illustration on the right of the first seven panels indicates how the fundamental region, in which is drawn a 'p', is transformed by the group.

Dual(G) $\cong$ Dual(G') but Aut(G) $\ncong$ Aut(G'). For a complete structure classification, therefore, it is necessary to consider the entire pairing.

The remainder of this section is devoted to a proof of Theorem 1, which enumerates the possible pairings Dual(G)$\rhd$Aut(G), and we will show how each pairing may be conveniently denoted by marking the schematic for the fundamental region with double circles, lines and arrows if the corresponding rotation, reflection or flip rotation is a self-duality.

Theorem 1. *Every self-dual pairing belongs to one of the 5 families* $[2,q]\rhd[q]$, $[2,q]^+\rhd[q]^+$, $[2^+,2q]\rhd[2q]$, $[2,q^+]\rhd[q]^+$ *or* $[2^+,2q^+]\rhd[2q]^+$; *or is one of the* 19 *pairings* $[2]\rhd[1]$; $[2]\rhd[2]^+$, $[4]\rhd[2]$, $[2]^+\rhd[1]^+$, $[4]^+\rhd[2]^+$, $[2,2]\rhd[2,2]^+$, $[2,4]\rhd[2^+,4]$, $[2,2]\rhd[2,2^+]$, $[2,4]\rhd[2,2]$, $[2,4]^+\rhd[2,2]^+$, $[2^+,4]\rhd[2,2]^+$, $[2^+,4]\rhd[2^+,4^+]$, $[2,4^+]\rhd[2^+,4^+]$, $[2,2^+]\rhd[2^+,2^+]$, $[2,4^+]\rhd[2,2^+]$, $[2,2^+]\rhd[1]$, $[3,4]\rhd[3,3]$, $[3,4]^+\rhd[3,3]^+$, *or* $[3^+,4]\rhd[3,3]^+$.

Lemma 2. *Suppose* Dual(G) *contains two reflections, at least one of which is a self-duality. The dihedral pairing they generate is either*

1. $[2] \rhd [1]$, *where exactly one of the two reflections is a self-duality, their fixed sets intersecting in two points lying in the interior of two quadrilaterals; or*

2. $[2] \rhd [2]^+$, *where both reflections are self-dualities and their fixed sets intersect at two vertices of the form* $e \cap e^*$; *or*

3. $[4] \rhd [2]$, *where two of the four reflections are self-dualities, the fixed sets intersecting at two vertices of the form* $e \cap e^*$.

Proof. If a reflection is a self-duality, then the equator cannot cross any edge of G_2, and may only pass through vertices of the form $e \cup e^*$, so the fixed cells of the reflections appear as illustrated in Fig. 2. If there is a second self-duality reflection, its equator must cross the first at two vertices of the form $e \cup e^*$, as in Fig. 3. If there are also color preserving reflections, they must have fixed sets bisecting the fixed sets of the self-dualities, so Dual(G) is either [4] or [2]. Thus if there are no color preserving reflections there can be only two color reversing reflections meeting at right angles, their product is a color reversing rotation of order 2 and the pairing is $[2] \rhd [2]^+$, as in Fig. 3(b). If there are color preserving reflections, the pairing is either $[2] \rhd [1]$, as in Fig. 3(a), or $[4] \rhd [2]$ as in Fig. 3(c). □

Lemma 3. *Suppose* Dual(G) *contains a cyclic group generated by a color reversing rotation. The cyclic sub-pairing is either*

1. $[2]^+ \rhd [1]^+$, *a color reversing rotation of order 2 whose poles are in the interior of quadrilaterals; or*

2. $[4]^+ \rhd [2]^+$, *generated by a color reversing rotation of order four with poles on vertices in* $E \cap E^*$.

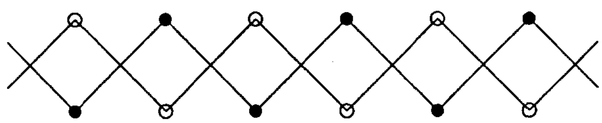


Fig. 2. The equator of a reflection.

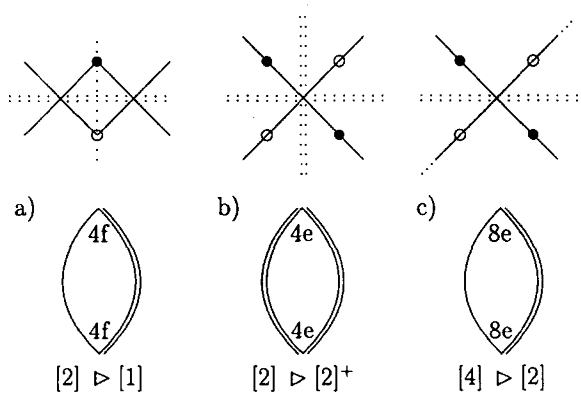


Fig. 3. Dual(G) = [q].

Proof. Given a color reversing rotation of G_2, the only pairs of points which are candidates for poles are either in the interior of quadrilatrals, so that the rotation must be of order 2, or vertices of the form $e \cap e^*$, so that the rotation must be of order four, as in Fig. 4. □.

Lemma 4. *The groups* [3, 3], $[3,3]^+$, [3, 5], *and* $[3,5]^+$ *do not occur as* Dual(*G*) *in any self-duality pairing. Conversely, the remainder of the groups listed in Fig.* 1 *do occur.*

Proof. [3, 3] is generated by two dihedral subgroups of order six, and [3, 5] is generated by dihedral subgroups of order six and ten, all of which must be color preserving by Lemma 2. $[3,3]^+$ is generated by cyclic rotation subgroups of order three, and $[3,5]^+$ is generated by cyclic rotation subgroups of order three and five, all of which must be color preserving by Lemma 3. Thus every element of Dual(*G*) is color preserving.

