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Quantization dimension for infinite self-similar probabilities

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ABSTRACT

The quantization dimension function for a probability measure induced by a set of infinite contractive similarity mappings and a given probability vector is determined. A relationship between the quantization dimension function and the temperature function of the thermodynamic formalism arising in multifractal analysis is established. The result in this paper is an infinite extension of Graf and Luschgy [S. Graf, H. Luschgy, The quantization dimension of self-similar probabilities, Math. Nachr. 241 (2002) 103–109].

1. Introduction

The term 'quantization' in the title originates in the theory of signal processing and denotes a process of discretizing signals. As a mathematical theory quantization concerns the best approximation of probabilities by discrete probabilities with a given number of points in their support. A detailed account of this theory can be found in [3]. Given a Borel probability measure μ on $\mathbb{R}^d$, a number $r \in (0, +\infty)$ and a natural number $n \in \mathbb{N}$, the nth *quantization error of order* r for μ is defined by

$$V_{n,r}(\mu) := \inf\left\{\int d(x,\alpha)^r \, d\mu(x) \colon \alpha \subset \mathbb{R}^d,\ \operatorname{card}(\alpha) \leqslant n\right\},$$

where $d(x,\alpha)$ denotes the distance from the point x to the set α with respect to a given norm $\|\cdot\|$ on $\mathbb{R}^d$. We note that if $\int \|x\|^r \, d\mu(x) < \infty$ then there is some set α for which the infimum is achieved (cf. [3]). The set α for which the infimum is achieved is called the optimal set of n-means or n-optimal set of order r for $0 < r < +\infty$. The upper and lower quantization dimension of order r for μ is defined to be

$$\overline{D}_r(\mu) := \limsup_{n\to\infty} \frac{r \log n}{-\log V_{n,r}(\mu)}; \qquad \underline{D}_r(\mu) := \liminf_{n\to\infty} \frac{r \log n}{-\log V_{n,r}(\mu)}.$$

If $\overline{D}_r(\mu)$ and $\underline{D}_r(\mu)$ coincide, we call the common value the *quantization dimension of order* r for the probability measure μ, and is denoted by $D_r := D_r(\mu)$. One sees that the quantization dimension is actually a function $r \mapsto D_r$ which measures the asymptotic rate at which $V_{n,r}$ goes to zero. If D_r exists, then one can write

$$\log V_{n,r} \sim \log\left(\frac{1}{n}\right)^{r/D_r}.$$

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Let $S_1, S_2, \ldots, S_N$ be a set of contractive similarity mappings on $\mathbb{R}^d$ with the similarity ratios respectively $s_1, s_2, \ldots, s_N$ for $N \geqslant 2$. Then for a given probability vector $(p_1, p_2, \ldots, p_N)$ there exists a unique Borel probability measure μ (cf. [5]) satisfying the condition

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1}.$$

Let the iterated function system $\{S_1, S_2, \ldots, S_N\}$ satisfy the open set condition: there exists a bounded nonempty open set $U \subset \mathbb{R}^d$ such that $\bigcup_{j=1}^N S_j(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for $1 \leqslant i \neq j \leqslant N$. The iterated function system satisfies the strong open set condition if U can be chosen such that $U \cap J \neq \emptyset$, where J is the limit set of the iterated function system. Under the open set condition, Graf and Luschgy showed that the quantization dimension function $D_r := D_r(\mu)$ for the probability measure μ exists, and satisfies the following relation (cf. [3,4]):

