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Mixed Affine Surface Area

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Various means (over the unit sphere) of powers of the curvature function of a convex body arise in geometric convexity. For example, both the surface area and the affine surface area of sufficiently smooth bodies can be expressed as such means. The survey of Gruber [11] on approximation of convex bodies contains a number of examples of means of powers of the curvature function arising in approximation problems (see, pp. 137–141, 143, and 146–148 of [11]). These means arise, surprisingly, in questions related to approximating convex bodies by polytopes (see, e.g., McClure and Vitale [18], Schneider [24], and Schneider and Wieacker [27]). An interesting conjecture of Firey [9, p. 8] (also [26, p. 257]), which was recently proved by Gage [10] for plane convex bodies, is that a certain weighted mean of the curvature function lies between the arithmetic and harmonic means of the curvature function.

It turns out that there is also a close connection between various means of the curvature function of a convex body and some mixed volumes involving the body.

In this note we investigate means of the curvature function of a convex body which arise as coefficients in the expansion of the affine surface area of outer and inner Blaschke parallel bodies. For a variety of reasons we will take a general approach to the study of these means, an approach similar to that taken in [14] (see also [4, pp. 164–167]) in presenting dual mixed volumes.

Let $\mathscr{K}^n$ denote the set of convex bodies (compact convex sets with interior points) in Euclidean n-space $\mathbb{R}^n$. We use $h(K, \cdot)$ to denote the support function of a convex body K; i.e.,

$$h(K, v) = \operatorname{Max}\{x \cdot v : x \in K\},$$

where $x \cdot v$ is the usual inner product of x and v in $\mathbb{R}^n$.

We will be concerned mainly with convex bodies that have positive con-

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tinuous curvature functions. A convex body $K \in \mathscr{K}^n$ is said to have a positive continuous curvature function (see [3, p. 115]),

$$f(K, \cdot): \quad S^{n-1} \to (0, \infty),$$

if for each $L \in \mathscr{K}^n$ the mixed volume $V_1(K, L) = V(K, ..., K, L)$ has the integral representation:

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} f(K, u)\, h(L, u)\, dS(u), \tag{1}$$

where the integration is over the unit sphere, S^{n-1}, in $\mathbb{R}^n$, and $dS(u)$ denotes the area element at $u \in S^{n-1}$. We note, as an aside, that if K is of class C^2, and has everywhere positive Gaussian curvature, then the Gaussian curvature of K, when viewed as a function of the outer normals of K, is the reciprocal of the curvature function of K.

A convex body can have only one curvature function associated with it (see [3, p. 115]). From the Minkowski mixed volume inequality it follows (see [3, p. 115]) that if two convex bodies have the same curvature function, then they must be translates of each other.

The subset of $\mathscr{K}^n$ consisting of bodies with positive continuous curvature functions will be denoted by $\mathscr{F}^n$. Blaschke addition and scalar multiplication (soon to be defined) provide a natural algebraic structure for $\mathscr{F}^n$. Blaschke addition of convex bodies has been considered recently by a number of investigators. (See the references in the survey of Schneider [23, p. 48], and also [12, 16, 19] which since have appeared.)

We recall (see [3, p. 115]) that the curvature function f of a convex body in $\mathscr{F}^n$ satisfies

$$\int_{S^{n-1}} u f(u)\, dS(u) = 0. \tag{2}$$

Conversely, if f is a positive continuous function which satisfies (2), then there exists a convex body in $\mathscr{F}^n$, unique up to translation, whose curvature function is f (see [3, p. 121]).

To define Blaschke addition and scalar multiplication, we observe that for $K, K' \in \mathscr{F}^n$ and positive scalars α, α', it follows from (2) that

$$\int_{S^{n-1}} u[\alpha f(K, u) + \alpha' f(K', u)]\, dS(u) = 0.$$

Hence, there exists a convex body in $\mathscr{F}^n$, which is unique up to translation, and which we denote by $\alpha K + \alpha' K'$, such that

$$f(\alpha K + \alpha' K', \cdot) = \alpha f(K, \cdot) + \alpha' f(K', \cdot). \tag{3}$$

Our notation for Blaschke addition and scalar multiplication is one commonly used for Minkowski addition and scalar multiplication; however, since Minkowski addition and scalar multiplication will not be used in this note, no confusion should arise.

