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Exponential solutions of equation $\dot{y}(t) = \beta(t)[y(t-\delta) - y(t-\tau)]$

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Abstract

Asymptotic behaviour of solutions of first-order differential equation with two deviating arguments of the form

$$\dot{y}(t) = \beta(t)\big[y(t-\delta) - y(t-\tau)\big]$$

is discussed for $t \to \infty$. A criterion for representing solutions in exponential form is proved. As consequences, inequalities for such solutions are given. Connections with known results are discussed and a sufficient condition for existence of unbounded solutions, generalizing previous ones, is derived. An illustrative example is considered, too.

Keywords: Discrete delay; Two deviating arguments; Exponential solution; Unbounded solution

1. Introduction

In this paper we discuss for $t \to \infty$ asymptotic behaviour of solutions to a linear homogeneous differential equation with two delayed terms containing discrete delays

$$\dot{y}(t) = \beta(t)\big[y(t-\delta) - y(t-\tau)\big], \tag{1}$$

where $\delta, \tau \in \mathbb{R}^+$, $\mathbb{R}^+ := (0, +\infty)$, $\tau > \delta$, $\beta : I_{-1} \to \mathbb{R}^+$ is a continuous function and $I_{-1} := [t_0 - \tau, \infty)$, $t_0 \in \mathbb{R}$. The symbol "$\dot{}$" or "$'$" denotes (at least) the *right-hand* deriv-

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ative. Similarly, if necessary, the value of a function at a point of I_{-1} is understood (at least) as value of the corresponding limit *from the right*. Denote $I := [t_0, \infty)$.

Problems of asymptotic behaviour of linear delayed functional differential equations have been intensively studied recently due to their numerous applications. In the paper presented we are trying to represent solutions of Eq. (1) by means of exponential-like functions

$$\exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,ds\right] \tag{2}$$

with a continuous function $\tilde{\varepsilon} : I_{-1} \setminus \{t_0\} \to (0, 1)$. We call representation (2) *exponential representation* (being aware of that, e.g., for functions $\tilde{\varepsilon}$ close to 0, this form can give a different kind of a function than just exponential ones).

Let $\mathcal{C} := C([-\tau, 0], \mathbb{R})$ be Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$ equipped with the supremum norm.

A function $y(t)$ is said to be a *solution of Eq.* (1) *on* $[\nu - \tau, \nu + A)$ with $\nu \in I$ and $A > 0$, if $y \in C([\nu - \tau, \nu + A), \mathbb{R}) \cap C^1([\nu, \nu + A), \mathbb{R})$, and $y(t)$ satisfies the Eq. (1) for $t \in [\nu, \nu + A)$.

For given $\nu \in I$, $\varphi \in \mathcal{C}$, we say that $y(\nu, \varphi)$ is a *solution* of Eq. (1) *through* (ν, φ) (or that $y(\nu, \varphi)$ *corresponds to the initial point* ν), if there is $A > 0$ such that $y(\nu, \varphi)$ is a solution of Eq. (1) on $[\nu - \tau, \nu + A)$ and $y(\nu, \varphi)(\nu + \theta) = \varphi(\theta)$ for $\theta \in [-\tau, 0]$.

Due to linearity of Eq. (1), the solution $y(\nu, \varphi)$ is unique and is defined on $[\nu - \tau, \infty)$, i.e., we can put $A := \infty$.

Let us note that close investigation of asymptotic behaviour of a solution of delayed functional differential equations is performed, e.g., in the papers [1–21]. The studied Eq. (1) occurs, e.g., in number theory [20].

The paper is organized as follows. In Section 2 a basic auxiliary inequality is studied and the relationship of its solutions with solutions of Eq. (1) is established. Several necessary auxiliary lemmas are given in Section 3. Section 4 contains main results of the paper as well as comparisons with the known results. An illustrative example is considered in Section 5 and a still unsolved problem is formulated in Section 6.

2. An auxiliary inequality

The inequality

$$\dot{\omega}(t) \leqslant \beta(t)\big[\omega(t-\delta) - \omega(t-\tau)\big], \tag{3}$$

which formally copies Eq. (1), will play a basic role in our investigation. A function $\omega(t)$ is said to be a *solution of* (3) *on* $[\nu - \tau, \nu + A)$ with $\nu \in I$ and $A > 0$, if $\omega \in C([\nu - \tau, \nu + A), \mathbb{R}) \cap C^1([\nu, \nu + A), \mathbb{R})$, and $\omega(t)$ satisfies the inequality (3) for $t \in [\nu, \nu + A)$.

2.1. Relationship between solutions of Eq. (1) and inequality (3)

Below we will discuss relations between solutions of Eq. (1) and inequality (3).

Theorem 1.

(a) *Suppose that* $y(t)$ *is a solution of Eq.* (1) *on* I_{-1}. *Then there exists a solution* $\omega(t)$ *of inequality* (3) *on* I_{-1} *such that an inequality*

$$y(t) > \omega(t) \tag{4}$$

holds on I_{-1}.

(b) *Suppose that* $\Omega(t)$ *is a solution of inequality* (3) *on* I_{-1}. *Then there exists a solution* $Y(t)$ *of Eq.* (1) *on* I_{-1} *such that an inequality*

$$Y(t) > \Omega(t) \tag{5}$$

holds on I_{-1}.