To see that the other groups all occur, consult the list of figures in the appendix. □

Lemma 5. *The self-dual pairings with* Dual(*G*) = [2, *q*] *are* $[2,q] \rhd [q]$, $[2,2] \rhd [2,2]^+$, $[2,4] \rhd [2^+,4]$, $[2,2] \rhd [2,2^+]$, *and* $[2,4] \rhd [2,2]$.

Proof. If [*q*] is made up of only color preserving transformations, the equatorial reflection must be color reversing, and the pairing is $[2,q] \rhd [q]$. On the other hand if at least one of the reflections in [*q*] is color reversing, then by Lemma 2 the angles between the fixed sets must be either $\pi/2$ or $\pi/4$. If the angle is $\pi/2$, then the pairing is generated by the reflections in the sides of a spherical triangle with three right angles. The case where one side only is color reversing has been enumerated as $[2,2] \rhd [2]$ above. The cases where two or all three are color reversing are $[2,2] \rhd [2,2]^+$ and $[2,2] \rhd [2,2^+]$ respectively, indicated in Fig. 5(b) and (d). If the angle is $\pi/4$, then depending on whether the equatorial reflection is color reversing or not, we have $[2,4] \rhd [2^+,4]$ or $[2,4] \rhd [2,2]$, indicated in Fig. 5(c) and (e). □

Note that the pairings $[2,2] \rhd [2,2]^+$, $[2,2] \rhd [2,2^+]$, and $[2,2] \rhd [2]$ are all distinct, even though they all have the same group for Dual(*G*).

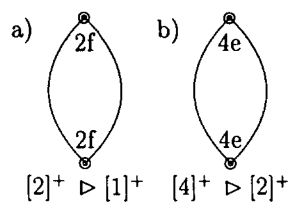


Fig. 4. Dual(*G*) = $[q]^+$.

Lemma 6. *The self-dual pairings with* $\mathrm{Dual}(G)=[2,q]^+$ *are* $[2,q]^+ \rhd [q]^+$ *and* $[2,4]^+ \rhd [2,2]^+$.

Proof. If the transformations in $[q]^+$ are all color preserving, the q equatorial rotations of order two must be color reversing, and the pairing is $[2,q]^+ \rhd [q]^+$. The fundamental region is indicated in Fig. 6(a). If the rotational subgroup $[q]$ is generated by a color reversing rotation, it must be of order two or four, by Lemma 3. If it is of order two, then Dual(G) consists of the identity together with three order two rotations about the coordinate axes, and it is impossible to tell the 'north' pole from the others, and since one order two rotation is color preserving, this situation is in fact the same as the previous case, $[2,2] \rhd [2]^+$. If, however, the generating color reversing rotation of $[q]^+$ is of order four with poles on a vertex in $E \cap E^*$, half of the order two equatorial rotations will be color preserving and half color reversing, the pairing is $[2,4]^+ \rhd [2,2]^+$ and we can indicate the fundamental region as Fig. 6(b). □

Lemma 7. *The self-dual pairings with* $\mathrm{Dual}(G)=[2^+,2q]$ *are* $[2^+,2q] \rhd [2q]$, $[2^+,4] \rhd [2,2]^+$ *and* $[2^+,4] \rhd [2^+4^+]$.

Proof. $[2^+,2q]$ is of order $4q$ consisting of q reflections about circles equally inclined about the poles, q rotations about those poles, q order two rotations whose poles lie

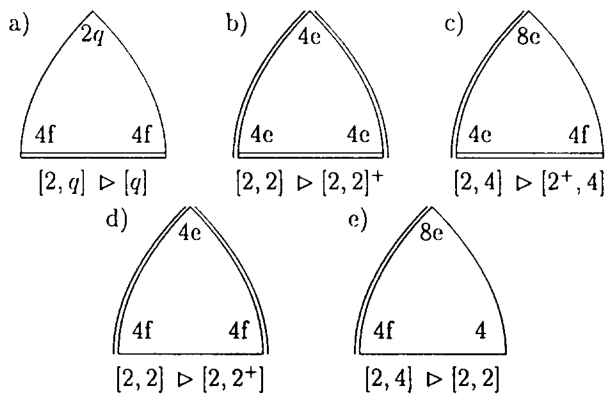


Fig. 5. $\mathrm{Dual}(G)=[2,q]$.

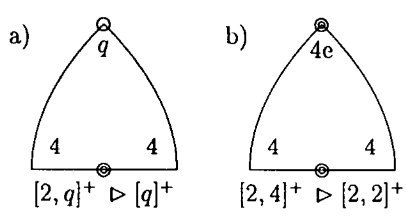


Fig. 6. $\mathrm{Dual}(G)=[2,q]^+$.

equally about the equator, and q flip rotations about that equator. The group is generated by one of the reflections in $[2q]$ and one of the order two rotations on the equator, the other reflections in $[2q]$ being conjugate to the generating reflection. If all reflections in $[2q]$ are color preserving, then the pairing is $[2^+,2q] \rhd [2q]$ and the fundamental region is indicated in Fig. 7(a). If the reflections in $[2q]$ are all color reversing, hence generating the dihedral group $[4]$ with poles on vertices of $E \cap E^*$ as in Lemma 2, then the pairing is $[2^+,4] \rhd [2,2]^+$ or $[2^+,4] \rhd [2^+,4^+]$ depending on whether the equatorial rotations are color preserving or reversing, and the fundamental regions are indicated in Figs. (7b) and (c) respectively. □

Lemma 8. *The self-dual pairings with* $\mathrm{Dual}(G)=[2,q^+]$ *are* $[2,q^+] \rhd [q]^+$; $[2,4^+] \rhd [2^+,4^+]$; $[2,2^+] \rhd [2^+,2^+]$; $[2,4^+] \rhd [2,2^+]$; *and* $[2,2^+] \rhd [1]$ (*Fig.* 8).