$$\sum_{j=1}^{N} \left(p_j s_j^r\right)^{\frac{D_r}{r+D_r}} = 1.$$

Note that from the above relation it is clear that the quantization dimension function for a self-similar probability has a relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis. Lindsay and Mauldin extended the above result to the F-conformal measure with finitely many conformal mappings (cf. [6]). The quantization dimension and its relationship with the temperature function for some other probability measures were also investigated, for example one could see [9–11,13]. But in each case, the number of mappings considered was finite. Determination of the quantization dimension for a probability measure generated by an infinite iterated function system associated with a probability vector is a long-time open problem. The work in this paper is the first advance in this direction. The probability measure μ considered here is induced by a set of infinite contractive similarity mappings $(S_n)_{n\geqslant 1}$ satisfying the strong open set condition with the similarity ratios respectively $(s_n)_{n\geqslant 1}$, and associated with a probability vector $(p_1, p_2, \ldots)$, that is, the probability measure μ satisfies

$$\mu = \sum_{j=1}^{\infty} p_j \mu \circ S_j^{-1}.$$

We have shown that for such a probability measure μ if the quantization dimension $D_r := D_r(\mu)$ exists, it satisfies the following relation:

$$\sum_{j=1}^{\infty} \left(p_j s_j^r\right)^{\frac{D_r}{r+D_r}} = 1. \tag{1}$$

Riedi and Mandelbrot showed that the multifractal formalism for a self-similar measure does indeed hold in the infinite case (cf. [12]). In particular, the singularity exponent $\beta(q)$ (also known as the temperature function) satisfies the usual equation

$$\sum_{j=1}^{\infty} p_j^q s_j^{\beta(q)} = 1, \tag{2}$$

and that the spectrum $f(\alpha)$ is the Legendre transform of $\beta(q)$. Comparing (1) and (2), we see that if $q_r = \frac{D_r}{r+D_r}$, then $\beta(q_r) = rq_r$, that is, the quantization dimension function for an infinite self-similar probability has a relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis (for thermodynamic formalism, multifractal analysis and Legendre transform one could see [2]). The result in this paper is an infinite extension of Graf and Luschgy (cf. [4]).

2. Basic definitions and results

In this paper, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space equipped with a metric d, and $\mathbb{R}_+$ represents the set of all nonnegative real numbers. Let us write,

$$V_{n,r} = V_{n,r}(\mu) := \inf\left\{\int d(x,\alpha)^r \, d\mu(x) \colon \alpha \subset \mathbb{R}^d,\ \operatorname{card}(\alpha) \leqslant n\right\}.$$

A set $\alpha \subset \mathbb{R}^d$ with $\operatorname{card}(\alpha) \leqslant n$ is called an *n-optimal set of centers for μ of order r* or *$V_{n,r}(\mu)$-optimal set* if

$$V_{n,r}(\mu) = \int d(x,\alpha)^r \, d\mu(x).$$

As stated before, n-optimal sets exist when $\int \|x\|^r \, d\mu(x) < \infty$.

Let X be a nonempty compact subset of $\mathbb{R}^d$ with $X = \mathrm{cl}(\mathrm{int}\, X)$. We call $f : X \to \mathbb{R}^d$ a *Lipschitz function* if there exists a number c such that

$$d\big(f(x), f(y)\big) \leqslant c d(x, y) \quad \text{for all } x, y \in X.$$

The infimum value of c for which such an inequality holds is called the *Lipschitz constant* of f, written as Lip f. A Lipschitz function $f : X \to \mathbb{R}^d$ is called a *contractive mapping* if $0 < \mathrm{Lip}\, f < 1$. Let $(S_n)_{n\geqslant 1}$ be an infinite set of contractive similarity mappings on X whose contraction ratios are respectively $(s_n)_{n\geqslant 1}$, i.e., $d(S_n(x), S_n(y)) = s_n d(x, y)$ for all $x, y \in X$, $0 < s_n < 1$, $n \geqslant 1$. Moreover, $\sup_n s_n < 1$. Let $\mathcal{K}(X)$ denote the class of all nonempty compact subsets of X. If we define a function $h : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathbb{R}_+$ by

$$h(A, B) = \max\big\{D(A, B), D(B, A)\big\},$$

where

$$D(A, B) = \sup_{x\in A}\Big(\inf_{y\in B} d(x, y)\Big) \quad \text{for all } A, B \in \mathcal{K}(X),$$