It is obvious that $\mathscr{F}^n$ is closed under Blaschke addition and (positive) scalar multiplication. It follows trivially (from the fact that a body in $\mathscr{F}^n$ has a unique curvature function associated with it) that Blaschke addition is both associative and commutative.

It is easy to verify that, for $K \in \mathscr{F}^n$ and $\lambda > 0$, the Blaschke scalar product λK is (up to translation) just a dilation of K by a factor $\lambda^{1/(n-1)}$. From this observation, and the fact that two convex bodies with the same curvature function must be translates of each other, we conclude that the curvature functions of a pair of convex bodies in $\mathscr{F}^n$ are proportional if and only if the bodies are homothetic.

To denote the volume and surface area of a convex body K we use $V(K)$ and $S(K)$, respectively. We shall use ω_n to denote the volume of the unit ball, B, in $\mathbb{R}^n$.

Petty [22] has extended the definition of affine surface area to bodies in $\mathscr{F}^n$. For $K \in \mathscr{F}^n$, the affine surface area of K, $\Omega(K)$, is defined by

$$\Omega(K) = \int_{S^{n-1}} f(K, u)^{n/(n+1)} \, dS(u). \tag{4}$$

Petty [22] also shows that, with this extended definition, the affine isoperimetric inequality of affine differential geometry, along with the conditions for equality, continues to hold. Specifically, for $K \in \mathscr{F}^n$,

$$\Omega(K)^{n+1} \leqslant n^{n+1} \omega_n^2 V(K)^{n-1}, \tag{5}$$

with equality if and only if K is an ellipsoid.

We define the mixed affine area of $K_1, ..., K_n \in \mathscr{F}^n$, $\Omega(K_1, ..., K_n)$, by

$$\Omega(K_1, ..., K_n) = \int_{S^{n-1}} [f(K_1, u) \cdots f(K_n, u)]^{1/(n+1)} \, dS(u). \tag{6}$$

Since the curvature function of a convex body is unaffected by translations of the body, it follows that $\Omega(K_1, ..., K_n)$ is invariant under translations of the K_i. Obviously, one also has

$$\Omega(K, ..., K) = \Omega(K).$$

We will need a form of the Hölder integral inequality [13, p. 140] (see also [6, p. 88]) which states that for positive continuous functions $f_1, ..., f_r$

on S^{n-1} and positive reals $\alpha_1,..., \alpha_r$, the sum of whose reciprocals is unity, one has

$$\int_{S^{n-1}} f_1(u) \cdots f_r(u)\, dS(u) \leqslant \prod_{i=1}^{r} \left[\int_{S^{n-1}} f_i^{\alpha_i}(u)\, dS(u) \right]^{1/\alpha_i}, \tag{7}$$

with equality if and only if the $f_i^{\alpha_i}$ are proportional to each other.

A general version of the Aleksandrov–Fenchel mixed volume inequality (see [1, p. 1220] or [5, p. 50]) can be written as

$$\prod_{i=0}^{m-1} V(K_1,..., K_{n-m}, K_{n-i},..., K_{n-i}) \leqslant V^m(K_1,..., K_n). \tag{8}$$

The conditions for equality in this general inequality are unknown (see Schneider [25] for a discussion). However, the special case where $m=n$ and $K_1 = \cdots = K_{n-1}$, is the Minkowski mixed volume inequality, and here it is known that equality can occur if and only if the bodies (K_1 and K_n) are homothetic (see [3, p. 91]).

For the mixed affine areas we have the complementary inequality:

THEOREM 1. *If $K_1,..., K_n \in \mathscr{F}^n$ and $1 < m \leqslant n$, then*

$$\prod_{i=0}^{m-1} \Omega(K_1,..., K_{n-m}, K_{n-i},..., K_{n-i}) \geqslant \Omega^m(K_1,..., K_n)$$

with equality if and only if $K_{n-m+1},..., K_n$ are homothetic.

To prove this we use the Hölder integral inquality (7) with $r=m$, $\alpha_1 = \cdots = \alpha_m = m$, and

$$f_i = [f(K_1, \cdot) \cdots f(K_{n-m}, \cdot)]^{1/m(n+1)} f(K_{n-i+1}, \cdot)^{1/(n+1)}.$$

The equality conditions follow from the equality conditions of the Hölder inequality, and the already noted observation that a pair of convex bodies have proportional curvature functions if and only if they are homothetic.