Proof. $[2,q^+]$ is of order $2q$, containing q rotations and q flip rotations, generated by one rotation and the equatorial reflection. If the rotation generating $[q]^+$ is color preserving, then the equatorial reflection is color reversing, and the pairing is $[2,q^+] \rhd [q]^+$. If the rotation is color reversing, hence of order two or four with poles in the interior of a quadrilateral or at vertices of $E \cap E^*$ respectively, the equatorial reflection may or may not be color reversing. If the equatorial reflection is color

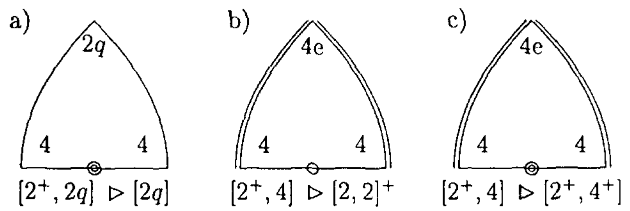


Fig. 7. $\mathrm{Dual}(G)=[2^+,2q]$.

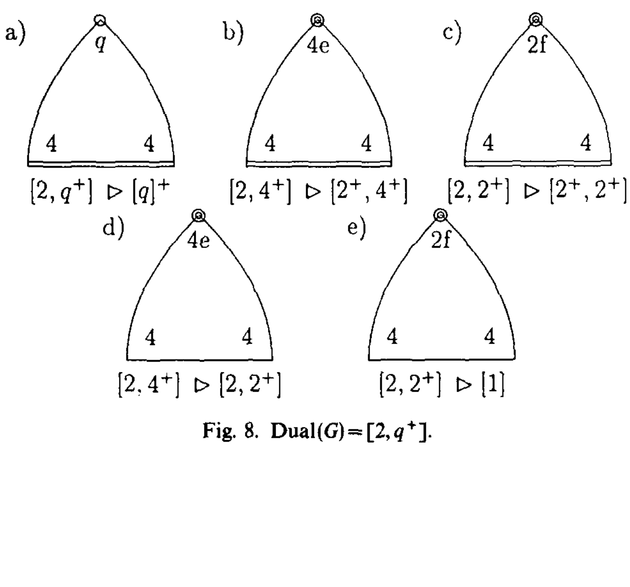


Fig. 8. $\mathrm{Dual}(G)=[2,q^+]$.

reversing, then the color preserving transformations are $q/2$ rotations and $q/2$ flip rotations, with no equatorial reflection, so the pairing is $[2,4^+] \rhd [2^+,4^+]$ or $[2,2^+] \rhd [2^+,2^+]$. If the equatorial reflection is color preserving, then the color preserving transformations are $q/2$ rotations and $q/2$ flip rotations, one of them being an equatorial reflection, so the pairing is $[2,4^+] \rhd [2,2^+]$ or $[2,2^+] \rhd [1]$. □

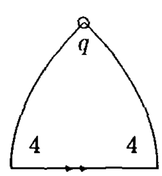
Lemma 9. *The self-dual pairing with* $\mathrm{Dual}(G) = [2^+,2q^+]$ *is* $[2^+,2q^+] \rhd [2q]^+$.

Proof. Since Dual(G) is cyclic, the generator must be color reversing, and so the pairing is $[2^+,2q] \rhd [2q]$ and the fundamental region is indicated in Fig. 9. □

Lemma 10. *The self-dual pairings with* Dual(G) *one of* $[3,4]$, $[3,4]^+$ *and* $[3^+,4]$ *are* $[3,4] \rhd [3,3]$; $(3,4]^+ \rhd [3,3]^+$; *and* $[3^+,4] \rhd [3,3]^+$ (*Fig.* 10).

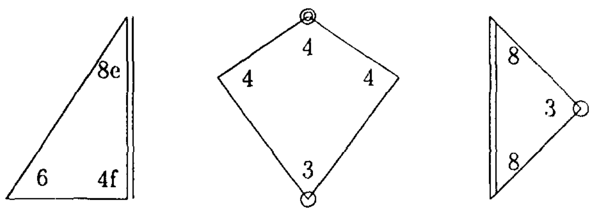
Proof. The symmetry group of the cube, $[3,4]$, is generated by the reflections in the sides of a spherical triangle with angles $\pi/2$, $\pi/3$ and $\pi/4$. Two reflections whose equators meet at angle $\pi/3$ must both be color preserving by Lemma 2, so the reflection in the opposite side must be color reversing and the pairing is $[3,4] \rhd [3,3]$. The rotational group of the cube $[3,4]^+$ is generated by a rotation of order three about two opposite corners and a rotation of order four about the centers of two opposite faces. The rotation of order three is color preserving by Lemma 3, so the rotation of order four must be color reversing, and the pairing is $[3,4]^+ \rhd [3,3]^+$. The group $[3^+,4]$ is generated by reflections in the $x=0$, $y=0$ and $x=0$ planes, together with rotations of order three with poles $(\pm 1, \pm 1, \pm 1)$. The fundamental region is a spherical triangle with angles $\pi/4$, $\pi/4$, and $2\pi/3$, with $[3^+,4]$ generated by a rotation of order three about the vertex with angle $2\pi/3$, which by Lemma 3 is color preserving, and the reflection in the opposite side, which therefore must be color reversing. The color preserving transformations are generated by the order three rotations which generate the rotational group of the tetrahedron, and the pairing is $[3^+,4] \rhd [3,3]^+$. □

Note that the pairings $[3,4]^+ \rhd [3,3]^+$ and $[3^+,4] \rhd [3,3]^+$ describe two distinct self-dual pairings corresponding to maps with isomorphic automorphism groups.