Proof. (a) Proof of this part is trivial. Let $y(t)$ be a solution of Eq. (1) on I_{-1}. Let us put, e.g., $\omega(t) := y(t) - 1$. Obviously, $\omega(t)$ solves the inequality (3) on I_{-1} and satisfies here the inequality (4), too.

(b) Let $\Omega(t)$ be a solution of inequality (3) on I_{-1}. Define $\Phi \in \mathcal{C}$ by means of the relation

$$\Phi(\theta) := \Omega(t_0 + \theta), \quad \theta \in [-\tau, 0]. \tag{6}$$

Let us show that the corresponding solution $y(t) := Y_1(t_0, \Phi)(t)$ of Eq. (1) satisfies inequality

$$Y_1(t_0, \Phi)(t) \geqslant \Omega(t) \tag{7}$$

on I_{-1}. Suppose the contrary. Then there exists a point $t^* \in (t_0, \infty)$ such that $Y_1(t^*) < \Omega(t^*)$. We show that the last statement leads to the following conclusion: There exist points $t_1 \in I$, $t_2 \in I$, $t_0 \leqslant t_1 < t_2 < t^*$ such that

$$Y_1(t) = \Omega(t), \quad t \in [t_0, t_1], \tag{8}$$

$$Y_1(t) > \Omega(t), \quad t \in (t_1, t_2), \tag{9}$$

$$Y_1(t_2) = \Omega(t_2). \tag{10}$$

Define on I_{-1} a function

$$W(t) := Y_1(t) - \Omega(t).$$

Then due to inequality (3), we have on I,

$$\dot{W}(t) = \dot{Y}_1(t) - \dot{\Omega}(t) \geqslant \beta(t)\big[Y_1(t-\delta) - Y_1(t-\tau)\big] - \beta(t)\big[\Omega(t-\delta) - \Omega(t-\tau)\big].$$

Moreover, due to initial data (6), we conclude that at least on an interval $[t_0, t_0 + \delta]$, the inequality $\dot{W}(t) \geqslant 0$ holds since $Y_1(t-\delta) \equiv \Omega(t-\delta)$ and $Y_1(t-\tau) \equiv \Omega(t-\tau)$ for $t \in [t_0, t_0 + \delta]$. If $\dot{W}(t) = 0$ on the interval $[t_0, t_0 + \delta]$ then $Y_1(t) \equiv \Omega(t)$ on $[t_0 - \tau, t_0 + \delta]$. In this case we can identify a new point of the type t_0 with the point $t_0 + \delta$ and repeat the previous reasoning. If our conclusions can be repeated step by step infinitely many times and remain the same, then the inequality (7) holds on I_{-1} and turns into an identity. Now, since we were supposing the contrary, we conclude that $\dot{W}(t) \not\equiv 0$ on I. Then the above indicated points t_1 and t_2 exist and relations (8)–(10) hold. Then $W(t_1) = W(t_2) = 0$. In view of Rolle's theorem there exists a point $t_{12} \in (t_1, t_2)$ such that $\dot{W}(t_{12}) = 0$, i.e., $\dot{Y}_1(t_{12}) = \dot{\Omega}(t_{12})$. Without loss of generality we can suppose that $\dot{W}(t) > 0$ on (t_1, t_{12}). Moreover, due to (10) there exists a point $t_{122} \in [t_{12}, t_2)$ such that $\dot{W}(t) \geqslant 0$ on $[t_{12}, t_{122}]$ and

$$\dot{W}(t) < 0 \tag{11}$$

on $(t_{122}, t_{122} + \varepsilon)$ with sufficiently small positive ε. Suppose $\varepsilon < \delta$ and choose $t^\diamond \in (t_{122}, t_{122} + \varepsilon)$. Then with the aid of Lagrange's theorem

$$\begin{aligned}
\dot{W}(t^\diamond) &= \dot{Y}_1(t^\diamond) - \dot{\Omega}(t^\diamond) \\
&\geqslant \beta(t^\diamond)\big[Y_1(t^\diamond - \delta) - Y_1(t^\diamond - \tau)\big] - \beta(t^\diamond)\big[\Omega(t^\diamond - \delta) - \Omega(t^\diamond - \tau)\big] \\
&= \beta(t^\diamond)\big[W(t^\diamond - \delta) - W(t^\diamond - \tau)\big] = \beta(t^\diamond)(\tau - \delta)\dot{W}(c),
\end{aligned}$$

where $c \in (t^\diamond - \tau, t^\diamond - \delta)$. Since $\dot{W}(t) \geqslant 0$ on $[t_0 - \tau, t_{122}]$ and $(t^\diamond - \tau, t^\diamond - \delta) \subset [t_0 - \tau, t_{122})$ we get

$$\dot{W}(t^\diamond) \geqslant \beta(t^\diamond)(\tau - \delta)\dot{W}(c) \geqslant 0.$$

This is a contradiction with (11). Consequently, the inequality (7) is valid on I_{-1}. Now let us put

$$Y(t) := Y_1(t) + 1. \tag{12}$$

This function is a solution of Eq. (1), satisfying the inequality (5) on I_{-1}. □

The following corollary follows easily from the proof of Theorem 1.