The special case $m=n$ of Theorem 1 is

$$\Omega^n(K_1,..., K_n) \leqslant \Omega(K_1) \cdots \Omega(K_n), \tag{9}$$

with equality if and only if the K_i are homothetic. If we take $m=n-1$ in the general Aleksandrov–Fenchel inequality (8), and then apply the

Minkowski mixed volume inequality (where the equality conditions are known), the result is

$$V(K_1)\cdots V(K_n)\leqslant V^n(K_1,\dots,K_n), \tag{10}$$

with equality if and only if the K_i are homothetic. If (9), (10), and the affine isoperimetric inequality (5) are combined the result is

THEOREM 2. *If $K_1,\dots,K_n\in\mathcal{F}^n$, then*

$$\Omega(K_1,\dots,K_n)^{n+1}\leqslant n^{n+1}\omega_n^2 V(K_1,\dots,K_n)^{n-1},$$

with equality if and only if the K_i are homothetic ellipsoids.

Naturally, the case of Theorem 2 where the K_i are equal reduces to the affine isoperimetric inequality.

If $K, L\in\mathcal{K}^n$ and i is an integer such that $0\leqslant i\leqslant n$, then, as usual, $V_i(K,L)$ denotes the mixed volume $V(K,\dots,K,L,\dots,L)$, with $n-i$ copies of K, and i copies of L. Similarly, if $K, L\in\mathcal{F}^n$ and $0\leqslant i\leqslant n$, we will use $\Omega_i(K,L)$ to denote the mixed affine area $\Omega(K,\dots,K,L,\dots,L)$, with $n-i$ copies of K, and i copies of L. This definition can easily be extended so that $\Omega_i(K,L)$ is defined for all real i. Specifically, for $i\in\mathbb{R}$, $\Omega_i(K,L)$ is given by

$$\Omega_i(K,L)=\int_{S^{n-1}} f(K,u)^{(n-i)/(n+1)}f(L,u)^{i/(n+1)}\,dS(u). \tag{11}$$

Just as the ith projection measure (Quermassintegral) $W_i(K)$ can be defined as $V_i(K,B)$, we define the ith affine area of $K\in\mathcal{F}^n$, $\Omega_i(K)$, to be $\Omega_i(K,B)$. Since $f(B,\cdot)=1$, one has

$$\Omega_i(K)=\int_{S^{n-1}} f(K,u)^{(n-i)/(n+1)}\,dS(u), \tag{12}$$

for all $i\in\mathbb{R}$. Obviously, $\Omega_0(K)=\Omega(K)$, and $\Omega_n(K)=n\omega_n$, for all $K\in\mathcal{F}^n$.

There are close connections between certain mixed affine areas and some mixed volumes (and dual mixed volumes). To see some of these connections we recall some well-known results.

For a body $K\in\mathcal{K}^n$, containing the origin in its interior, let K^* denote the polar body of K, with respect to the unit sphere centered at the origin (see [3] for definitions). For $K\in\mathcal{K}^n$ and $x\in\mathbb{R}^n$, let $x+K$ denote the translate of K by x; i.e., $x+K=\{x+y\colon y\in K\}$ (where the addition is ordinary vector addition). Associated with a convex body $K\in\mathcal{K}^n$ is its Santaló point, $s=s(K)$, which can be defined (see [15] for a discussion) as the unique point $s\in\operatorname{int}K$ such that

$$\int_{S^{n-1}} uh(-s+K,u)^{-(n+1)}\,dS(u)=0. \tag{13}$$

We shall use K^s to denote the polar body of K with the Santaló point of K taken as origin; i.e., $K^s = s + (-s + K)^*$.

The restricted (to S^{n-1}) support function of a convex body, containing the origin in its interior, is a positive continuous function. From this, and (13), it follows that, associated with a convex body $K \in \mathscr{K}^n$, there is a convex body $\Lambda K \in \mathscr{F}^n$, unique up to translation, such that

$$f(\Lambda K, \cdot) = h(-s + K, \cdot)^{-(n+1)}. \tag{14}$$

It is easy to see that $\Lambda B = B$, up to translation. (The definition of ΛK is somewhat different than that given in [17].)