$[2^+,2q^+] \rhd [2q]^+$

Fig. 9. $\mathrm{Dual}(G) = [2^+,2q^+]$.



$[3,4] \rhd [3,3]$ $[3,4]^+ \rhd [3,3]^+$ $[3^+,4] \rhd [3,3]^+$

Fig. 10. Pairings with Dual(G) acting on the cube.

Using the marked diagrams, it is straightforward to construct all self-dual maps with a given pairing. The diagram represents one region of a spherical tessellation, inside of which one may place any planar graph, and then place copies of the graph and the dual graph in the other regions according to the action of the group. If the graph in the marked diagram itself has some symmetry, the resulting self-dual map may have a larger pairing. The only difficulty is to determine the boundary conditions, which vary from pairing to pairing. The constructions of [11] were of this type for the pairing $[2^+, 2q^+] \rhd [1]^+$ under different boundary conditions, and this method was used to construct the maps in the appendix.

3. The rank of self-dual pairings

We can now easily compute the ranks of self-dual maps and polyhedra, which clearly depend only on the pairing.

Theorem 2. *The rank of a self-dual map whose pairing is either* $[4]^+ \rhd [2]^+$ *or* $[2, 4^+] \rhd [2, 2^+]$ *is four. If the pairing is* $[2^+, 2q^+] \rhd [q]^+$ *then the rank is* $2q/s$, *where* s *is the largest odd divisor of* q. *For any other pairing, the rank is* 2.

Proof. A brief inspection of the marked diagrams for the pairings not specifically mentioned in the theorem reveals that each has either a color reversing reflection, or a color reversing rotation of order two. $[4]^+ \rhd [2]^+$ is generated by a color reversing rotation α of order four, so the color reversing map automorphisms are α and α^3. For $[2, 4^+] \rhd [2, 2^+]$, there is a color reversing rotation α of order four and the color preserving equatorial reflection gives additionally rotatory reflections of the same orders. If the pairing is $[2^+, 2q^+] \rhd [q]^+$, then $[2^+, 2q^+]$ is a cyclic group whose generator α is color reversing. If α^j is an odd power then the order of α^j is $2q/\gcd(q,j)$. $\square$

Appendix

In the appendix we give maps which illustrate each of the 24 self-dual pairings (Figs. 11–34). The maps are drawn on unfolded cubes and hexagonal bipyramids, with each fold line indicated by a dashed line. The vertices of the map and dual map are distinguished by solid and hollow vertices. We note that these maps are vertex minimal except for Figs. 16, 21 and 26, where slightly larger maps are drawn for convenience.

Polyhedral examples of each pairing may be obtained similarly, and can be found in [10]. It is not clear whether or not each pairing has a *harmonious* polyhedral realization, i.e., such that the polyhedron is congruent to its polar with respect to a suitable sphere.

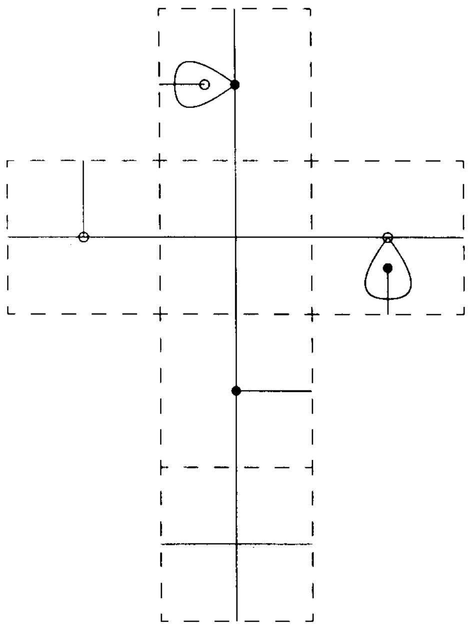


Fig. 11. $[2] \rhd [1]$.

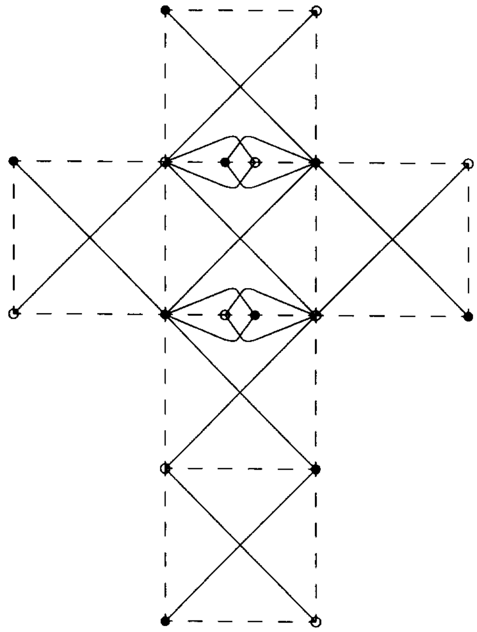


Fig. 12. $[2] \rhd [2]^+$.