we obtain a metric, namely the Hausdorff metric. If (X, d) is a complete metric space, then $\mathcal{K}(X)$ is a complete metric space with respect to the metric h. Also $(\mathcal{K}(X), h)$ is a compact metric space provided that (X, d) is compact (cf. [1]). Let us now define a set function $\mathcal{S} : \mathcal{K}(X) \to \mathcal{K}(X)$, by

$$\mathcal{S}(E) = \overline{\bigcup_{n\geqslant 1} S_n(E)},$$

where $\overline{A}$ of a set A represents the closure of the set A. Then for any two sets $E, F \in \mathcal{K}(X)$, we have

$$h\big(\mathcal{S}(E), \mathcal{S}(F)\big) = h\bigg(\overline{\bigcup_{n\geqslant 1} S_n(E)}, \overline{\bigcup_{n\geqslant 1} S_n(F)}\bigg) = h\bigg(\bigcup_{n\geqslant 1} S_n(E), \bigcup_{n\geqslant 1} S_n(F)\bigg),$$

which implies

$$h\big(\mathcal{S}(E), \mathcal{S}(F)\big) \leqslant \sup_{n\geqslant 1} h\big(S_n(E), S_n(F)\big) = \Big(\sup_{n\geqslant 1} s_n\Big) h(E, F),$$

i.e., $\mathcal{S}$ is a contractive mapping on $(\mathcal{K}(X), h)$ with contraction ratio $s \leqslant \sup_{n\geqslant 1} s_n$. Hence by the contraction mapping theorem, there exists a unique nonempty compact set $J \subset X$, which is known as the *attractor* or the *invariant set* of the family $(S_n)_{n\geqslant 1}$, i.e., J satisfies

$$J = \mathcal{S}(J) = \overline{\bigcup_{n\geqslant 1} S_n(J)}.$$

The set J is called the *infinite self-similar set* corresponding to the infinite iterated function system $(S_n)_{n\geqslant 1}$ considered in this paper. Let $(p_1, p_2, \ldots)$ be a probability vector with $p_n > 0$ for all $n \geqslant 1$. Then there exists a unique Borel probability measure μ on $\mathbb{R}^d$ such that

$$\mu = \sum_{n=1}^{\infty} p_n \mu \circ S_n^{-1},$$

which has the support the compact set J (cf. [8]). We call μ the *infinite self-similar probability* or the *infinite self-similar measure* induced by the similarity mappings $(S_n)_{n\geqslant 1}$ and the probability vector $(p_1, p_2, \ldots)$. The iterated function system $(S_n)_{n\geqslant 1}$ is said to satisfy the *open set condition* (OSC) if there exists a bounded nonempty open set $U \subset X$ (in the topology of X) such that $S_j(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all $i \neq j$, $i, j \geqslant 1$, and the *strong open set condition* (SOSC) if, in addition, U can be chosen such that $U \cap J \neq \emptyset$. In the paper, we assume that the set of infinite similarity mappings satisfies the strong open set condition.

Let us now consider the auxiliary function:

$$P(q, t) = \log \sum_{j=1}^{\infty} p_j^q s_j^t \tag{3}$$

for $q, t \in \mathbb{R}$. For a given $q \in \mathbb{R}$, let $\theta(q) = \inf\{t \in \mathbb{R}\colon\ P(q, t) < \infty\}$. Then $\theta(q) \geqslant -\infty$. For a given $q \in \mathbb{R}$ the function $P(q, t)$ is strictly decreasing, convex and hence continuous in $(\theta(q), +\infty)$. Its value ranges from $-\infty$ (when $t \to +\infty$) to $+\infty$ (when $t \to \theta(q)$). Therefore, by the intermediate value theorem there is a real number t such that $P(q, t) = 0$. The solution t is unique as $P(q, \cdot)$ is strictly decreasing in t. This defines t implicitly as a function of q: for each q there is a unique $t = \beta(q)$ such that $P(q, \beta(q)) = 0$. $P(q, t)$ is called the *topological pressure* corresponding to the given infinite iterated function system. The function $\beta(q)$ is sometimes denoted by $T(q)$, and called the *temperature function*.

