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The X-Ray Transform*

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INTRODUCTION

The purpose of this paper is to study k-plane integral transformations on the spaces $L^1(R^n)$ and $L^2(R^n)$. Such transformations arise naturally in electron microscopy [5, 7], (crystallography [2], biochemistry [12], molecular biology [23]), aerodynamics [22], radio astronomy [3, 30], radiography [4, 16], and in various areas of pure mathematics such as partial differential equations [18], and integral geometry [9]. Particularly significant advances have been made in radiography in developing new methods for detecting brain tumors [10, 15]. Of practical importance is the problem of reconstructing a three-dimensional object from certain projections. The three-dimensional reconstruction problem can be formulated as follows.

An object in three-dimensional space is determined by a density function f on the space R^3, $f(x)$ being the density at the point x. An X-ray picture taken in the direction θ provides a function $L_\theta f$ on the plane orthogonal to θ whose value at a point x on this plane is the total mass along the line through x in the direction θ;

$$L_\theta f(x) = \int_{-\infty}^{\infty} f(x + t\theta)\, dt, \qquad x \in \theta^\perp.$$

The reconstruction problem is to recover f from a finite number of the X rays $L_\theta f$. (Technically speaking $L_\theta f$ is the radiograph of f. Radiologists refer to the photon beam as the X ray and the picture as the radiograph. However, we shall not make this distinction.) See [28] for some of the results that have been obtained on phantoms and on actual brain tumor patients. Reference [8] contains a discussion of the many algorithms currently in use.

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More generally, let π be a k-dimensional subspace of R^n. The X ray of a function f on R^n in the direction π at the point x'' in $\pi^\perp$ is defined by

$$L_\pi f(x'') = Lf(\pi, x'') = \int_\pi f(x', x'')\, dx',$$

provided that the integral exists in the Lebesgue sense. Here, and in general, once a subspace π is fixed we write $x = (x', x'')$ where x' and x'' are the orthogonal projections of x on π and $\pi^\perp$, respectively. The k-dimensional subspaces of R^n form the Grassmann manifold $G_{n,k}$. The X ray of f is a function $Lf(\pi, x'')$ on a fiber bundle $T(G_{n,k})$ with base space $G_{n,k}$ and fibers isomorphic to R^{n-k}. When $k = 1$, or $k = n - 1$, $G_{n,k} = S^{n-1}$. The purpose of this paper is to discuss the transformations L and L^{-1}.

The topics discussed are given in the following table of contents.

These topics have received a great deal of study in the case $k = n - 1$. In this case, the X-ray transform is the same as the Radon transform. Indeed, the Radon transform is defined by

$$R_\theta f(t) \int_{\langle x,\theta\rangle = t} f(x)\, d\alpha_{n-1}(x),$$

where θ is a direction on the unit sphere S^{n-1}, $t \in R^1$, and α_{n-1} is the $(n-1)$-dimensional surface area measure in R^n. So $L_\pi f = R_\theta f$ with $\pi = \theta^\perp$. Radon [25] and John [17] proved that a differentiable function with compact support in R^n is uniquely determined by means of its integrals over the hyperplanes in the space. Radon and John also give inversion formulas. Ludwig [20], characterized the range of the Radon transform on various function spaces and spaces of distributions. The lower dimensional cases, $k \leqslant n - 2$, have received less attention. However, Helgason [13] has given inversion formulas for the space of infinitely differentiable functions with compact support when k is even, and for a special subspace of the Schwartz space of rapidly decreasing functions for all k [14].

As one might expect, the results depend on the value of k. However, often there is a critical value of k at which results either become false, or become much more difficult to prove. For example, a square integrable function on R^n is actually integrable on almost every translate of almost every k-dimensional subspace when $k < n/2$ [26]. This fails when $k \geqslant n/2$. Also, the X-ray transform with domain $C_0^\infty(R^n)$, the infinitely differentiable functions with compact support, has a closure in all dimensions, but the closure is given by the defining integral only when $k < n/2$. In studying the supports of f and Lf, it was discovered that when $k \leqslant n - 2$, f has compact support if and only if Lf has compact support. Moreover, if $k \leqslant n - 2$, it is possible, in some cases, to get inside the convex hull of the support of f from a knowledge of the support of Lf.

A few remarks on notation are needed before beginning. The symbol L_π will be used when a subspace π is fixed. In the case $k = 1$, L_π will sometimes be written L_θ with $\theta \in S^{n-1}$ the direction of π, and $L_\theta f = L_\pi f$ will be referred to as the *ordinary* X *ray* of f. Functions on the fiber bundle $T(G_{n,k})$ will be denoted $g(\pi, x'')$ or $g_\pi(x'')$. The latter notation will be used when looking at g as a function on the fiber over a fixed k-space π. Finally, all inner products will be designated by $\langle , \rangle$.

The work in this paper stems from joint research on the practical and mathematical aspects of the three-dimensional reconstruction problem with Guenther, Smith, and Wagner [10, 28]. The author is especially indebted to Smith for many helpful suggestions in both the research and writing of this paper.

1. The Determination of an Integrable Function by X Rays

In this section the X-ray transform is defined and a few simple, but useful, formulas are developed. These lead to two interesting results concerning the determination of a compactly supported integrable function by ordinary X rays. First, such a function is uniquely determined by any infinite set of ordinary X rays. Second, finitely many ordinary X rays tell nothing about the function in the interior of the support.

If π is a k-dimensional subspace of R^n, the X ray of the function f in the direction π at the point x'' in $\pi^\perp$ is defined by

$$L_\pi f(x'') = Lf(\pi, x'') = \int_\pi f(x', x'')\, dx', \tag{1.1}$$

provided that the integral exists in the Lebesgue sense. Here, and in general, once a subspace π is fixed we write $x = (x', x'')$ where x' and x'' are the orthogonal projections of x on π and $\pi^{\perp}$, respectively. The k-dimensional subspaces of R^n form the Grassmann manifold $G_{n,k}$. The X ray of f is a function $Lf(\pi, x'')$ on a fiber bundle $T(G_{n,k})$ with base space $G_{n,k}$ and fibers isomorphic to R^{n-k}. When $k = 1$, or $k = n - 1$, then $G_{n,k} = S^{n-1}$.