From (1), (11), and the translation invariance of mixed volumes, it follows that for $K \in \mathscr{F}^n$ and $L \in \mathscr{K}^n$,

$$\Omega_{-1}(K, \Lambda L) = nV_1(K, L). \tag{15}$$

Since $S(K) = nW_1(K) = nV_1(K, B)$, it follows that for $K \in \mathscr{F}^n$,

$$\Omega_{-1}(K) = S(K). \tag{15$'$}$$

From (1), (11), and the translation invariance of mixed volumes, it also follows that for $K \in \mathscr{K}^n$ and $L \in \mathscr{F}^n$,

$$\Omega_{n+1}(\Lambda K, L) = nV_{n-1}(K, L). \tag{16}$$

Since $V_{n-1}(K, B) = W_{n-1}(K)$, we have for $K \in \mathscr{K}^n$,

$$\Omega_{n+1}(\Lambda K) = nW_{n-1}(K). \tag{16$'$}$$

From (4), (14), and the polar coordinate formula for volume, we conclude that for $K \in \mathscr{K}^n$,

$$\Omega(\Lambda K) = nV(K^s). \tag{17}$$

The general Aleksandrov–Fenchel inequality (8) tells us that for K, $L \in \mathscr{K}^n$ and integers i, j, k such that $0 \leqslant i \leqslant j \leqslant k \leqslant n$,

$$V_i(K, L)^{k-j}\, V_k(K, L)^{j-i} \leqslant V_j(K, L)^{k-i}.$$

A similar inequality can be obtained for mixed affine areas from Theorem 1; however, this inequality will hold even without the restriction that i, j, k must assume values between 0 and n.

THEOREM 3. *If K, $L \in \mathscr{F}^n$ and $i, j, k \in \mathbb{R}$ such that $i < j < k$, then*

$$\Omega_i(K, L)^{k-j}\, \Omega_k(K, L)^{j-i} \geqslant \Omega_j(K, L)^{k-i},$$

with equality if and only if K and L are homothetic.

To prove this we use Hölder integral inequality (7) with $r=2$, $\alpha_1 = (k-i)/(j-i)$, $\alpha_2 = (k-i)/(k-j)$,

$$f_1 = f(K,\cdot)^{(j-i)(n-k)/(k-i)(n+1)}\, f(L,\cdot)^{k(j-i)/(k-i)(n+1)}$$

and

$$f_2 = f(K,\cdot)^{(k-j)(n-i)/(k-i)(n+1)}\, f(L,\cdot)^{i(k-j)/(k-i)(n+1)}.$$

The equality conditions follow, as before, from the equality conditions of the Hölder inequality, and the fact that the curvature functions of a pair of bodies are proportional if and only if the bodies are homothetic.

The special case of Theorem 3 with $i=-1$, $j=0$, $k=n$, and $L=B$, states that for $K\in\mathscr{F}^n$,

$$\Omega(K)^{n+1} \leqslant n\omega_n S(K)^n,$$

with equality if and only if K is a ball. This is (an extension to $\mathscr{F}^n$ of) Berwald's generalization of an inequality of Winternitz [2, p. 206] (see also Petty [21, p. 94]).

The special case of Theorem 3, with $i=0$, $j=n$ $k=n+1$, $L=B$ and ΛK taken for K, states that for $K\in\mathscr{K}^n$,

$$V(K^s) \geqslant \omega_n^{n+1} W_{n-1}(K)^{-n},$$

with equality if and only if K is a ball. This is the dual of the Urysohn Inequality obtained in [14].

The Knesser–Süss inequality [3, p. 124] states that $V^{(n-1)/n}$ is concave with respect to Blaschke addition; i.e.,

$$V(K+L)^{(n-1)/n} \geqslant V(K)^{(n-1)/n} + V(L)^{(n-1)/n}.$$

From this and the affine isoperimetric inequality one might suspect that $\Omega^{(n+1)/n}$ is also concave with respect to Blaschke addition. This is the special case $i=0$ of the following:

THEOREM 4. *If $K, L\in\mathscr{F}^n$ and $i\in\mathbb{R}^n$, then for $i>-1$*

$$\Omega_i(K+L)^{(n+1)/(n-i)} \geqslant \Omega_i(K)^{(n+1)/(n-i)} + \Omega_i(L)^{(n+1)/(n-i)},$$

with equality if and only if K and L are homothetic, while for $i<-1$

$$\Omega_i(K+L)^{(n+1)/(n-i)} \leqslant \Omega_i(K)^{(n+1)/(n-i)} + \Omega_i(L)^{(n+1)/(n-i)},$$

with equality if and only if K and L are homothetic.