Corollary 1. *Let us suppose that $\Omega(t)$ is a solution of inequality* (3) *on I_{-1}. Then the solution $y = Y_1(t_0, \Phi)(t)$ of Eq.* (1) *with initial data* (6) *satisfies the inequality $Y_1(t_0, \Phi)(t) \geqslant \Omega(t)$ on I_{-1}.*

2.2. A comparison lemma

Let us consider, together with the inequality (3), an inequality

$$\dot{\omega}^*(t) \leqslant \beta_1(t)\big[\omega^*(t - \delta) - \omega^*(t - \tau)\big], \tag{13}$$

where $\beta_1 : I_{-1} \to \mathbb{R}^+$ is a continuous function satisfying inequality $\beta_1(t) \leqslant \beta(t)$ on I_{-1}. The following comparison lemma will be used below.

Lemma 1. *Let the inequality* (13) *have a nondecreasing solution on I_{-1}. Then this solution is a solution of the inequality* (3) *on I_{-1}, too.*

Proof. Let $\omega^* := \varphi(t)$ be a nondecreasing solution of inequality (13) on I_{-1}. Then

$$\dot{\varphi}(t) \leqslant \beta_1(t)\big[\varphi(t - \delta) - \varphi(t - \tau)\big] \leqslant \beta(t)\big[\varphi(t - \delta) - \varphi(t - \tau)\big].$$

The last inequality means that the function $\omega := \varphi(t)$ solves the inequality (3), too. □

2.3. A possible form of solution of the inequality (3)

Let us show that a solution of inequality (3) can be found in an exponential form.

Lemma 2. *Suppose there exists a continuous function $\varepsilon : I_{-1} \setminus \{t_0\} \to \mathbb{R}$ with at most first-order discontinuity at the point $t = t_0$ satisfying on I the inequality*

$$\varepsilon(t) + \exp\left[\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,ds\right] \geqslant \exp\left[\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,ds\right]. \tag{14}$$

Then on I_{-1}, there exists a solution $\omega(t) = \omega_e(t)$ of inequality (3) *having the form*

$$\omega_e(t) := \exp\left[-\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,ds\right]. \tag{15}$$

Proof. Inequality (14) follows immediately from inequality (3) if a possible solution $\omega(t)$ is taken in the form (15). □

3. Auxiliary lemmas

In this part we prove auxiliary results concerning solutions of Eq. (1).

Lemma 3. *Let $\varphi \in \mathcal{C}$ increases* (*decreases*) *on* $[-\tau, 0]$. *Then the corresponding solution $y(\nu, \varphi)(t)$ of Eq.* (1) *with $\nu \in I$ is increasing* (*decreasing*) *in* $[\nu - \tau, \infty)$.

Proof. As follows from Eq. (1), $\operatorname{sign} \dot{y}(\nu, \varphi)(\nu) = +1$ in the case when the function φ increases on $[-\tau, 0]$ and $\operatorname{sign} \dot{y}(\nu, \varphi)(\nu) = -1$ in the case when the function φ decreases on $[-\tau, 0]$. The case $\dot{y}(\nu, \varphi)(t^*) = 0$ for $t^* \in (\nu, \infty)$ and simultaneously $\operatorname{sign} \dot{y}(\nu, \varphi)(t) \neq 0$ on interval $t \in (\nu, t^\star)$ is impossible because, as follows from Eq. (1) and from the properties of function φ, the inequality $y(t^* - \delta) \neq y(t^* - \tau)$ holds. □

Lemma 4. *Let us suppose that $\Omega(t)$ is a solution of inequality* (3) *on I_{-1}*, increasing *on* $[t_0 - \tau, t_0]$. *Then on I_{-1} there exists an* increasing *solution $Y(t)$ of Eq.* (1) *on I_{-1} satisfying the inequality* (5). *If, moreover, $\Omega(t)$ is continuously differentiable on* $[t_0 - \tau, t_0]$, *then the solution $Y(t)$ is continuously differentiable on $I_{-1} \setminus \{t_0\}$.*

Proof. The proof is based on the proof of Theorem 1. Since the solution $y(t) = Y_1(t_0, \Phi)(t)$ of Eq. (1) with Φ given by (6) is a function increasing on $[t_0 - \tau, t_0]$, then (by Lemma 3) it is increasing on the whole interval I_{-1}. It is obvious that the solution $Y(t)$ given by (12) has the same property. Similarly, continuous differentiability of $\Omega(t)$ on $[t_0 - \tau, t_0]$, leads to continuous differentiability of $Y(t)$ on $I_{-1} \setminus \{t_0\}$. □

Remark 1. Suppose that Lemma 2 holds with a function ε *negative* on $[t_0 - \tau, t_0]$. Then on I_{-1}, the solution ω_e of inequality (3) satisfies all assumptions of Lemma 4 with respect to $\Omega(t)$, i.e., we can put $\Omega(t) := \omega_e(t)$. Moreover, the solution Y of Eq. (1) satisfying on I_{-1} Lemma 4 is of the form $Y(t) := Y(t_0, \omega_e)(t)$.