LEMMA 1.2. *If ρ is a locally integrable function of one variable and for fixed $y'' \in \pi^{\perp}$, $\rho(\langle x'', y''\rangle) L_\pi |f| (x'')$ is integrable on $\pi^{\perp}$, then*

$$\int_{\pi^{\perp}} \rho(\langle x'', y''\rangle) L_\pi f(x'')\, dx'' = \int_{R^n} \rho(\langle x, y''\rangle) f(x)\, dx. \tag{1.3}$$

Proof. Fubini's theorem gives that

$$\begin{aligned}\int_{\pi^{\perp}} \rho(\langle x'', y''\rangle) L_\pi f(x'')\, dx'' &= \int_{\pi^{\perp}}\int_{\pi} \rho(\langle x, y''\rangle) f(x', x'')\, dx'\, dx'' \\ &= \int_{R^n} \rho(\langle x, y''\rangle) f(x)\, dx.\end{aligned}$$

The Fourier transform of an integrable function on R^n is given by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{-i\langle x,\xi\rangle} f(x)\, dx.$$

Thus if π is a k-space and $g \in L^1(\pi^{\perp})$, it is natural to define

$$\hat{g}(\xi'') = (2\pi)^{(k-n)/2} \int_{\pi^{\perp}} e^{-i\langle x'',\xi''\rangle} g(x'')\, dx'', \qquad \xi'' \in \pi^{\perp}.$$

Lemma 1.2 leads immediately to a relationship between the Fourier transform of f and the Fourier transform of $L_\pi f$.

LEMMA 1.4. *For each k-space π and integrable function f,*

$$(L_\pi f)^{\wedge}(\xi'') = (2\pi)^{k/2} \hat{f}(\xi'') \qquad \textit{for } \xi'' \in \pi^{\perp}.$$

Proof. The proof is immediate if one takes $\rho(t) = e^{-it}$ in Lemma 1.2.

If $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and Q is a polynomial in n variables, it follows immediately that

$$(L_\pi(Q(D)f))^{\wedge}(\xi'') = (2\pi)^{k/2} Q(i\xi'') \hat{f}(\xi'') \qquad \text{for } \xi'' \in \pi^{\perp}. \tag{1.5}$$

If V is a $(k+1)$-dimensional subspace of R^n and ξ is an arbitrary point in R^n, then $\xi \in \pi^{\perp}$ for some k-space π contained in V. Lemma 1.4 tells how to compute $\hat{f}(\xi)$ from $L_\pi f$. Since an integrable function f is uniquely determined by its Fourier transform, f is uniquely determined by the X rays in the directions $\pi \subset V$. This establishes the following result, $G_{n,k}(V)$ being the submanifold of $G_{n,k}$ consisting of the k-spaces contained in V.

COROLLARY 1.6. *An integrable function on R^n is uniquely determined by the* X *rays in the directions $\pi \in G_{n,k}(V)$ for any $(k+1)$-dimensional subspace V of R^n.*

Note that if $k = 1$, then $G_{n,k}(V)$ is a great circle on the sphere S^{n-1}.

Much more can be said in the case of ordinary X rays when f is assumed to have compact support.

THEOREM 1.7. *If $f \in L^1(R^n)$ has compact support, then f is uniquely determined by any infinite set of ordinary X rays.*

Proof. Suppose that $L_{\theta_j} f = 0$ for an infinite set of directions θ_j, $j = 1, 2, \ldots$. Lemma 1.4 implies that $\hat{f}$ vanishes identically on the hyperplanes $\theta_j^{\perp}$ for all j. Since f has compact support, $\hat{f}$ is an analytic function on R^n and thus cannot vanish identically on infinitely many hyperplanes through the origin unless $\hat{f}$ is identically zero. Thus $f = 0$ almost everywhere.

In the practical reconstruction problem the functions do have compact support but only a finite number of X rays can be taken. Thus, it is useful to know the amount of information given by finitely many ordinary X rays. The following, which is a joint result with Smith and was announced in [10, 27], gives a rather pessimistic answer to this question.

THEOREM 1.8. *Let $f_0 \in C_0^\infty(R^n)$, A be any compact set in the interior of the support of f_0, and $\theta_1, \ldots, \theta_N$ be a finite set of directions. Then there is a new infinitely differentiable function f with the same shape, the same ordinary X rays in the given directions, and completely arbitrary on A.*

Proof. Let Q be a polynomial such that $Q(i\xi)$ vanishes identically on the hyperplanes $\theta_1^{\perp}, \ldots, \theta_N^{\perp}$, and $f_1 \in C^\infty(R^n)$ be arbitrary. The theorem of Malgrange on the existence of solutions to constant coefficient partial differential equations [21], guarantees the existence of functions u_0 and u_1 in $C^\infty(R^n)$ such that

$$Q(D)\, u_i = f_i, \qquad i = 0, 1. \tag{1.9}$$

Choose $\varphi \in C_0^\infty(R^n)$ such that $\varphi \equiv 1$ in a neighborhood of A and vanishes outside the support of f_0. Now, let

$$v_0 = Q(D)\,(\varphi u_0) \tag{1.10}$$

and

$$v_1 = \underset{\sim}{Q}(D)\,(\varphi u_1). \tag{1.11}$$

Now, (1.9) and (1.10) show that $v_0 = f_0$ in a neighborhood of A, while (1.9) and (1.11) show that $v_1 = f_1$ in a neighborhood of A. Moreover v_0 and v_1 vanish outside the support of f_0. Also, (1.5) shows that for each θ_j, $j = 1,\ldots, N$,

$$(L_{\theta_j} v_0)^\wedge\,(\xi'') = (2\pi)^{1/2} \underset{\sim}{Q}(i\xi'')\,(\varphi u_0)^\wedge\,(\xi'') = 0, \qquad \xi'' \in \theta_j^\perp$$

since $\underset{\sim}{Q}(i\xi)$ vanishes identically on $\theta_j^\perp$. Thus

$$L_{\theta_j} v_0 = 0, \qquad j = 1,\ldots, N. \tag{1.12}$$

Similarly

$$L_{\theta_j} v_1 = 0, \qquad j = 1,\ldots, N. \tag{1.13}$$

Finally, define $f = f_0 - v_0 + v_1$. In a neighborhood of A, $f = f_1$, and moreover

$$L_{\theta_j} f = L_{\theta_j} f_0\,, \qquad j = 1,\ldots, N,$$

from (1.12) and (1.13).

This theorem has very practical consequences which are discussed in Section 7 and in [28].

2. Lower Dimensional Integrability of L^2 Functions

In this section a formula relating integrals over R^n with integrals over the fiber bundle $T(G_{n,k})$ is used to find an isometry from $L^2(R^n)$ into the square integrable functions on this fiber bundle. Also a theorem on the lower dimensional integrability of L^2 functions is stated. This theorem gives the first example of a critical value of k at which results become false.

The symbol $L^2(T)$ will be used to designate the measurable functions on the fiber bundle $T(G_{n,k})$ which satisfy

$$\int_{G_{n,k}} \int_{\pi^\perp} |\, g(\pi, x'')|^2 \, dx''\, d\mu < \infty,$$

where μ is the finite measure on $G_{n,k}$ invariant under orthogonal transformations [24], and normalized so that

$$\mu(G_{n,k}) = |\, S^{n-1} \,|/|\, S^{n-k-1} \,|\,,$$

the bars denoting the appropriate area measures on the spheres.