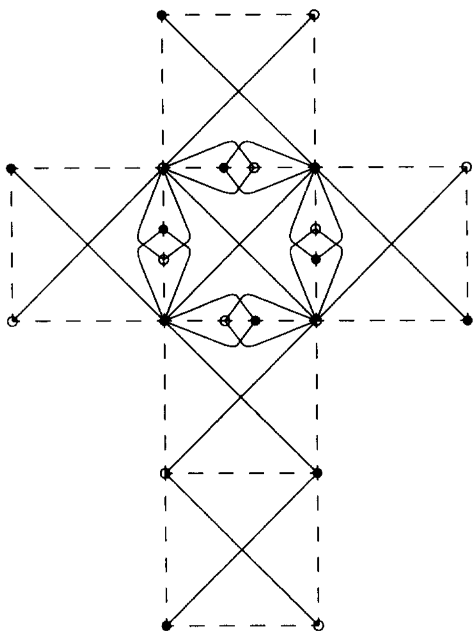


Fig. 13. $[4] \rhd [2]$.

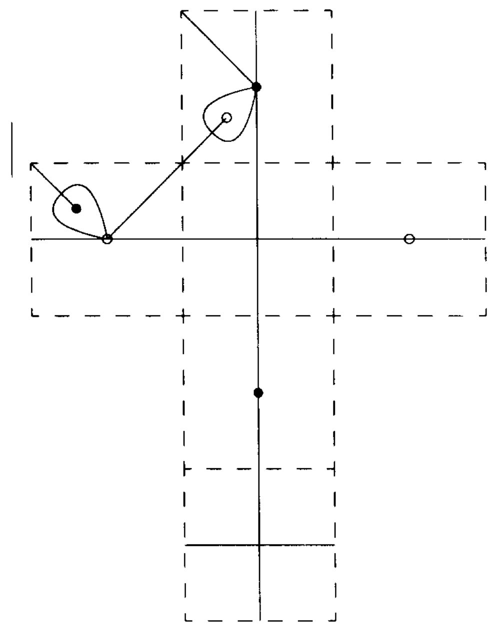


Fig. 14. $[2]^+ \rhd [1]^+$.

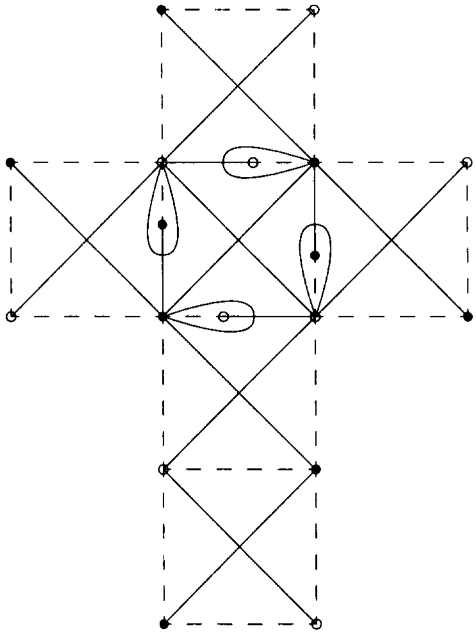


Fig. 15. $[4]^+ \rhd [2]^+$.

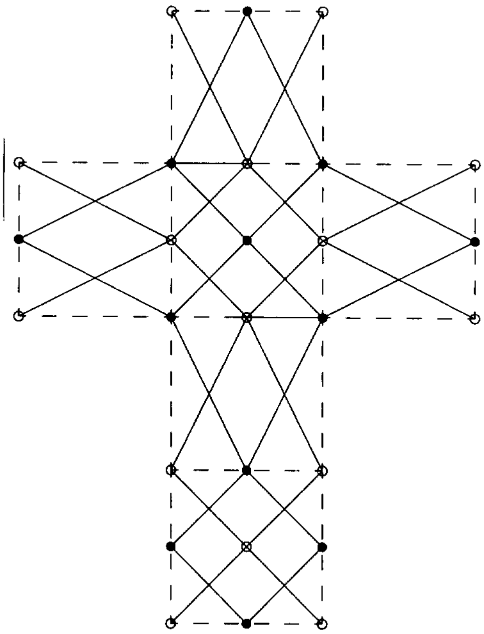


Fig. 16. $[2,6] \rhd [6]$.

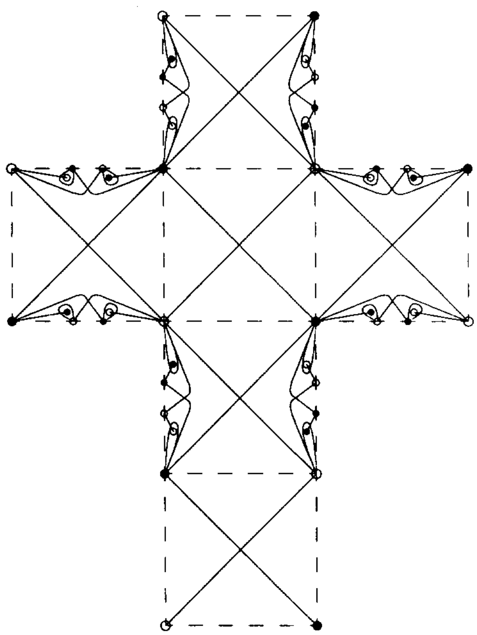


Fig. 17. $[2,2] \rhd [2,2]^+$.