Lemma 5. *Let $y(t)$ be a nondecreasing positive* (*a nonincreasing negative*) *solution of Eq.* (1) *on I_{-1}. Then the expression*

$$V(t) := \exp\left[-\int_{t_0-\tau}^{t} \beta(s)\,ds\right] \cdot y(t)$$

is a decreasing (*an increasing*) *function on I.*

Proof. Let us investigate the sign of the derivative $\dot{V}(t)$ on I. We get

$$\begin{aligned}\dot{V}(t) &= \exp\left[-\int_{t_0-\tau}^{t} \beta(s)\,ds\right] \cdot \left[-\beta(t)y(t) + \dot{y}(t)\right]\\ &= \exp\left[-\int_{t_0-\tau}^{t} \beta(s)\,ds\right] \cdot \left[-\beta(t)y(t) + \beta(t)y(t-\delta) - \beta(t)y(t-\tau)\right].\end{aligned}$$

Now it is clear that

$$\operatorname{sign} \dot{V}(t) = \operatorname{sign}\left[-y(t) + y(t-\delta) - y(t-\tau)\right] = -1$$

in the case when $y(t)$ is a nondecreasing and positive solution of Eq. (1) and, similarly, $\operatorname{sign} \dot{V}(t) = 1$ when $y(t)$ is a nonincreasing and negative solution of Eq. (1). □

4. Existence of a solution of Eq. (1) having an exponential form

In this part the main results are formulated and proved. We declare that the existence of a solution of Eq. (1) having and exponential form is equivalent with the existence of a solution of integral inequality (14). This affirmation is then modified—stronger conditions permit to estimate such solution. Moreover, it is showed that results obtained serve as a source for deriving sufficient conditions for the existence of unbounded solutions of Eq. (1). Connections with known results are discussed in Section 4.3.

4.1. Main result

Theorem 2 (Main result). *The following two statements are equivalent*:

(a) *There exists solution $y = y(t)$ of Eq.* (1), *continuously increasing on I_{-1}, continuously differentiable on $I_{-1} \setminus \{t_0\}$ and representable in exponential form*

$$y(t) = \exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,ds\right] \tag{16}$$

on the interval I_{-1}, where $\tilde{\varepsilon} : I_{-1} \setminus \{t_0\} \to (0,1)$ is a continuous function with at most first-order discontinuity at t_0.

(b) *There exists a continuous function $\varepsilon : I_{-1} \setminus \{t_0\} \to (-1,0)$ with at most first-order discontinuity at the point $t = t_0$ satisfying integral inequality* (14) *on I.*

Proof. (b) $\Rightarrow$ (a) In this case there exists (by Lemma 2) a solution of inequality (3) $\omega(t) \equiv \omega_e(t)$ given by the formula (15). Moreover, in accordance with Corollary 1, there exists a solution $y(t) = Y(t_0, \omega_e)(t)$ of Eq. (1) on I_{-1} satisfying the inequality

$$Y(t_0, \omega_e)(t) \geqslant \Omega(t) \equiv \omega_e(t) = \exp\left[-\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,ds\right]. \tag{17}$$

Since $\varepsilon(t) < 0$, $\omega_e(t)$ is an increasing solution of inequality (3). Then we can, in accordance with Lemma 4 and Remark 1, improve the last statement with a statement that the solution $Y(t_0, \omega_e)(t)$ is *increasing* on I_{-1} and *continuously differentiable* on $I_{-1} \setminus \{t_0\}$, too. Solution $Y(t_0, \omega_e)(t)$ is obviously *positive*. Therefore we can apply the auxiliary Lemma 5 in order to conclude that the corresponding expression

$$V(t) := \exp\left[-\int_{t_0-\tau}^{t} \beta(s)\,ds\right] \cdot Y(t_0, \omega_e)(t) \tag{18}$$

is a decreasing (positive) function on I. Moreover, the expression (18) is a decreasing positive function on $[t_0 - \tau, t_0)$, too. Indeed, in this case

$$\begin{aligned}\dot{V}(t) &= \exp\left[-\int_{t_0-\tau}^{t} \beta(s)\,ds\right] \cdot \left[-\beta(t)\omega_e(t) - \varepsilon(t)\beta(t)\omega_e(t)\right] \\ &= \beta(t) \cdot \omega_e(t) \cdot \left[-1 - \varepsilon(t)\right] \cdot \exp\left[-\int_{t_0-\tau}^{t} \beta(s)\,ds\right] < 0\end{aligned}$$

since $\varepsilon(t) > -1$. Obviously $Y(t_0, \omega_e)(t_0 - \tau) = \omega_e(t_0 - \tau) = 1$. Then $V(t) \leqslant V(t_0 - \tau) = 1$ and

$$Y(t_0, \omega_e)(t) \leqslant \exp\left[\int_{t_0-\tau}^{t} \beta(s)\,ds\right] \tag{19}$$

holds on I_{-1}. Using inequalities (17) and (19), we conclude

$$\exp\left[-\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,ds\right] \leqslant Y(t_0, \omega_e)(t) \leqslant \exp\left[\int_{t_0-\tau}^{t} \beta(s)\,ds\right]. \tag{20}$$

On the basis of (20), we can expect that it is possible to represent the solution $Y(t_0, \omega_e)(t)$ on I_{-1} in the form

$$Y(t_0, \omega_e)(t) = \exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,ds\right] \tag{21}$$

with a function $\tilde{\varepsilon}(t)$ continuous on $I_{-1} \setminus \{t_0\}$ and satisfying inequalities

$$0 < \tilde{\varepsilon}(t) < 1. \tag{22}$$

Let us prove it. At first, define on $I_{-1} \setminus \{t_0\}$,

$$\tilde{\varepsilon}(t) := \frac{\dot{Y}(t_0, \omega_e)(t)}{\beta(t)Y(t_0, \omega_e)(t)}. \tag{23}$$