LEMMA 2.1. *If g is nonnegative and measurable on the sphere S^{n-1}, then*

$$\int_{G_{n,k}} \int_{S^{n-1} \cap \pi^\perp} g(\omega)\, d\omega\, d\mu = \int_{S^{n-1}} g(\theta)\, d\theta.$$

Proof. The integral on the left defines a continuous linear form on the space $C(S^{n-1})$ (hence a measure on S^{n-1}). This form is obviously finite and rotation invariant and there is only one such up to a constant factor, namely the integral on the right. The normalization of μ is chosen to make the constant 1. Once the formula is established for continuous functions it extends immediately to nonnegative functions by the standard arguments of measure theory.

LEMMA 2.2. *If g is nonnegative and measurable on R^n, then*

$$\int_{G_{n,k}} \int_{\pi^\perp} | x'' |^k g(x'')\, dx''\, d\mu = \int_{R^n} g(x)\, dx.$$

Proof. Write the integral over $\pi^\perp$ in polar coordinates and use Lemma 2.1.

In terms of Fourier transforms the operator Λ is defined by

$$(\Lambda f)\hat{}\,(\xi) = | \xi | \hat{f}(\xi). \tag{2.3}$$

To avoid cumbersome notation, the same symbol is used irrespective of the space R^n, or subspace of R^n, in which Λ acts.

THEOREM 2.4. *The map $(2\pi)^{-k/2} (\Lambda^{k/2} L)$ extends to an isometry V from $L^2(R^n)$ into $L^2(T)$.*

Proof. It suffices to show that $(2\pi)^{-k/2} (\Lambda^{k/2} L)$ maps $C_0^\infty(R^n)$ isometrically into $L^2(T)$. If $f \in C_0^\infty(R^n)$, the definition of Λ, the Parseval equality in $\pi^\perp$, Lemmas 1.4 and 2.2 show that

$$\int_{G_{n,k}} \int_{\pi^\perp} | \Lambda^{k/2} L_\pi f(x'')|^2\, dx''\, d\mu$$

$$= (2\pi)^k \int_{G_{n,k}} \int_{\pi^\perp} | \xi'' |^k | \hat{f}(\xi'')|^2\, d\xi''\, d\mu = (2\pi)^k \| f \|^2_{L^2(R^n)}\,.$$

The next theorem, which is a rather surprising one, is proved in a recent paper [26].

THEOREM 2.5. *For $k < n/2$ there is a constant c (depending on k and n) such* *that if $f \in L^2(R^n)$, then for almost every $\pi \in G_{n,k}$, f is integrable on almost every k-plane parallel to π and*

$$\int_{G_{n,k}} \| L_\pi f \|^2_{L^q(\pi^\perp)} \, d\mu \leqslant c^2 \| f \|^2_{L^2(R^n)}, \qquad q = 2(n-k)/(n-2k).$$

Remark 2.6. The function

$$\begin{aligned} f(x) &= |x|^{-n/2} (\log |x|)^{-1}, && |x| \geqslant 2, \\ &= 0, && \text{otherwise} \end{aligned}$$

is square integrable on R^n, but is not integrable over any plane of dimension $\geqslant n/2$.

3. The X-Ray Transform as an Unbounded Operator on L^2

Now the X-ray transform is considered as an unbounded operator from $L^2(R^n)$ into $L^2(T)$. There are two seemingly natural choices for the domain. The most direct is

$$D_k = \left\{ f \in L^2(R^n) : \int_{G_{n,k}} \| L_\pi f \|^2_{L^2(\pi^\perp)} \, d\mu < \infty \right\}.$$

However, the Fourier transform relationship of Lemma 1.4 and the integration formula of Lemma 2.2 suggest the indirect definition

$$\tilde{D}_k = \{ f \in L^2(R^n) : |\xi|^{-k/2} \hat{f} \in L^2(R^n) \}.$$

It is shown that $D_k = \tilde{D}_k$ if and only if $k < n/2$. Moreover the X-ray transform with domain $C_0^\infty(R^n)$ has a closure with domain $\tilde{D}_k$ for all k. (It is not known whether the operator is even closable with domain D_k when $k \geqslant n/2$.)

Lemma 3.1. *If $f \in L^1(R^n) \cap L^2(R^n)$ then $f \in D_k$ and*

$$\| Lf \|^2_{L^2(T)} \leqslant (2\pi)^k \left(|S^{n-1}| \, \| f \|^2_{L^1} + \| f \|^2_{L^2} \right). \tag{3.2}$$

Proof. Since $f \in L^1 \cap L^2$, it follows that $|\xi|^{-k/2} \hat{f} \in L^2(R^n)$ and an easy calculation shows that

$$\| |\xi|^{-k/2} \hat{f} \|^2_{L^2} \leqslant |S^{n-1}| \, \| \hat{f} \|^2_{L^\infty} + \| \hat{f} \|^2_{L^2}. \tag{3.3}$$

But Parseval's relation on $\pi^\perp$, Lemmas 1.4 and 2.2 give that

$$\| Lf \|^2_{L^2(T)} = (2\pi)^k \| \, | \xi |^{-k/2} \hat{f} \|^2_{L^2} .$$

The result follows from (3.3), Parseval's relation on R^n, and the fact that $\| \hat{f} \|_{L^\infty} \leqslant \| f \|_{L^1}$.

THEOREM 3.4. *Let $k < n/2$. Then $D_k = \tilde{D}_k$ and if $f \in D_k$, then for almost every k-space π*

$$(L_\pi f)\hat{} = (2\pi)^{k/2} \hat{f} \quad \text{a.e.} \qquad \textit{on} \qquad \pi^\perp. \tag{3.5}$$

If $k \geqslant n/2$, then $D_k \neq \tilde{D}_k$.

Proof. Let $0 < k < n$ and assume that $f \in \tilde{D}_k$. Choose $\hat{f}_n \in C_0^\infty(R^n)$ such that $\hat{f}_n \to \hat{f}$ in L^2 and $| \xi |^{-k/2} \hat{f}_n \to | \xi |^{-k/2} \hat{f}$ in L^2. Let f_n be the inverse Fourier transform of $\hat{f}_n$. Lemmas 1.4 and 2.2 show that

$$(L_\pi f_n)\hat{} \to (2\pi)^{k/2} \hat{f} \qquad \text{in} \qquad L^2(\pi^\perp) \quad \text{for a.e. } \pi.$$

Choosing a suitable subsequence if necessary, we obtain for almost every π

$$(L_\pi f_n) \to g_\pi \quad \text{a.e.} \qquad \text{on} \qquad \pi^\perp, \tag{3.6}$$

where g_π is defined by

$$\hat{g}_\pi = (2\pi)^{k/2} \hat{f} \qquad \text{on} \qquad \pi^\perp. \tag{3.7}$$

However, when $k < n/2$, Theorem 2.5 shows that for almost every π, and again a suitable subsequence

$$L_\pi f_n \to L_\pi f \quad \text{a.e.} \qquad \text{in} \qquad \pi^\perp.$$

Thus for almost every π

$$g_\pi = L_\pi f \quad \text{a.e.} \qquad \text{on} \qquad \pi^\perp. \tag{3.8}$$

Hence $f \in D_k$. Moreover (3.7) and (3.8) show that (3.5) is valid.