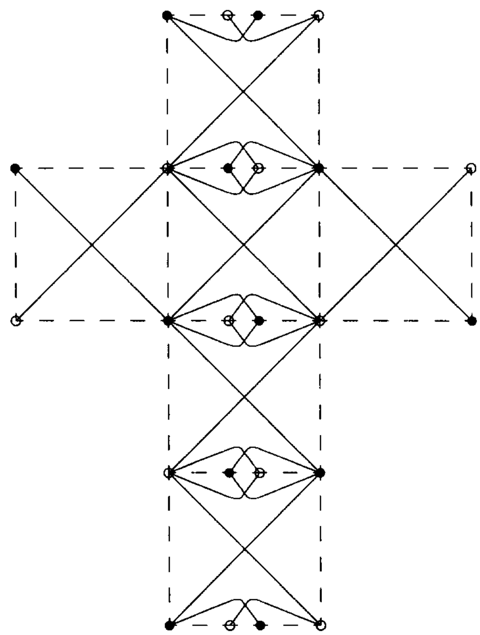


Fig. 18. $[2,4] \rhd [2^+,4]$.

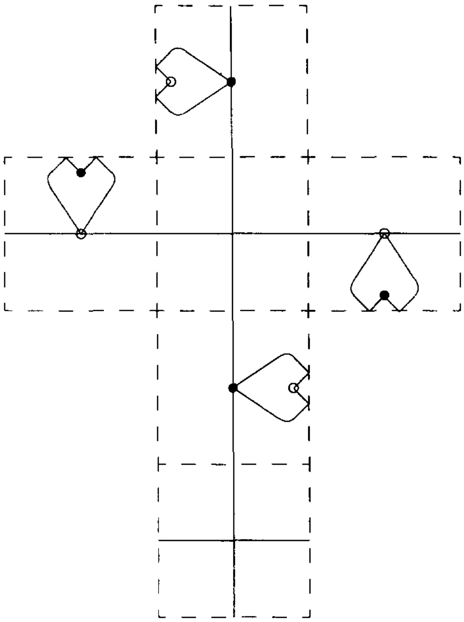


Fig. 19. $[2,2] \rhd [2,2^+]$.

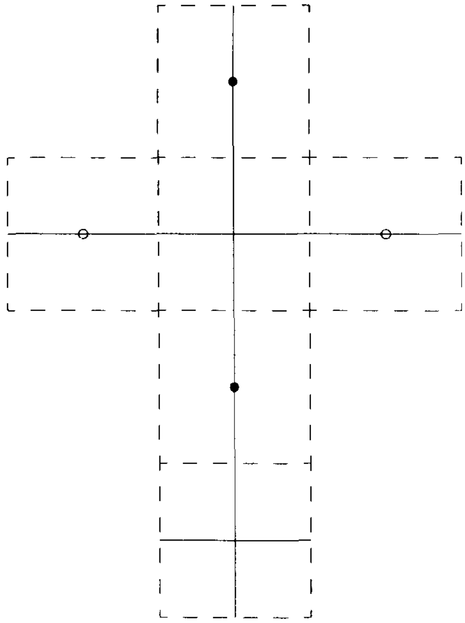


Fig. 20. $[2,4] \rhd [2,2]$.

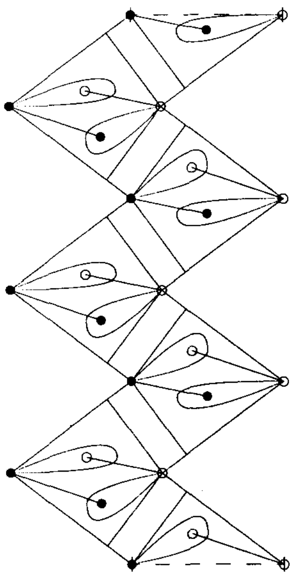


Fig. 21. $[2,6]^+ \rhd [6]^+$.

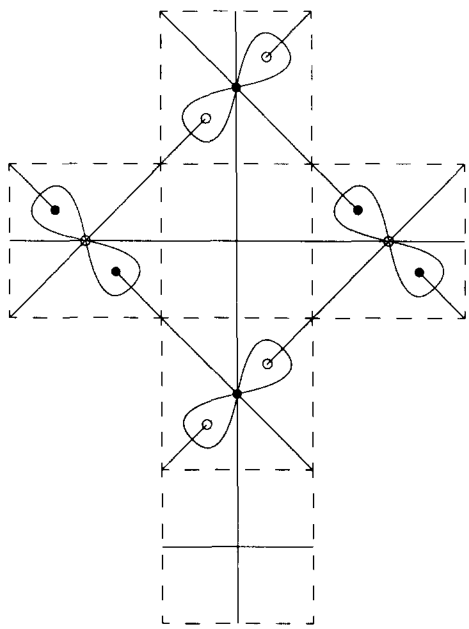


Fig. 22. $[2,4]^+ \rhd [2,2]^+$.

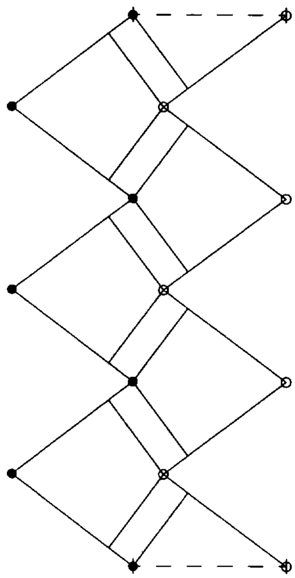


Fig. 23. $[2^+,12] \rhd [12]$.