Then the representation (21) turns into an identity. Let us verify that inequalities (22) hold, too. Inequalities (22) are valid on interval $[t_0 - \tau, t_0)$ since in this case $Y(t_0, \omega_e)(t) \equiv \omega_e(t)$ and $\tilde{\varepsilon}(t) \equiv -\varepsilon(t)$. The left-hand side of inequality (22) is, on interval (t_0, ∞), an obvious consequence of the relation (23) and properties of $Y(t_0, \omega_e)$. Let us verify on (t_0, ∞) the right-hand side of (22). Using (21) and the statement that the expression (18) is a decreasing function we have

$$0 > \dot{V}(t) = \left(\exp\left[-\int_{t_0-\tau}^{t} \beta(s)\,ds\right] \cdot Y(t_0, \omega_e)(t)\right)' = \big(\tilde{\varepsilon}(t) - 1\big) \cdot V(t) \cdot \beta(t),$$

i.e., $\tilde{\varepsilon}(t) < 1$ and the right-hand side of (22) holds. Finally, it is easy to show that the values $\tilde{\varepsilon}(t_0 \pm 0)$ exist and are finite. The part (b) $\Rightarrow$ (a) is proved.

(a) $\Rightarrow$ (b) Let $y(t)$ be a solution of Eq. (1) on I_{-1} with properties indicated in the part (a). Then on $I_{-1} \setminus \{t_0\}$,

$$\dot{y}(t) = \tilde{\varepsilon}(t)\beta(t) \cdot \exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,ds\right].$$

Let us put the solution $y(t)$ having the form (16) into Eq. (1). Then on I,

$$\tilde{\varepsilon}(t) = \exp\left[-\int_{t-\delta}^{t} \tilde{\varepsilon}(s)\beta(s)\,ds\right] - \exp\left[-\int_{t-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,ds\right].$$

Let us define function $\varepsilon : I_{-1} \setminus \{t_0\} \to (-1, 0)$,

$$\varepsilon(t) := -\tilde{\varepsilon}(t). \tag{24}$$

Then the last equality turns into

$$\varepsilon(t) + \exp\left[\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,ds\right] = \exp\left[\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,ds\right],$$

i.e., the integral inequality (14) holds on I. This ends the proof. $\square$

4.2. Modifications of the main result

The main result, formulated above, gives an equivalence of an exponential behaviour of a solution of Eq. (1) with the existence of a solution of the integral inequality (14). Since function $\tilde{\varepsilon} : I_{-1} \setminus \{t_0\} \to (0, 1)$, then as a consequence we get inequalities for such solution $y = y(t)$ on $I_{-1} \setminus \{t_0\}$,

$$1 \leqslant y(t) \leqslant \exp\left[\int_{t_0-\tau}^{t} \beta(s)\, ds\right]. \tag{25}$$

The left-hand side of the inequality (25) is obviously not satisfactory if it is necessary to get more concrete qualitative information. A small modification of assumptions in Theorem 2 leads to more exact left-hand side in (25). It is formulated in the following theorem. The scheme of its proof copies exactly the proof of Theorem 2 and therefore is omitted, except for the several modified points indicated.

Theorem 3. *Let q be a constant, $q \in (0, 1)$. Then the following two statements are equivalent*:

(a) *There exists a continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y = y(t)$ of Eq.* (1) *representable on the interval I_{-1} in the form* (16), *where $\tilde{\varepsilon} : I_{-1} \setminus \{t_0\} \to (1 - q, 1)$ is a continuous function with at most first-order discontinuity at the point $t = t_0$, and satisfying, on the interval I_{-1}, the inequalities*

$$\exp\left[(1-q) \cdot \int_{t_0-\tau}^{t} \beta(s)\, ds\right] \leqslant y(t) \leqslant \exp\left[\int_{t_0-\tau}^{t} \beta(s)\, ds\right]. \tag{26}$$

(b) *There exists a continuous function $\varepsilon : I_{-1} \setminus \{t_0\} \to (-1, q - 1)$ with at most first-order discontinuity at the point $t = t_0$ satisfying the integral inequality* (14) *on I.*

Proof. (b) $\Rightarrow$ (a) Obviously, a solution, having the form (16) exists due to Theorem 2. The left-hand side of inequality (26) follows from the inequality (20), since

$$Y(t_0, \omega_e)(t) \geqslant \exp\left[-\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\, ds\right] \geqslant \exp\left[(1-q) \cdot \int_{t_0-\tau}^{t} \beta(s)\, ds\right].$$

(a) $\Rightarrow$ (b) Since, by (24), $\tilde{\varepsilon} = -\varepsilon \in (1 - q, 1)$, we get $\varepsilon < q - 1$. □

If the function ε is known then the following result which will be used in the proof of Theorem 7 below, follows immediately from the proof of Theorem 2 (part (b) $\Rightarrow$ (a), inequalities (20)).