Conversely, suppose $f \in D_k$, $k < n/2$, and define $f_\rho(x) = e^{-\rho^2 |x|^2} f$. Lemma 2.2 implies (for a suitable sequence of ρ's) that

$$\lim_{\rho \to 0} \int_{\pi^\perp} | \xi'' |^k \, | \hat{f}_\rho - \hat{f} |^2 \, d\xi'' = 0 \qquad \text{for a.e. } \pi, \tag{3.9}$$

and Theorem 2.5 gives (for a suitable sequence of ρ's) that

$$L_\pi f_\rho \to L_\pi f \qquad \text{in} \qquad L^q(\pi^\perp) \quad \text{for a.e. } \pi,$$

and hence that

$$L_\pi f_\rho \to L_\pi f \quad \text{in} \quad \mathscr{S}'(\pi^\perp) \quad \text{for a.e. } \pi, \tag{3.10}$$

where $\mathscr{S}'(\pi^\perp)$ denotes the tempered distributions in $\pi^\perp$. Since the Fourier transform is a topological isomorphism on $\mathscr{S}'$, and $f_\rho \in L^1$, Lemma 1.4 and 3.10 show that

$$(2\pi)^{k/2} \hat{f}_\rho \to (L_\pi f)\hat{} \quad \text{in} \quad \mathscr{S}'(\pi^\perp) \quad \text{for a.e. } \pi. \tag{3.11}$$

Now it follows from (3.9) that

$$(L_\pi f)\hat{} = (2\pi)^{k/2} \hat{f} \ \text{ a.e.} \quad \text{on} \quad \pi^\perp$$

for any π for which (3.9) and (3.11) hold. Since $f \in D_k$, Lemma 2.2 shows that $| \xi |^{-k/2} \hat{f} \in L^2$, and the theorem is proved for the case $k < n/2$.

To prove the second part of the theorem, we construct a function $g \in L^2(R^n)$ such that $| \xi |^{-k/2} \hat{g} \in L^2$ but g is not Lebesgue integrable over any plane of dimension $k \geqslant n/2$. Let f be the function in Remark 2.6. Define g_0 by

$$\begin{aligned} g_0(x) &= f(x), 2m_j \leqslant x_j \leqslant 2m_j + 1, \qquad j = 1,\ldots, n, \\ &= 0, \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise} \end{aligned}$$

where the m_j's run through the integers. (For example, on the line, $g_0 = 0$ on alternate intervals between integers.) Let $e_1, \ldots, e_n$ be the unit vectors along the axes, and put

$$g_l(x) = g_{l-1}(x + e_l) - g_{l-1}(x).$$

One can easily check that g_n is not Lebesgue integrable over any plane of dimension $\geqslant n/2$. But

$$\hat{g}_n(\xi) = (e^{i\xi_1} - 1) \cdots (e^{i\xi_n} - 1)\, \hat{g}_0(\xi),$$

which gives the desired result since

$$(e^{i\xi_1} - 1) \cdots (e^{i\xi_n} - 1) \mid \xi \mid^{-n}$$

is bounded.

THEOREM 3.12. *The X-ray transform with domain $C_0^\infty(R^n)$ admits a closure $\bar{L}$ with domain $\tilde{D}_k$ ($= D_k$ for $k < n/2$ and $\neq D_k$ for $k \geqslant n/2$).*

Proof. Assume that $f_m \in C_0^\infty(R^n)$, $f_m \to f$ in $L^2(R^n)$, and that $Lf_m \to g$

in $L^2(T)$. Lemma 2.2 and Parseval's relation on $\pi^\perp$ show that for almost every π (and an appropriate subsequence if necessary),

$$(2\pi)^{k/2} \hat{f}_m \to \hat{g}_\pi \quad \text{a.e.} \quad \text{in} \quad \pi^\perp.$$

Also, for an appropriate sequence of m

$$\hat{f}_m \to \hat{f} \quad \text{a.e.} \quad \text{in} \quad R^n.$$

It follows that

$$\hat{g}_\pi = (2\pi)^{k/2} \hat{f} \quad \text{a.e.} \quad \text{on} \quad \pi^\perp \quad \text{for a.e. } \pi. \tag{3.13}$$

In particular, if $g = 0$, then $f = 0$, so $\bar{L}$ exists. Moreover, since $g \in L^2(T)$, it follows from Lemma 2.2 and 3.13 that $|\xi|^{-k/2} \hat{f} \in L^2(R^n)$. Thus $f \in \tilde{D}_k$.

Finally suppose that $f \in \tilde{D}_k$. The proof of Theorem 3.4 (up to (3.7) which is valid for all k) shows that there is a sequence f_m in the Schwartz space $\mathscr{S}(R^n)$ such that $f_m \to f$ in $L^2(R^n)$ and Lf_m converges in $L^2(T)$. Thus it suffices to show that $\mathscr{S}(R^n)$ is in the domain of $\bar{L}$. But this is immediate from (3.2).

Note that if $k < n/2$, then the closure of the X-ray transform is defined by the integral (1.1) for almost every π. However, when $k \geqslant n/2$, the integral may not exist and in this case $Lf = g$, where g is defined by (3.13). From now on the same symbol L wil be used to denote both the X-ray transform and its closure.

The Sobolev spaces $H^s(T)$ on the fiber bundle $T(G_{n,k})$ consist of the measurable functions g which satisfy

$$\| g \|_s^2 = \int_{G_{n,k}} \int_{\pi^\perp} (1 + |\xi''|^2)^s \, |\hat{g}_\pi(\xi'')|^2 \, d\xi'' \, d\mu < \infty.$$

COROLLARY 3.14. *If $f \in \tilde{D}_k$, $k > n/2$, then for almost every k-space π, $L_\pi f$ is continuous on $\pi^\perp$ (after being altered on a set of measure zero).*

Proof. Lemmas 1.4 and 2.2 and Theorem 3.12 show that Lf is in $H^{k/2}(T)$, and thus is in $H^{k/2}(\pi^\perp)$ for almost every π. Since $k > n/2$, it follows that $k/2 > (n - k)/2$ and the theorem of Sobolev [29] shows that $L_\pi f$ can be made continuous by a change on a set of measure zero for such π.