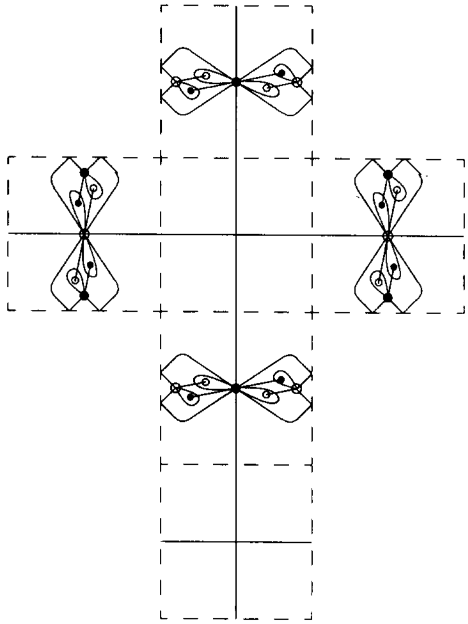


Fig. 24. $[2^+,4] \rhd [2,2]^+$.

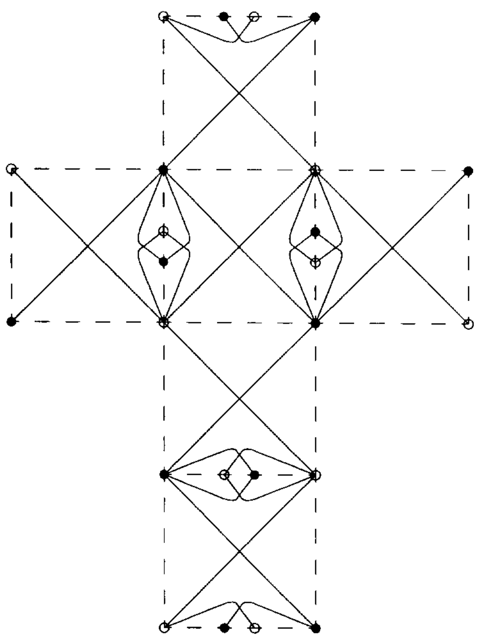


Fig. 25. $[2^+,4] \rhd [2^+,4^+]$.

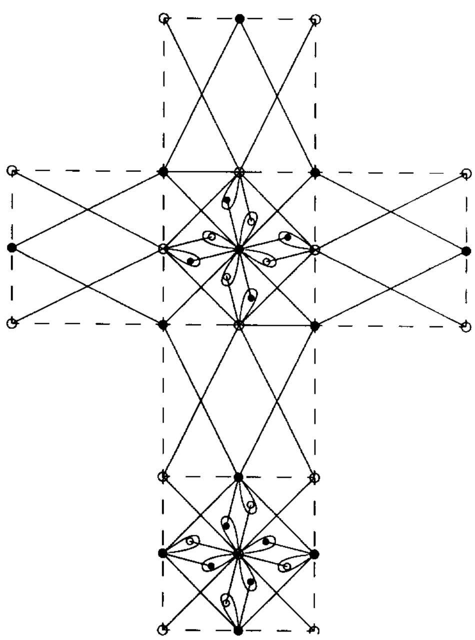


Fig. 26. $[2,6^+] \rhd [6]^+$.

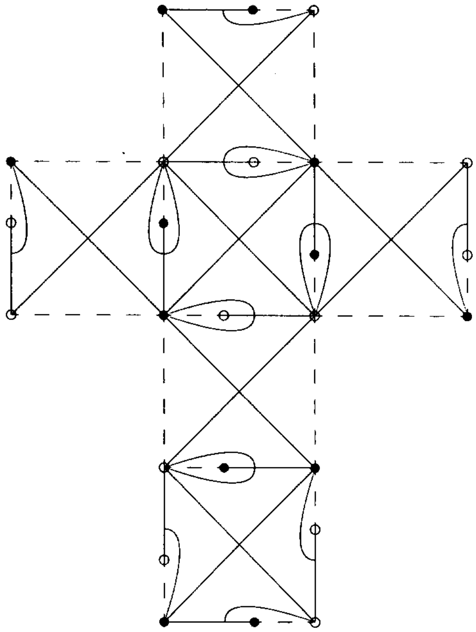


Fig. 27. $[2,4^+] \rhd [2^+,4^+]$.

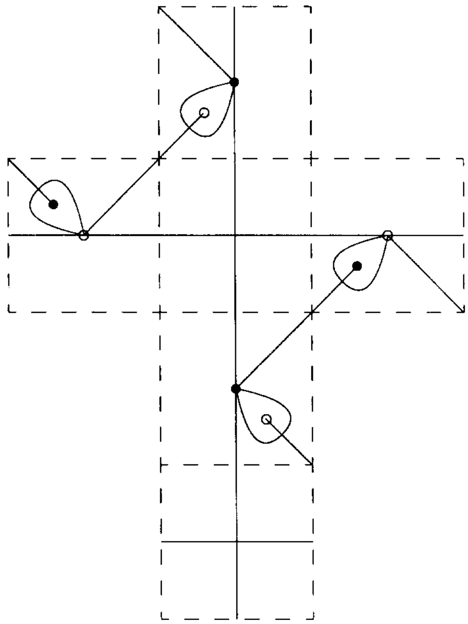


Fig. 28. $[2,2^+] \rhd [2^+,2^+]$.