Theorem 4. *Let a continuous function $\varepsilon : I_{-1} \setminus \{t_0\} \to (-1, 0)$ with at most first-order discontinuity at the point $t = t_0$ satisfying the integral inequality* (14) *on I exist. Then there* *exists a solution* $y = y(t)$ *of Eq.* (1), *continuously increasing on* I_{-1} *and differentiable on* $I_{-1} \setminus \{t_0\}$, *such that*

$$\exp\left[-\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,ds\right] \leqslant y(t) \leqslant \exp\left[\int_{t_0-\tau}^{t} \beta(s)\,ds\right]. \tag{27}$$

4.3. *Sufficient conditions for existence of divergent unbounded solutions*

In this part we give an easily verifiable sufficient condition for the existence of unbounded solutions. The proof is based on Theorem 4. Corresponding results imply that Eq. (1) admits an unbounded solution with infinite limit. First let us discuss several connections with the known results. Suppose $\delta = 0$ in Eq. (1) and consider the equation

$$\dot{y}(t) = \beta(t)\big[y(t) - y(t-\tau)\big]. \tag{28}$$

In paper [22, Corollary 2 and Theorem 5] a criterion for convergence of all solutions of Eq. (28) and a point test of convergence are given. A partial case of this test is formulated as the first statement of following theorem. The second part of it follows from results given in [3,6].

Theorem 5. *Let for all* $t \in I_{-1}$ *and a constant* $p > 1$,

$$\beta(t) \leqslant \frac{1}{\tau} - \frac{p}{2t}. \tag{29}$$

Then each solution of Eq. (28) *corresponding to the initial point* t_0 *converges. Let for all* $t \in I_{-1}$ *exist a constant* ρ *such that*

$$\beta(t) \leqslant \rho < \frac{1}{\tau - \delta}. \tag{30}$$

Then each solution of Eq. (1) *corresponding to the initial point* t_0 *converges.*

Therefore in constructing sufficient condition we can expect in some sense opposite inequalities with respect to (29), (30). Moreover, in [11,21] the following result is given for the case of Eq. (1) with $\beta(t) \equiv \beta = \text{const}$.

Theorem 6. *If*

$$\beta > \frac{1}{\tau - \delta}, \tag{31}$$

there are solutions of Eq. (1) *which are unbounded as* $t \to \infty$.

Let us remark that the equality is admissible in the inequality (31), too since in this case Eq. (1) admits an unbounded solution $y(t) = t$. Comparing inequalities (29)–(31) we imagine conditions generalizing the last one. The following sufficient condition (Theorem 7 below) and corollary (Corollary 2 below) generalize the previous result for variable coefficient $\beta(t)$.

Theorem 7. *If on I_{-1} with sufficiently large t_0 the inequality*

$$\beta(t) \geqslant \frac{1}{\tau-\delta} + \frac{1}{2t}\left(-1 + \frac{\nu}{\tau-\delta}\right) \tag{32}$$

holds with $\nu \in (0, \tau-\delta)$, then Eq. (1) admits an increasing unbounded as $t \to \infty$ solution $y = y(t)$ satisfying inequalities

$$k(t) \cdot t^{\nu/(\tau-\delta)} \leqslant y(t) \leqslant \exp\left[\int_{t_0-\tau}^{t} \beta(s)\,ds\right] \tag{33}$$

on I, with a positive bounded function $k(t)$, $k(\infty) > 0$, defined as

$$k(t) := (t_0-\tau)^{-\nu/(\tau-\delta)} \cdot \exp\left[\frac{\nu}{2}\left(1 - \frac{\nu}{\tau-\delta}\right)\left(\frac{1}{t} - \frac{1}{t_0-\delta}\right)\right].$$

Proof. In the proof we employ Theorem 4. Let us verify that the integral inequality (14) holds with $\varepsilon(t) := -a/t$, $0 < a \leqslant \nu$ and

$$\beta(t) := \frac{1}{\tau-\delta} + \frac{1}{2t}\left(-1 + \frac{\nu}{\tau-\delta}\right). \tag{34}$$

Then the left-hand side of (14) equals

$$\begin{aligned}
\mathcal{L}(t) &\equiv \varepsilon(t) + \exp\left[\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,ds\right] \\
&= -\frac{a}{t} + \exp\left[-\int_{t-\delta}^{t} \frac{a}{s}\left[\frac{1}{\tau-\delta} + \frac{1}{2s}\left(-1 + \frac{\nu}{\tau-\delta}\right)\right]ds\right] \\
&= -\frac{a}{t} + \left(1 - \frac{\delta}{t}\right)^{a/(\tau-\delta)} \cdot \exp\left[\frac{-a\delta}{t(t-\delta)} \cdot \frac{1}{2}\left(-1 + \frac{\nu}{\tau-\delta}\right)\right].
\end{aligned}$$

Let us develop the asymptotic decomposition of $\mathcal{L}(t)$ for $t \to \infty$ with sufficient accuracy for further application. We get

$$\begin{aligned}
\mathcal{L}(t) &= -\frac{a}{t} + \left[1 - \frac{1}{\tau-\delta} \cdot \frac{\delta a}{t} + \frac{\delta^2}{2} \cdot \frac{a}{\tau-\delta} \cdot \left(\frac{a}{\tau-\delta} - 1\right)\frac{1}{t^2} + O\left(\frac{1}{t^3}\right)\right] \\
&\quad \times \left[1 - \frac{\delta a}{2} \cdot \left(-1 + \frac{\nu}{\tau-\delta}\right)\frac{1}{t^2} + O\left(\frac{1}{t^3}\right)\right] \\
&= 1 + \left[-a - \frac{\delta a}{\tau-\delta}\right]\frac{1}{t} + \left[\frac{\delta^2}{2} \cdot \frac{a}{\tau-\delta} \cdot \left(\frac{a}{\tau-\delta} - 1\right)\right. \\
&\quad \left. - \frac{\delta a}{2} \cdot \left(-1 + \frac{\nu}{\tau-\delta}\right)\right]\frac{1}{t^2} + O\left(\frac{1}{t^3}\right),
\end{aligned}$$