Theorem 4 is an immediate consequence of the Minkowski integral inequality, [13, pp. 146]. The conditions for equality follow from the equality conditions of the Minkowski integral inequality and the fact that the curvature functions of two bodies are proportional if and only if the bodies are homothetic.

For the case where $i=-1$, Ω_{-1} is just ordinary surface area, which is linear with respect to Blaschke addition. For an application of the case $i=0$ of Theorem 4, see Petty [20, p. 240].

From Theorem 2 we know that for $K\in\mathscr{F}^n$,

$$\Omega_{n-1}^{n+1}(K)\leqslant n^{n+1}\omega_n^2 W_{n-1}^{n-1}(K),$$

with equality if and only if K is a ball. Theorem 4 tells us that Ω_{n-1}^{n+1} is concave with respect to Blaschke addition. A still-unanswered question of Firey [7, p. 100, 101] (see also [8]) asks about the behavior of W_{n-1}^{n-1} with respect to Blaschke addition.

For $K\in\mathscr{K}^n$, we define $\eta(K)$ by

$$\eta(K)=\mathrm{Min}\{f(K,u): u\in S^{n-1}\}.$$

If $\lambda>-\eta(K)$, then $f(K,\cdot)+\lambda$ is a positive continuous function that satisfies (2). Hence, there exists a convex body, K_λ, which is unique up to translation, such that

$$f(K_\lambda,\cdot)=f(K,\cdot)+\lambda. \tag{18}$$

We call K_λ a Blaschke outer, or inner, parallel body of K, depending on whether λ is positive or negative. The Blaschke outer parallel body K_λ is, of course, just $K+\lambda B$. See Firey [7] for a discussion of the analogy between Blaschke and Minkowski parallel bodies.

Analogous to the Steiner polynomial for the volume of a Minkowski parallel body, we have the following:

THEOREM 5. *If $K\in\mathscr{F}^n$ and $|\lambda|<\eta(K)$, then*

$$\Omega(K_\lambda)=\sum_{i=0}^{\infty}\binom{n/(n+1)}{i}\Omega_{i(n+1)}(K)\lambda^i.$$

Proof. From the definition of K_λ, (18), we have

$$f(K_\lambda,u)^{n/(n+1)}=f(K,u)^{n/(n+1)}(1+\lambda f(K,u)^{-1})^{n/(n+1)}.$$

The Taylor series expansion of $(1+x)^{n/(n+1)}$, for $|x|<1$, is

$$(1+x)^{n/(n+1)}=\sum_{i=0}^{\infty}\binom{n/(n+1)}{i}x^i,$$

and, for any $\delta > 0$, the convergence is uniform when x is restricted to the closed interval $[-1+\delta, 1-\delta]$. Since $|\lambda f(K, u)^{-1}| \leqslant |\lambda| \, \eta(K)^{-1} < 1$, for all $u \in S^{n-1}$, we have

$$f(K_\lambda, u)^{n/(n+1)} = \sum_{i=0}^{\infty} f_i(u), \tag{19}$$

where

$$f_i(u) = \binom{n/(n+1)}{i} f(K, u)^{(n-i(n+1))/(n+1)} \lambda^i,$$

and the convergence in (19) is uniform on S^{n-1}. The desired expansion of $\Omega(K_\lambda)$ is now obtained by integrating (19) over S^{n-1}.

We note, as an aside, that if $L \in \mathscr{K}^n$, and we take ΛL for K in Theorem 5, we obtain

$$\Omega(\Lambda L + \lambda B) = \Omega(\Lambda L) + \frac{n}{n+1} \Omega_{n+1}(\Lambda L) \lambda + \cdots.$$

The first two coefficients in this expansion have simple geometric interpretations. From (17) we have $\Omega(\Lambda L) = nV(L^*)$, while from (16′) we know that $\Omega_{n+1}(\Lambda L) = nW_{n-1}(L)$.

An expansion for $\Omega_j(K_\lambda)$ also is obtained easily for all $j \in \mathbb{R}$. Specifically, for $K \in \mathscr{F}^n$, and $|\lambda| < \eta(K)$, one has

$$\Omega_j(K_\lambda) = \sum_{i=0}^{\infty} \binom{(n-j)/(n+1)}{i} \Omega_{j+i(n+1)}(K) \lambda^i,$$

for all real j.

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