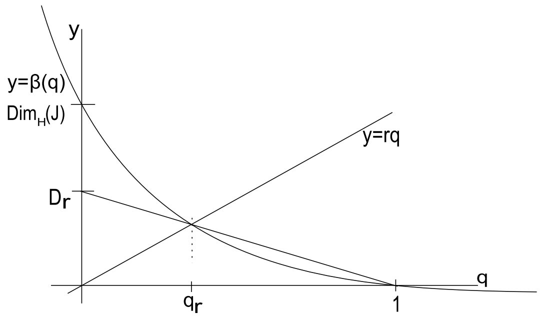

Fig. 1. To determine D_r first find the point of intersection of $y = \beta(q)$ and the line $y = rq$. Then D_r is the y-intercept of the line through this point and the point $(1, 0)$.

Note 2.1. If $q = 0$ then from (3), we have

$$\sum_{j=1}^{\infty} s_j^{\beta(0)} = 1,$$

i.e., $\beta(0)$ gives the Hausdorff dimension $\dim_{\mathrm{H}}(J)$ of the infinite self-similar set J (cf. [7]). Moreover, $P(1, 0) = 0$, which gives $\beta(1) = 0$ (see Fig. 1).

3. Main result

The relationship between the quantization dimension function and the temperature function $\beta(q)$ for the probability measure μ, where the temperature function is the Legendre transform of the $f(\alpha)$ curve (the definitions of the $f(\alpha)$ and the Legendre transform are given in [2]) is given by the following theorem. For a graphical description see Fig. 1.

Theorem 3.1. *Let μ be the infinite self-similar probability induced by the infinite iterated function system $(S_n)_{n\geqslant 1}$ satisfying the strong open set condition, and associated with the probability vector $(p_1, p_2, \ldots)$. Let $\beta = \beta(q)$ be the temperature function of the thermodynamic formalism. For each $r \in (0, +\infty)$ choose q_r such that $\beta(q_r) = rq_r$. Then the quantization dimension for the probability measure μ is given by*

$$D_r = \frac{\beta(q_r)}{1 - q_r}.$$

Lemma 3.2. *Let $0 < r < +\infty$. Then there exists exactly one number $\kappa_r \in (0, +\infty)$ such that*

$$\sum_{j=1}^{\infty} \bigl(p_j s_j^r\bigr)^{\frac{\kappa_r}{r+\kappa_r}} = 1.$$

Proof. The function $P(t, rt) := \log \sum_{j=1}^{\infty} (p_j s_j^r)^t$ is strictly decreasing, convex and hence continuous in the interval $(0, +\infty)$. Moreover, $P(0, 0) = \infty$ and $P(1, r1) = \log \sum_{j=1}^{\infty} p_j s_j^r < \log \sum_{j=1}^{\infty} p_j = 0$. Therefore, by the intermediate value theorem, there exists a unique t which lies between 0 and 1, such that $P(t, rt) = 0$. Take $\kappa_r = \frac{rt}{1-t}$, and then $\log \sum_{j=1}^{\infty} (p_j s_j^r)^{\frac{\kappa_r}{r+\kappa_r}} = 0$, which implies $\sum_{j=1}^{\infty} (p_j s_j^r)^{\frac{\kappa_r}{r+\kappa_r}} = 1$, and thus the lemma is obtained. □

For every $M \geqslant 2$, let us consider the partial iterated function system $(S_n)_{n=1}^{M}$ defined on X associated with the probability vector $(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_M)$, where $\hat{p}_j = p_j$ for $1 \leqslant j \leqslant M - 1$ and $\hat{p}_M = \sum_{j=M}^{\infty} p_j$. Then for every $M \geqslant 2$, there exists a unique Borel probability measure μ_M (cf. [5]) on X with the support J_M such that

$$J_M = \bigcup_{j=1}^{M} S_j(J_M) \quad \text{and} \quad \mu_M = \sum_{j=1}^{M} \hat{p}_j \mu_M \circ S_j^{-1}.$$

Let us now state the following lemma.