4. INVERSION FORMULAS

In Theorem 2.4 it was shown that the extension V of $(2\pi)^{-k/2} (\Lambda^{k/2} L)$ is an isometry from $L^2(R^n)$ into $L^2(T)$. It is well known that for any isometry $V^* V = I$. Thus if $Lf = g$, then

$$f = (2\pi)^{-k} (\Lambda^{k/2} L)^* \Lambda^{k/2} g.$$

Formally the operators L and Λ commute. The purpose of this section is to show that for a proper permutation of the operators L^* and Λ the domains are correct and meaningful inversion formulas can be given for all $f \in \tilde{D}_k$. An inversion formula was given by Helgason [13] for $f \in C_0^\infty(R^n)$ when k is even. Also, in the proof of [14, Theorem 8.2], Helgason derived an inversion formula which is valid in all dimensions for a special subspace of $\mathscr{S}(R^n)$.

Suppose that $f \in L_0^2(R^n)$, $g \in L^2(T)$ are nonnegative functions and that $P_{\pi^\perp}$ is the projection on $\pi^\perp$. Then Fubini's theorem gives that

$$\langle Lf, g\rangle = \int_{G_{n,k}} \int_{\pi^\perp} L_\pi f(x'') \overline{g(\pi, x'')}\, dx''\, d\mu$$

$$= \int_{R^n} f(x) \overline{\int_{G_{n,k}} g(\pi, P_{\pi^\perp} x)\, d\mu}\, dx = \langle f, L^\# g\rangle,$$

where

$$L^\# g(x) = \int_{G_{n,k}} g(\pi, P_{\pi^\perp} x)\, d\mu.$$

The following result has been established.

LEMMA 4.1. *For every $g \in L^2(T)$, $L^\# g$ is defined almost everywhere and is locally square integrable. Moreover, g is in the domain of the adjoint of the X-ray transform L^* if and only if $L^\# g$ is globally square integrable, in which case $L^* g = L^\# g$.*

It is worth noting that the domain of L^* is rather peculiar, especially when $k \geqslant n/2$. Indeed, if g is a nontrivial nonnegative function and $k \geqslant n/2$, then there exists a nonnegative $f \in L^2(R^n)$ which is not integrable over any k-plane (Remark 2.6). Choose a sequence of nonnegative functions $f_n \in L_0^\infty(R^n)$ such that $f_n \nearrow f$. Then

$$\infty = \langle Lf, g\rangle = \lim_{n\to\infty} \langle Lf_n, g\rangle = \lim_{n\to\infty} \langle f_n, L^\# g\rangle = \langle f, L^\# g\rangle.$$

Thus $L^\# g$ cannot be globally square integrable. Hence the condition that $L^\# g$ be square integrable depends entirely on cancellation. On the other hand, if $k < n/2$, then it follows from Theorem 2.5 that any $g \in L^2(T)$ with compact support is in the domain of L^*.

THEOREM 4.2. *If $f \in \tilde{D}_k$ and $Lf = g$, then*

$$(2\pi)^{-k} \Lambda^{k/2} L^* \Lambda^{k/2} g = f.$$

(Note that the $\Lambda^{k/2}$ on the left acts in R^n while the one on the right acts on each fiber in $T(G_{n,k})$.)

Proof. First it is necessary to show that $\Lambda^{k/2}g$ is in the domain of L^*. Let $h \in C_0^\infty(R^n)$. Parseval's relation, Theorem 3.4, and Lemma 2.2 give that

$$\langle Lh, \Lambda^{k/2}g\rangle_{L^2(T)} = (2\pi)^k \langle \hat{h}, | \xi'' |^{k/2} \hat{f}\rangle_{L^2(T)}$$
$$= (2\pi)^k \langle \hat{h}, | \xi |^{-k/2} \hat{f}\rangle_{L^2(R^n)} = (2\pi)^k \langle h, u\rangle_{L^2(R^n)},$$

where

$$\hat{u} = | \xi |^{-k/2} \hat{f}. \tag{4.3}$$

Note that the last equality holds by Parseval's relation on R^n and the fact that $f \in \tilde{D}_k$. Thus $\Lambda^{k/2}g$ is in the domain of L^* and

$$L^*\Lambda^{k/2}g = (2\pi)^k u.$$

The theorem follows from the definition of Λ and (4.3).

Actually there are several variations of the inversion formula given above depending on the regularity of f. Let us define

$$\mathscr{D}_s = \{f \in L^2(R^n) : | \xi |^s \hat{f} \in L^2(R^n)\}.$$

Notice that $\mathscr{D}_{-k/2} = \tilde{D}_k$. For $s \geqslant 0$, $\mathscr{D}_s$ is simply the Sobolev space H^s, but for $s < 0$ this is not so. In general $\mathscr{D}_s$ is a strictly decreasing function of $| s |$. The following can be established by the methods used in proving Theorem 4.2.

THEOREM 4.4. *If $f \in \mathscr{D}_{(s-k)/2} \cap \mathscr{D}_{-k/2}$ for $s \geqslant k$, or $f \in \mathscr{D}_{(s/2)-k}$ for $s < k$, and $Lf = g$, then*

$$(2\pi)^{-k} \Lambda^{k-s/2} L^* \Lambda^{s/2} g = f. \tag{4.5}$$

Setting $s/2 = k = n - 1$ in (4.5) gives the classical inversion formula for the Radon transform.

5. The Support of f and Lf

Another critical value of k appears in the relationship between the supports of f and Lf. When $k \leqslant n - 2$ and f is integrable, then f has compact support if and only if Lf has compact support. Moreover, in this case one can sometimes get inside the convex hull of the support of f from a knowledge of the support of Lf.

Until further notice K will designate an n-dimensional compact, convex subset of R^n.

THEOREM 5.1. *Let $f \in L^1(R^n)$ and $k \leqslant n-2$. Then f has support in K if and only if $L_\pi f(x'') = 0$ whenever $x'' + \pi$ does not intersect K. The result fails when $k = n-1$ unless f is assumed a priori to have compact support.*

Proof. Assuming that $L_\pi f(x'') = 0$ for almost every $x'' \in \pi^\perp$ such that $x'' + \pi$ does not intersect K and that $k \leqslant n-2$, we shall conclude that f has support in K (the converse being obvious). Let H be any half-space not meeting K and let V be the subspace parallel to the boundary. Since K is convex it is sufficient to show that

$$\begin{aligned} h(x) &= f(x) \qquad \text{if} \qquad x \in H, \\ &= 0 \qquad\quad\ \ \text{otherwise} \end{aligned}$$

is identically 0; and for this it is sufficient, according to Corollary 1.6, to show that $L_\pi h = 0$ for all $\pi \subset V$. Now, if $x'' \notin H$ then $x'' + \pi$ does not meet H, so $L_\pi h(x'') = 0$, while if $x'' \in H$, then $x'' + \pi \subset H$, so $L_\pi h(x'') = L_\pi f(x'') = 0$.