where O is the Landau order symbol. Similarly for the right-hand side of (14) we get

$$\begin{aligned}
\mathcal{R}(t) &\equiv \exp\left[\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,ds\right] = \exp\left[-\int_{t-\tau}^{t} \frac{a}{s}\left[\frac{1}{\tau-\delta}+\frac{1}{2s}\left(-1+\frac{\nu}{\tau-\delta}\right)\right]ds\right] \\
&= \left(1-\frac{\tau}{t}\right)^{a/(\tau-\delta)} \cdot \exp\left[\frac{-a\tau}{t(t-\tau)}\cdot\frac{1}{2}\left(-1+\frac{\nu}{\tau-\delta}\right)\right] \\
&= \left[1-\frac{1}{\tau-\delta}\cdot\frac{\tau a}{t}+\frac{\tau^2}{2}\cdot\frac{a}{\tau-\delta}\cdot\left(\frac{a}{\tau-\delta}-1\right)\frac{1}{t^2}+O\left(\frac{1}{t^3}\right)\right] \\
&\quad \times \left[1-\frac{\tau a}{2}\cdot\left(-1+\frac{\nu}{\tau-\delta}\right)\frac{1}{t^2}+O\left(\frac{1}{t^3}\right)\right] \\
&= 1-\frac{\tau a}{\tau-\delta}\cdot\frac{1}{t}+\left[\frac{\tau^2}{2}\cdot\frac{a}{\tau-\delta}\cdot\left(\frac{a}{\tau-\delta}-1\right)-\frac{\tau a}{2}\cdot\left(-1+\frac{\nu}{\tau-\delta}\right)\right]\frac{1}{t^2} \\
&\quad + O\left(\frac{1}{t^3}\right).
\end{aligned}$$

Comparing the coefficients of identical functional terms of $\mathcal{L}(t)$ and $\mathcal{R}(t)$, we see that for $\mathcal{L}(t) \geqslant \mathcal{R}(t)$ it is sufficient to compare coefficients of the terms t^{-2} since coefficients of the terms t^0 and t^{-1} are equal. That is, we need

$$\begin{aligned}
&\frac{\delta^2 a}{2(\tau-\delta)}\cdot\left(\frac{a}{\tau-\delta}-1\right)-\frac{\delta a}{2}\cdot\left(-1+\frac{\nu}{\tau-\delta}\right) \\
&\quad > \frac{\tau^2 a}{2(\tau-\delta)}\cdot\left(\frac{a}{\tau-\delta}-1\right)-\frac{\tau a}{2}\cdot\left(-1+\frac{\nu}{\tau-\delta}\right)
\end{aligned}$$

or, after simplifying,

$$\tau(a-\nu)+\delta\cdot\bigl(a+\nu-2(\tau-\delta)\bigr)<0.$$

This inequality obviously holds, since $0<\delta<\tau$, $0<a\leqslant\nu<\tau-\delta$. So the integral inequality (14) for $t\to\infty$ holds and, consequently, Theorem 4 holds, too. The left-hand side of inequality (33) is a straightforward consequence of inequality (27). Really, computing the left-hand side of (27) with $\varepsilon(t)$ as above and with $a=\nu$ leads to

$$\begin{aligned}
&\exp\left[-\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,ds\right] = \exp\left[\int_{t_0-\tau}^{t} \frac{\nu}{s}\left[\frac{1}{\tau-\delta}+\frac{1}{2s}\left(-1+\frac{\nu}{\tau-\delta}\right)\right]ds\right] \\
&\quad = \left(\frac{t}{t_0-\tau}\right)^{\nu/(\tau-\delta)} \cdot \exp\left[\frac{\nu}{2}\cdot\left(-1+\frac{\nu}{\tau-\delta}\right)\cdot\left(-\frac{1}{t}+\frac{1}{t_0-\delta}\right)\right] \\
&\quad = k(t)\cdot t^{\nu/(\tau-\delta)}.
\end{aligned}$$

The above verification means that for the *fixed function* β given by relation (34), the left-hand side of inequality (33) holds. Let us show that it holds for *every function* β satisfying inequality (32). Put

$$\beta_1(t) := \frac{1}{\tau-\delta}+\frac{1}{2t}\left(-1+\frac{\nu}{\tau-\delta}\right)$$

and suppose $\beta(t) \geqslant \beta_1(t)$. It is easy to verify that inequality (13) has an increasing solution. This verification can be made with the aid of Lemma 2 since (as it was shown above) the integral inequality (14) holds with $\varepsilon(t) := -\nu/t$. Then by Comparison Lemma 1 the inequality (3) has the same solution and the conclusion of Theorem 7 is now a straightforward consequence of Theorem 1. □

Remark 2. Let us note that Theorem 7 improves (as it follows from inequality (32)) known results when $\nu < \tau - \delta$. This is taken into account in the following corollary which reformulates the affirmation of Theorem 7 concerning the existence of increasing unbounded solution.