Lemma 3.3. *(Cf. [3, Lemma 14.4].) Let $0 < r < +\infty$ and M is as before. Then there exists exactly one number $\kappa_r^{(M)} \in (0, +\infty)$ with*

$$\sum_{j=1}^{M} \bigl(\hat{p}_j s_j^r\bigr)^{\frac{\kappa_r^{(M)}}{r+\kappa_r^{(M)}}} = 1.$$

The above $\kappa_r^{(M)}$ is the quantization dimension for the probability measure μ_M generated by the set of self-similar mappings $S_1, S_2, \ldots, S_M$ associated with the probability vector $(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_M)$ (cf. [3,4]).

Let us now prove the following lemma.

Lemma 3.4. *Let $0 < r < +\infty$, and let $\kappa_r^{(M)}$ be as in Lemma 3.3. Then $\kappa_r^{(M)} \to \kappa_r$ as $M \to \infty$, where κ_r is as in Lemma 3.2.*

Proof. Let $q_r^{(M)} = \frac{\kappa_r^{(M)}}{r+\kappa_r^{(M)}}$ and $q_r = \frac{\kappa_r}{r+\kappa_r}$. First we prove $q_r^{(M)} \to q_r$ as $M \to \infty$. If possible, let $\lim_{M\to\infty} q_r^{(M)} > q_r$. Then there exist $\gamma > 0$ and a sequence of positive integers $\{M_n\}_{n\geqslant 1}$ such that $q_r^{(M_n)} \geqslant q_r + \gamma > q_r$ for all n. Using the fact that $\lim_{n\to\infty} \sum_{j=M_n}^{\infty} p_j = \lim_{n\to\infty}(1 - \sum_{j=1}^{M_n-1} p_j) = 1 - \sum_{j=1}^{\infty} p_j = 1 - 1 = 0$, Lemmas 3.2 and 3.3, we have

$$\begin{aligned}
1 &= \lim_{n\to\infty}\left[\sum_{j=1}^{M_n-1}\left(p_j s_j^r\right)^{q_r^{(M_n)}} + \left(\left(\sum_{j=M_n}^{\infty} p_j\right) s_{M_n}^r\right)^{q_r^{(M_n)}}\right] \\
&\leqslant \lim_{n\to\infty}\left[\sum_{j=1}^{M_n-1}\left(p_j s_j^r\right)^{q_r+\gamma} + \left(\left(\sum_{j=M_n}^{\infty} p_j\right) s_{M_n}^r\right)^{q_r+\gamma}\right] \\
&\leqslant \lim_{n\to\infty}\left[\sum_{j=1}^{M_n-1}\left(p_j s_j^r\right)^{q_r+\gamma} + \left(\sum_{j=M_n}^{\infty} p_j\right)^{q_r+\gamma}\right] \\
&\leqslant \sum_{j=1}^{\infty}\left(p_j s_j^r\right)^{q_r+\gamma} < \sum_{j=1}^{\infty}\left(p_j s_j^r\right)^{q_r} = 1,
\end{aligned}$$

which gives a contradiction. Hence, $\lim_{M\to\infty} q_r^{(M)} \leqslant q_r$. Similarly if we take $\lim_{M\to\infty} q_r^{(M)} < q_r$, a contradiction will arise. Hence, $q_r^{(M)} \to q_r$ as $M \to \infty$, and then $\kappa_r^{(M)} = \frac{r q_r^{(M)}}{1-q_r^{(M)}} \to \frac{r q_r}{1-q_r} = \kappa_r$ as $M \to \infty$, which yields the lemma. □