Now assume that $k = n-1$. Choose an integrable C^∞ function f, without compact support and with $\Delta f = 0$ for $|x| > 1$, (e.g., $f(x) = \partial^3 |x|^{2-n}/\partial x_n^3$ for $|x| > 1, n > 2$). Let π be an arbitrary $(n-1)$-dimensional subspace. We will show that $L_\pi f(x'') = 0$ if $|x''| > 1$. If t is the coordinate on the line orthogonal to π, it follows from (1.5) that

$$d^2 L_\pi f/dt^2 = L_\pi(\Delta f) = 0 \qquad \text{for } |t| > 1.$$

But, since $L_\pi f$ is an integrable function of t, it cannot be a nonzero linear function of t on $t > 1$, and $t < -1$. Thus

$$L_\pi f(t) = 0, \qquad |t| > 1,$$

and the theorem is proved, except for the final case where $k = n-1$ and f is assumed a priori to have compact support. This is covered in Corollary 5.5 below.

The idea for the proof of the next theorem comes from [20, Lemma 3.1].

THEOREM 5.2. *Let π_0 be a k-space, $0 < k \leqslant n-1$, and V be an open, connected, unbounded subset of $\pi_0^\perp$. If $f \in L^1(R^n)$ has compact support and Lf vanishes in a neighborhood of $(\pi_0, V) = \{(\pi_0, x'') \in T(G_{n,k}): x'' \in V\}$ then f vanishes on $\pi_0 + V$.*

Note that when $k = n-1$, $\pi_0^\perp$ is one-dimensional. Thus V is a half-line,

$\pi_0 + V$ is a half-space, and $R^n \backslash (\pi_0 + V)$ is convex. However, when $k \leqslant n-2$, $\pi_0{}^\perp$ has dimension $\geqslant 2$ and V can be more interesting. In particular, $R^n \backslash (\pi_0 + V)$ need not be convex. Thus, when $k \leqslant n-2$, it is possible, in some cases, to get inside the convex hull of the support of f.

Proof. First assume that $f \in C_0^\infty(R^n)$. Without loss of generality, assume that $\pi_0 = [e_1, \ldots, e_k]$, the k-space spanned by $e_1, \ldots, e_k$. Choose neighborhoods $N_1, \ldots, N_k$ of $e_1, \ldots, e_k$ such that whenever $w = (w_1, \ldots, w_k) \in N_1 \times \cdots \times N_k$, then $\pi_w = [w_1, \ldots, w_k]$ is a subspace of dimension k. For $y \in R^n$ define

$$\widetilde{L_w} f(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y + t_1 w_1 + \cdots + t_k w_k)\, dt_1 \cdots dt_k .$$

Note that

$$\widetilde{L_w} f(y) = \frac{1}{|w_1| \cdots |w_k|} L_{\pi_w} f(y'')$$

and $\widetilde{L_w} f$ vanishes in a neighborhood of each point $(y, w_1, \ldots, w_k)$ with $y \in \pi_0 + V$ and $w_i = b_i e_i \in N_i$. If $w_i = (w_{i1}, \ldots, w_{in}) \in N_i$ and $y \in \pi_0 + V$ is fixed, then

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial w_{1i}}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial w_{ki}}\right)^{\alpha_k} (\widetilde{L_w} f(y)) \Big|_{w=(e_1,\ldots,e_k)} \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} t_1^{\alpha_1} \cdots t_k^{\alpha_k} \frac{\partial^{|\alpha|} f}{\partial x_i^{|\alpha|}} (y + t_1 e_1 + \cdots + t_k e_k)\, dt_1 \cdots dt_k , \end{aligned}$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_k) \qquad \text{and} \qquad |\alpha| = \sum_{i=1}^{k} \alpha_i .$$

Letting y vary in $\pi_0 + V$ we see that for each i

$$\frac{\partial^{|\alpha|}}{\partial y_i^{|\alpha|}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} t_1^{\alpha_1} \cdots t_k^{\alpha_k} f(y + t_1 e_1 + \cdots + t_k e_k)\, dt_1 \cdots dt_k = 0.$$

Thus

$$P_\alpha(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} t_1^{\alpha_1} \cdots t_k^{\alpha_k} f(y + t_1 e_1 + \cdots + t_k e_k)\, dt_1 \cdots dt_k \tag{5.3}$$

is a polynomial of degree less than $|\alpha|$ in y in the domain $\pi_0 + V$. Since f has compact support $P_\alpha(y) = 0$ for sufficiently large $y \in \pi_0 + V$. Since $\pi_0 + V$ is connected P vanishes identically on $\pi_0 + V$. But then (5.3) shows that for all $y \in V$, the function $f(y + t_1 e_1 + \cdots + t_k e_k)$ is orthogonal to all polynomials in the variables $t_1, \ldots, t_k$. Since f has compact support f vanishes on $\pi_0 + V$.

To pass from $f \in C_0^\infty(R^n)$ to $f \in L^1(R^n)$, note that

$$L_\pi(f * h) = (L_\pi f) * (L_\pi h) \tag{5.4}$$

where $*$ denotes convolution, and use standard regularization techniques.

COROLLARY 5.5. *If f is an integrable function with compact support and for each $(n-1)$-dimensional subspace π, $L_\pi f(x'') = 0$ whenever $x'' + \pi$ does not intersect K, then f has support in K.*

Proof. If $k = n - 1$ in Theorem 5.2, then V is a half-line and $\pi_0 + V$ is a half-space. Theorem 5.2 implies that f vanishes on each half-space not intersecting K. Since K is convex f has support in K.

6. THE RANGE OF L

Ludwig [20] and Lax and Phillips [19] derived necessary and sufficient conditions that a measurable function $g(\theta, t)$, $\theta \in S^{n-1}$, $t \in R^1$, be the Radon transform of a square integrable function with support in an n-dimensional compact, convex set. The same is done here for the X-ray transform. As might be expected from Theorem 5.1, the theorem of Ludwig, $k = n - 1$, is more subtle than that of the lower dimensions. The proof given here is valid only for $k < n - 1$.

LEMMA 6.1. *If $f \in L^1(R^n)$ has compact support and*

$$P_{\pi,m}(\xi'') = \int_{\pi^\perp} \langle \xi'', x'' \rangle^m L_\pi f(x'')\, dx'', \qquad \xi'' \in \pi^\perp, \tag{6.2}$$

then for each nonnegative integer m there is a homogeneous polynomial P_m of degree m on R^n such that

$$P_m(\xi) = P_{\pi,m}(\xi), \qquad \textit{for } \xi \in \pi^\perp.$$

Proof. Letting $\rho(t) = t^m$ in Lemma 1.2, it follows that the polynomial

$$P_m(\xi) = P_m(f, \xi) = \int_{R^n} \langle x, \xi \rangle^m f(x)\, dx \tag{6.3}$$

does the job.

Note that the polynomials (6.3) can also be defined for $\zeta \in \mathbb{C}^n$ simply by replacing ξ by ζ in the formula. The following result shows the relationship between the Fourier transform of f and the polynomials P_m.