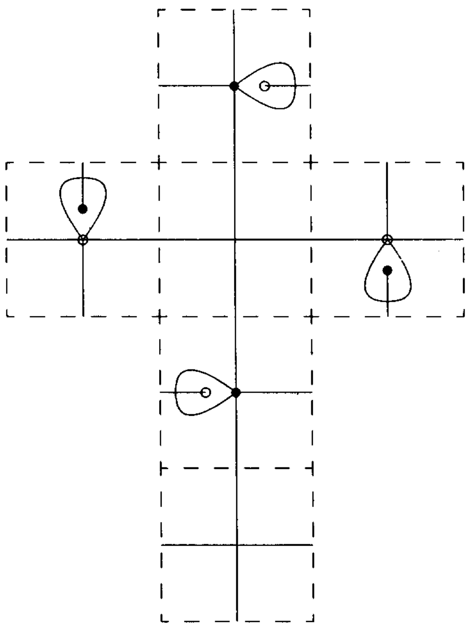


Fig. 29. $[2,4^+] \rhd [2,2^+]$.

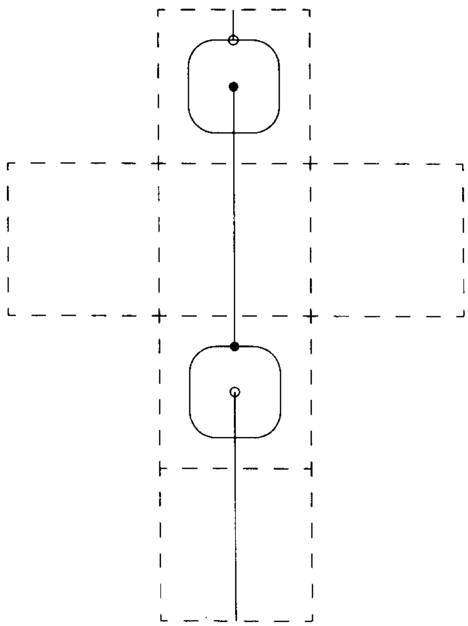


Fig. 30. $[2,2^+] \rhd [1]$.

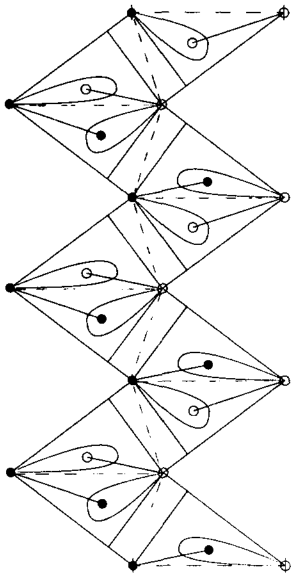


Fig. 31. $[2^+, 12^+] \rhd [12]^+$.

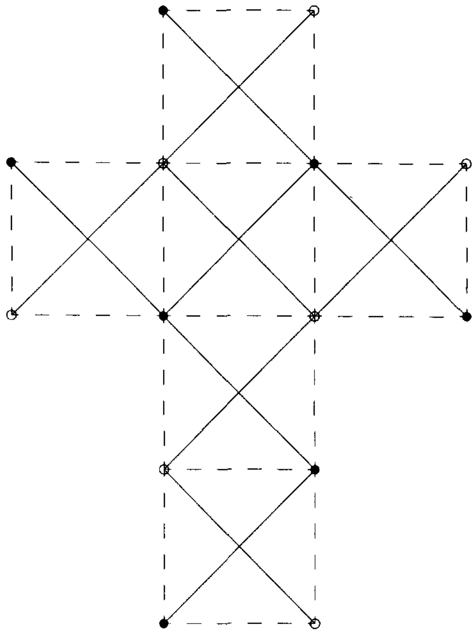


Fig. 32. $[3,4] \rhd [3,3]$.

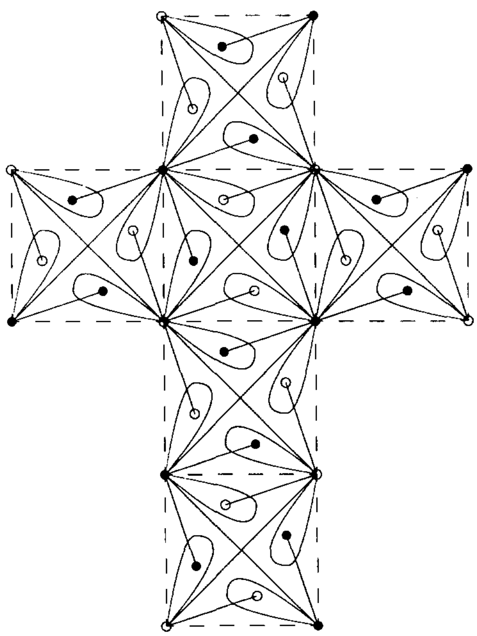


Fig. 33. $[3,4]^+ \rhd [3,3]^+$.

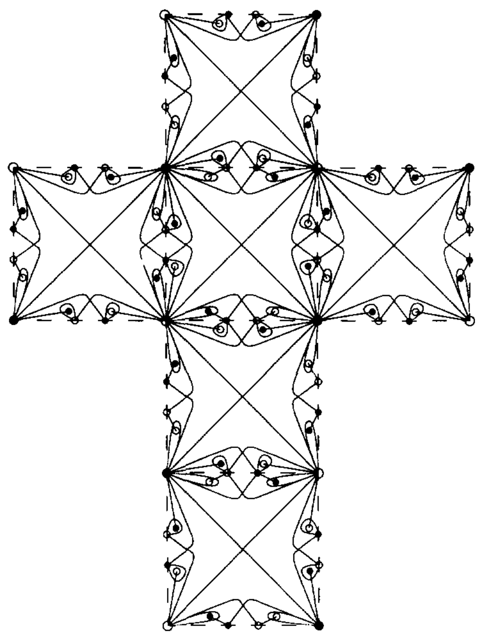


Fig. 34. $[3^+,4] \rhd [3,3]^+$.

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