Let $\mathcal{M}$ denote the set of all normalized Borel measures on X. The map $d_H : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ such that

$$d_H(\mu, \nu) = \sup\left\{\left|\int_X g\,d\mu - \int_X g\,d\nu\right| : \operatorname{Lip} g \leqslant 1\right\}$$

for all $\mu, \nu \in \mathcal{M}$, is a metric, namely the *Hutchinson metric*. $(\mathcal{M}, d_H)$ is a compact metric space (cf. [1]). With respect to the Hutchinson metric d_H, it is known that $\{\mu_M\}_{M\geqslant 2}$ tends to the probability measure μ as $M \to \infty$ (cf. [8, Theorem 3]). Again we know that in the weak topology on $\mathcal{M}$,

$$\mu_M \to \mu \quad \Leftrightarrow \quad \int_X f\,d\mu_M - \int_X f\,d\mu \to 0 \quad \text{for all } f \in \mathcal{C}(X),$$

where $\mathcal{C}(X) := \{f : X \to \mathbb{R} \colon f \text{ is continuous}\}$. Clearly, X being compact all measures in $\mathcal{M}$ have a compact support. It is a standard fact that d_H topology and the weak topology coincide on the space of Borel normalized measures with compact support. Using this fact, let us prove the following lemma.

Lemma 3.5. *Let $0 < r < +\infty$, and $\mu_M \to \mu$ with respect to the Hutchinson metric d_H. Then*

$$\lim_{M\to\infty} V_{n,r}(\mu_M) = V_{n,r}(\mu)$$

for every $n \geqslant 1$.

Proof. By our assumption, $\int \|x\|^r\,d\mu(x) < \infty$. Since the function $f : X \to \mathbb{R}$ defined by $f(x) = \|x\|^r$ is continuous, we have $\lim_{M\to\infty} \int \|x\|^r\,d\mu_M(x) = \int \|x\|^r\,d\mu(x)$, which yields that there exists a positive integer $M_0 \geqslant 2$ such that for all $M \geqslant M_0$, $\int \|x\|^r\,d\mu_M(x) < \infty$. Take $M \geqslant M_0$. Then n-optimal sets for both $V_{n,r}(\mu_M)$ and $V_{n,r}(\mu)$ exist. Let α_n be a $V_{n,r}(\mu_M)$-optimal set and β_n be a $V_{n,r}(\mu)$-optimal set for $n \geqslant 1$. As $\mu_M \to \mu$ (weakly), and for every $\alpha \subset \mathbb{R}^d$ the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, \alpha)^r$ is continuous, for every $n \geqslant 1$ we have

$$\lim_{M\to\infty} V_{n,r}(\mu_M) = \lim_{M\to\infty} \int d(x, \alpha_n)^r\,d\mu_M(x) = \int d(x, \alpha_n)^r\,d\mu(x) \geqslant V_{n,r}(\mu) \quad \text{and}$$

$$\lim_{M\to\infty} V_{n,r}(\mu_M) \leqslant \lim_{M\to\infty} \int d(x, \beta_n)^r\,d\mu_M(x) = \int d(x, \beta_n)^r\,d\mu(x) = V_{n,r}(\mu),$$

and thus

$$\lim_{M\to\infty} V_{n,r}(\mu_M) = V_{n,r}(\mu),$$

which yields the lemma. □

Proof of Theorem 3.1. To prove the theorem let us first prove

$$\kappa_r \leqslant \liminf_n \frac{r\log n}{-\log V_{n,r}(\mu)} \leqslant \limsup_n \frac{r\log n}{-\log V_{n,r}(\mu)} \leqslant \kappa_r. \tag{4}$$