LEMMA 6.4. *If $f \in L^1(R^n)$ has compact support, then the Fourier transform of f extends to a complex analytic function on $\mathbb{C}^n$ with the expansion*

$$\hat{f}(\zeta) = (2\pi)^{-n/2} \sum_{m=0}^{\infty} (i^{-m} P_m(\zeta)/m!). \tag{6.5}$$

Proof. The extension of the Fourier transform is given by the Laplace transform

$$\hat{f}(\zeta) = (2\pi)^{-n/2} \int_{R^n} e^{-i\langle x,\zeta\rangle} f(x)\, dx.$$

The integral converges absolutely for every $\zeta \in \mathbb{C}^n$ even after differentiating under the integral sign and does determine an entire function on $\mathbb{C}^n$.

On the other hand,

$$e^{-i\langle x,\zeta\rangle} = \sum_{m=0}^{\infty} (i^{-m} \langle x, \zeta\rangle^m / m!),$$

and term-by-term integration gives the right-hand side of (6.5).

In proving the theorem characterizing the range of L for dimensions $k < n - 1$, the classical Paley–Wiener theorem [29] will be used. An improved version of the Paley–Wiener theorem has been established by Smith [28], which allows the omission of condition (iv) in Theorem 6.6.

THEOREM 6.6. *Let $g(\pi, x'')$ be a measurable function on $T(G_{n,k})$. There exists a square integrable function f with support in K such that $Lf = g$ if and only if*

(i) $g \in H^{k/2}(T)$;

(ii) (*compatibility conditions*) *for each nonnegative integer m there exists a homogeneous polynomial P_m of degree m on R^n such that $P_m(\xi'') = P_{\pi,m}(\xi'')$ for each $\pi \in G_{n,k}$ and $\xi'' \in \pi^\perp$, where $P_{\pi,m}$ is defined as in* (6.2) *with $L_\pi f$ replaced by g_π;*

(iii) $g(\pi, x'') = 0$ *whenever $x'' + \pi$ does not meet K;*

(iv) *there exists a constant $c > 0$ such that $\| g_\pi(x'')\|_{L^1(\pi^\perp)} < c$ for all $\pi \in G_{n,k}$.*

Proof. When $k = n - 1$, this is the theorem of Ludwig and the proof given here fails. The necessity of the conditions follows from Theorem 2.4, Lemma 6.1, Theorem 5.1, and Fubini's theorem. It remains to establish the sufficiency. Assume that $k \leqslant n - 2$.

Condition (iii) and Lemma 6.4 show that $\hat{g}_\pi$ is given by the series (6.5) with P_m replaced by $P_{\pi,m}$ and the constant $(2\pi)^{-n/2}$ replaced by $(2\pi)^{(k-n)/2}$. Using the polynomials P_m provided by (ii), define $\hat{f}$ on R^n by the series (6.5). The relationship between P_m and $P_{\pi,m}$ guarantees that the series defining $\hat{f}$ converges absolutely for every $\xi \in R^n$ and that

$$\hat{f}(\xi) = (2\pi)^{-k/2}\, \hat{g}_\pi(\xi) \qquad \text{for } \xi \in \pi^\perp. \tag{6.7}$$

Condition (i) and Lemma 2.2 show that $\hat{f}$ is square integrable and that the inverse Fourier transform, f, is in $\tilde{D}_k$. Thus

$$(L_\pi f)\hat{} = (2\pi)^{k/2}\hat{f} = \hat{g}_\pi \quad \text{a.e.} \qquad \text{on} \qquad \pi^\perp \quad \text{for a.e. } \pi. \tag{6.8}$$

Thus $Lf = g$. It remains to show that f has support in K. It has already been established that $\hat{f}$ is square integrable and analytic on R^n. In order to apply the classical Paley–Wiener theorem, it is necessary to check the growth of the entire extension of $\hat{f}$ to $\mathbb{C}^n$. (It is here that the proof fails when $k = n - 1$.)

Conditions (iii), (iv), and the Paley–Wiener theorem show that for each π, $\hat{g}_\pi$ has an entire extension to

$$\pi_{\mathbb{C}}^\perp = \{\zeta \in \mathbb{C}^n : \zeta = \xi + i\eta,\ \xi \in \pi^\perp,\ \eta \in \pi^\perp\}$$

such that

$$|\hat{g}(\pi, \zeta)| \leqslant ce^{r|\eta|}, \tag{6.9}$$

where r is chosen so that K is contained in the ball $B(0, r)$. But, by (6.7) the analytic extension of $\hat{f}$ to $\mathbb{C}^n$ must agree with that of $(2\pi)^{-k/2}\hat{g}_\pi$ on $\pi_{\mathbb{C}}^\perp$. However, when $k \leqslant n - 2$, each $\zeta \in \mathbb{C}^n$ is in $\pi_{\mathbb{C}}^\perp$ for some π. Thus the analytic extension of $\hat{f}$ to $\mathbb{C}^n$ satisfies the inequality (6.9) and the classical Paley–Wiener theorem implies that f has support in the ball $B(0, r)$. Now, f has support in K by Theorem 5.1.

7. An Iterative Scheme and Some Comments on the Three-Dimensional Reconstruction Problem

The research in this paper was inspired by the work in [10, 28], on the detection of brain tumors with ordinary hospital equipment, and without the introduction of contrast material. This section deals with an iterative scheme for solving the n-dimensional reconstruction problem. In addition some practical comments on the brain tumor work and the three-dimensional reconstruction problem in general are made.

Let K be a fixed compact subset of R^n. (We no longer assume that K is convex.) The square integrable functions on R^n which vanish outside K will be denoted by $L_0^2(K)$. Now, we consider L_π as an operator on $L_0^2(K)$.

THEOREM 7.1. *The null space $\mathcal{N}_\pi$ of L_π is a closed subspace of $L_0^2(K)$ whose orthogonal complement consists of the functions that are constant on k-planes parallel to π.*

Proof. It can easily be checked that L_π is continuous on $L_0^2(K)$ and so $\mathcal{N}_\pi$ is closed. A function h is constant on k-planes parallel to π if and only if h is a function of x'' alone, where as usual x'' is the projection of x on $\pi^\perp$. Suppose that $f = \mathscr{X}h$ where $\mathscr{X}$ is the characteristic function of K and h is a function of x'' alone. If $g \in \mathcal{N}_\pi$, then

$$\langle g, f\rangle = \int_{R^n} g\overline{\mathscr{X}h}\, dx = \int_{\pi^\perp} h \int_\pi g\, dx'\, dx'' = 0$$

since the inner integral is $L_\pi g(x'') = 0$.