Corollary 2. *Let for all $t \in I_{-1}$ with sufficiently large t_0 and for a constant $p \in (0, 1)$,*

$$\beta(t) \geqslant \frac{1}{\tau - \delta} - \frac{p}{2t}.$$

Then there exists an increasing and unbounded solution of Eq. (1) *as $t \to \infty$.*

Remark 3. It is to be pointed out that Theorems 5–7 concern the so called "*critical*" case, since the value $\beta(t) \equiv 1/(\tau - \delta)$ separates the case when all solutions of Eq. (1) converge and the case when there are divergent solutions. Investigation of linear delay equations with more that one argument in a different "*critical*" state separating the case when all solutions are oscillatory and the case when there exists a positive solution was started in [12].

5. Example

Let us consider the inequality of the type (3) with $\beta(t) := \lambda \cdot (1 - 1/t)$, $\lambda = e^2/(e - 1) \doteq 4.30$, $\delta = 1$ and $\tau = 2$, i.e., the inequality

$$\dot{\omega}(t) \leqslant \frac{e^2}{e - 1}\left(1 - \frac{1}{t}\right) \cdot \bigl[\omega(t - 1) - \omega(t - 2)\bigr]. \tag{35}$$

Let us put $t_0 = 10$. Then it is easy to verify that the corresponding inequality (14) holds with $\varepsilon(t) \equiv \varepsilon = \text{const}$, $\varepsilon = -1/\lambda$ since it turns into an inequality

$$\frac{1}{e^2} - \frac{1}{e} + \exp\left[-\int_{t-1}^{t}\left(1 - \frac{1}{s}\right) ds\right] \geqslant \exp\left[-\int_{t-2}^{t}\left(1 - \frac{1}{s}\right) ds\right] \tag{36}$$

or, after simplifying,

$$t \geqslant \frac{2(e - 1)}{e - 2} \doteq 4.78.$$

Then a solution of (35) is expressed on $I_{-1} = [8, \infty)$ in the form (15), i.e.,

$$\omega_e(t) = \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\, ds\right] = \exp\left[\int_{8}^{t}\left(1 - \frac{1}{s}\right) ds\right] = \frac{8 \cdot e^t}{t \cdot e^8}. \tag{37}$$

Let us consider the equation of the type (1) with δ, τ, β and t_0 as above, i.e., the equation

$$\dot{y}(t) = \frac{e^2}{e-1}\left(1-\frac{1}{t}\right)\cdot\big[y(t-1)-y(t-2)\big]. \tag{38}$$

Then the corresponding inequality (14) holds (with ε indicated above), since it turns into inequality (36). Therefore by Theorem 2 there exists a continuously increasing on I_{-1} and continuously differentiable on $[8,10)\cup(10,\infty)$ solution $y = y(t)$ such that on I_{-1},

$$y(t) = \exp\left[\lambda\int_8^t \tilde{\varepsilon}(s)\left(1-\frac{1}{s}\right)ds\right],$$

where $\tilde{\varepsilon}: I_{-1}\setminus\{10\}\to(0,1)$ is a continuous function. Corollary 2 immediately gives the answer concerning the existence of a solution with infinite limit, since

$$\frac{1}{\tau-\delta}-\frac{p}{2t}\leqslant\frac{1}{\tau-\delta}=1<3<\lambda\cdot\left(1-\frac{1}{t}\right)=\beta(t).$$

More exact information concerning asymptotic behaviour of a solution of Eq. (38) can be obtained with the aid of Theorem 4. Since

$$\exp\left[\int_{t_0-\tau}^t \beta(s)\,ds\right] = \left[\frac{8\cdot e^t}{t\cdot e^8}\right]^\lambda,$$

then in view of (37) we conclude that there exists a solution $y = y(t)$ of Eq. (38) on I_{-1} satisfying the inequality

$$\frac{8\cdot e^t}{t\cdot e^8}\leqslant y(t)\leqslant\left[\frac{8\cdot e^t}{t\cdot e^8}\right]^\lambda.$$

6. Open problem

It is known, provided that there exists an increasing solution $y = Y(t)$ on I_{-1} of the Eq. (1) with $\delta = 0$ satisfying $Y(+\infty) = +\infty$, that the general structure of solutions can be clarified. Namely, in accordance with [23, Theorem 4] (see investigations [8–10,17,18], too) every solution $y = \tilde{y}(t)$ of the equation

$$\dot{y}(t) = \beta(t)\big[y(t)-y(t-\tau)\big]$$

can be expressed by the formula

$$\tilde{y}(t) = K\cdot Y(t)+\delta(t) \tag{39}$$

on $t\in I_{-1}$, where $K\in\mathbb{R}$ is a constant, dependent on $\tilde{y}(t)$, and $\delta(t)$ is a bounded solution of Eq. (1) on I_{-1} dependent on $\tilde{y}(t)$. This representation is unique (with respect to K and $\delta(t)$). Let us formulate the corresponding problem with respect to Eq. (1).

Problem 1. Let $y = Y(t)$ be an unbounded increasing solution of Eq. (1) with $\delta\neq 0$. Can every solution $y = \tilde{y}(t)$ of Eq. (1) be represented on I_{-1} by formula (39) with the above-indicated restrictions?

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