If possible, let $\liminf_n \frac{r\log n}{-\log V_{n,r}(\mu)} < \kappa_r$. Then there exists a subsequence $(\frac{r\log n_k}{-\log V_{n_k,r}(\mu)})_{k\geqslant 1}$ of the sequence $(\frac{r\log n}{-\log V_{n,r}(\mu)})_{n\geqslant 1}$ such that $\lim_{k\to\infty} \frac{r\log n_k}{-\log V_{n_k,r}(\mu)} < \kappa_r$, which implies that there exists a positive integer K_0 such that $\frac{r\log n_k}{-\log V_{n_k,r}(\mu)} < \kappa_r$ for all $k \geqslant K_0$. Thus for $k \geqslant K_0$, using Lemmas 3.4 and 3.5, we obtain

$$\lim_{M\to\infty} \frac{r\log n_k}{-\log V_{n_k,r}(\mu_M)} = \frac{r\log n_k}{-\log V_{n_k,r}(\mu)} < \kappa_r = \lim_{M\to\infty} \kappa_r^{(M)},$$

and so there exists a positive integer M' such that $\frac{r\log n_k}{-\log V_{n_k,r}(\mu_M)} < \kappa_r^{(M)}$ for all $M \geqslant M'$ and for all $k \geqslant K_0$. In particular, for all $k \geqslant K_0$ we have

$$\frac{r\log n_k}{-\log V_{n_k,r}(\mu_{M'})} < \kappa_r^{(M')}. \tag{5}$$

Note that $V_{n_k,r}(\mu_{M'}) \to 0$ as $k \to \infty$, and $V_{n_k,r}(\mu_{M'}) \geqslant V_{n_{k+1},r}(\mu_{M'}) > 0$ for all $k \geqslant 1$, and thus there exists a positive integer K_0' such that for all $k \geqslant K_0'$,

$$1 > V_{n_k,r}(\mu_{M'}) \geqslant V_{n_{k+1},r}(\mu_{M'}) > 0.$$

Hence for all $k \geqslant K_0'$, we have

$$\frac{r\log n_k}{-\log V_{n_k,r}(\mu_{M'})} \geqslant \frac{r\log n_{k+1}}{-\log V_{n_{k+1},r}(\mu_{M'})}, \tag{6}$$

i.e., $(\frac{r\log n_k}{-\log V_{n_k,r}(\mu_{M'})})_{k\geqslant K_0'}$ is a decreasing sequence of real numbers. Then by (5) and (6), we deduce

$$\lim_{k\to\infty} \frac{r\log n_k}{-\log V_{n_k,r}(\mu_{M'})} < \kappa_r^{(M')}, \quad \text{i.e.,} \quad \liminf_n \frac{r\log n}{-\log V_{n,r}(\mu_{M'})} \leqslant \lim_{k\to\infty} \frac{r\log n_k}{-\log V_{n_k,r}(\mu_{M'})} < \kappa_r^{(M')}.$$

$\kappa_r^{(M')}$ is the quantization dimension for the probability measure $\mu_{M'}$, and so by the preceding inequality, we obtain

$$\kappa_r^{(M')} = \lim_{n\to\infty} \frac{r\log n}{-\log V_{n,r}(\mu_{M'})} = \liminf_n \frac{r\log n}{-\log V_{n,r}(\mu_{M'})} < \kappa_r^{(M')},$$

which is a contradiction. Hence

$$\kappa_r \leqslant \liminf_n \frac{r\log n}{-\log V_{n,r}(\mu)}, \quad \text{and similarly,} \quad \limsup_n \frac{r\log n}{-\log V_{n,r}(\mu)} \leqslant \kappa_r.$$

Therefore, the inequalities in (4) are proved, and thus $\lim_{n\to\infty} \frac{r\log n}{-\log V_{n,r}(\mu)}$ exists and equals κ_r, i.e.,

$$D_r(\mu) = \lim_{n\to\infty} \frac{r\log n}{-\log V_{n,r}(\mu)} = \kappa_r.$$

Note that if $q_r = \frac{\kappa_r}{r+\kappa_r}$, by Lemma 3.2, we have $\beta(q_r) = rq_r$, and then $D_r = \frac{\beta(q_r)}{1-q_r}$. Thus the proof of the theorem is complete. □

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