Suppose that $f \in \mathcal{N}_\pi^\perp$. By the first half of the proof, it is sufficient to find a function h of x'' alone such that

$$L_\pi f = L_\pi \mathscr{X}h.$$

To find h, note that $L_\pi f = L_\pi \mathscr{X}h$ if and only if

$$\int_\pi f(x', x'')\, dx' = \int_\pi \mathscr{X}(x', x'')\, h(x'')\, dx' = \delta(x'')\, h(x'') \tag{7.2}$$

where $\delta(x'') = L_\pi \mathscr{X}(x'')$. Define $h(x'')$ by the above formula. It remains to show that $\mathscr{X}h \in L^2(K)$. Now

$$\int_{R^n} |\, \mathscr{X}h\,|^2\, dx = \int_\pi \delta(x'')\, |\, h(x'')|^2\, dx''. \tag{7.3}$$

But since $\mathscr{X}f = f$, (7.2) and the Cauchy–Schwarz inequality give that

$$(\delta(x''))^2\, |\, h(x'')|^2 = \left| \int_\pi f(x', x'')\, dx' \right|^2 \leqslant \delta(x'') \int_\pi |\, f\,|^2\, dx'.$$

It follows from (7.3) that $\|\mathscr{X}h\|_{L^2}^2 \leqslant \|f\|_{L^2}^2$ and the theorem is proved.

COROLLARY 7.2. *Let $f_0 \in L_0^2(K)$. The orthogonal projection P_π on the plane $f_0 + \mathcal{N}_\pi$ is given by the formula*

$$P_\pi g = g + \mathscr{X}(L_\pi f_0 - L_\pi g)/L_\pi \mathscr{X},$$

where $\mathscr{X}$ is the characteristic function of K.

In practice one is given a finite number of X rays, $L_{\pi_1} f_0, \ldots, L_{\pi_j} f_0$, of an unknown object f_0, and one would like to find f_0, or at least an object f with the correct X rays. The objects with the ith X ray correct are those in the plane $f_0 + \mathcal{N}_{\pi_i}$, and the objects with all j X rays correct are those in the plane $f_0 + \mathcal{N}_0$, $\mathcal{N}_0 = \bigcap_{i=1}^{j} \mathcal{N}_{\pi_i}$. According to Corollary 7.2 the orthogonal projection P_i on $f_0 + \mathcal{N}_{\pi_i}$ is computable, and there is the following theorem due to Kacmarz [6] in the finite-dimensional case and to Amemiya and Ando [1] in the infinite-dimensional case.

THEOREM 7.3. *Let $\mathcal{N}_1, \ldots, \mathcal{N}_J$ be closed subspaces of a Hilbert space $\mathcal{H}$ with intersection $\mathcal{N}_0$. Let P_i be the orthogonal projection on the plane $f_0 + \mathcal{N}_i$ and let $\underset{\sim}{Q} = P_J \cdots P_1$. For every $g \in \mathcal{H}$, $\underset{\sim}{Q}^m g \to P_0 g$.*

Thus $\underset{\sim}{Q}^m g$ is a computable approximate solution to the problem of finding an object, $P_0 g$, with the correct X rays. The initial guess g can be chosen arbitrarily, and poor choices of course provide poor solutions, far from the true solution f_0. In specific practical problems criteria for choosing g are lacking. In the brain tumor problem the best results have been obtained with $g = 0$, which leads to the solution with minimum norm in $L_0^2(K)$.

There are many reasonable reconstruction techniques [8], any one of which picks out an object with the same, or nearly the same, X rays. However, Theorem 1.8 shows that such objects are arbitrary, at least in the interior. Thus, it seems that the reconstruction technique must be suited to the problem so that it picks out a solution close to the solution of nature. For example, the EMI scanner initially used the technique of Theorem 7.3 with initial guess $g = 0$. The demonstrated effectiveness of the EMI scanner in detecting brain tumors [15] suggests that this is a good method *in the case of heads*. However, this technique may not be successful in other reconstruction problems.

The rate of convergence of the iterative scheme of projections in Theorem 7.3 is of practical interest. When the angles between the subspaces are positive, the following theorem gives an upper bound for the rate of convergence.

THEOREM 7.4. *For each i let α_i be the angle between $\mathcal{N}_i$ and $\mathcal{N}_{i+1} \cap \cdots \cap \mathcal{N}_J$. Then*

$$\| \underset{\sim}{Q}^m g - P_0 g \|^2 \leqslant \left(1 - \prod_{i=1}^{J-1} \sin^2 \alpha_i\right)^m \| g - P_0 g \|^2.$$

In practice the angles between the X rays are known and one would like to know the relationship between these angles and the angles between the corresponding null spaces. When K is the unit disc in R^2 the relationship between the angle between two X rays and the angle between the corresponding null spaces is given by the following theorem.

THEOREM 7.5. *Let K be the unit disc in R^2. If the angle between the X rays L_{θ_1} and L_{θ_2} is θ, then the angle α between the null spaces $\mathcal{N}_{\theta_1}$ and $\mathcal{N}_{\theta_2}$ in $L_0^2(K)$ is given by*

$$\cos \alpha = \sup_{n>0} \left| \frac{\sin(n+1)\,\theta}{(n+1)\sin\theta} \right|. \tag{7.6}$$

Note that if $\theta = \pi/2$, then $\cos \alpha = \frac{1}{3}$. Moreover, the sup in (7.6) need only be taken over finitely many n. The proof of Theorem 7.4 is given in [28], while that of Theorem 7.5 and a formula for all the angles α_i between the null spaces is given in [11].

We conclude this paper with two short remarks concerning some practical applications of Theorem 6.6 to the reconstruction problem.

A small, but useful, application of Theorem 6.6 arises in the problem of matching data from radiographs accurately. Consider the problem of reconstructing a cross section of a patient's head from ordinary X-ray data. The head may be considered as a positive density function with support in a ball B. It is of evident importance to match the several X rays accurately. With due clinical level precautions, the patient's head can still be expected to move around some, the films can be expected to vary in their cassettes, etc. Theorem 6.6 shows that the orthogonal projection of the center of gravity of the object f on the plane $\theta^\perp$ is the center of gravity of the projection $L_\theta f$. Thus, the center of gravity of each $L_\theta f$ can be shifted to the geometric center of the projection of B, and the effect will be to shift the center of gravity of f to the geometric center of B. This technique of matching the X rays accurately was used in [10, 28].

The compatibility conditions of Theorem 6.6 may also be of practical use. Data from X rays are invariably noisy, and the compatibility conditions may be useful in measuring the extent of the noise. The functions in practice are positive, and the compatibility condition for $m = 0$ simply says that all the L_{θ_i} should have the same L^1 norm. This can easily be checked. Alternatively, one might normalize the data so that the 0th compatibility condition is satisfied. (This normalization has been done in some of the joint work on detecting brain tumors with Guenther and Smith, but more work is necessary to determine the effect on the reconstruction.) Similar steps might be taken for any finite number of the compatibility conditions.

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