Algorithm for solving a new class of general mixed variational inequalities in Banach spaces

Fu-Quan Xia, Nan-Jing Huang∗

Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China

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Abstract

In this paper, a new concept of η-proximal mapping for a proper subdifferentiable functional (which may not be convex) on a Banach space is introduced. An existence and Lipschitz continuity of the η-proximal mapping are proved. By using properties of the η-proximal mapping, a new class of general mixed variational inequalities is introduced and studied in Banach spaces. An existence theorem of solutions is established and a new iterative algorithm for solving the general mixed variational inequality is suggested. A convergence criteria of the iterative sequence generated by the new algorithm is also given.

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Keywords: General mixed variational inequality; η-Proximal mapping; Strongly monotone mapping; Iterative algorithm; Convergence

1. Introduction

It is well known that the variational inequality theory is a very effective and a powerful tool for studying a wide class of linear and nonlinear problems arising in many diverse fields of pure and applied science, such as mathematical programming, optimization theory, engineering, elasticity theory and equilibrium problems of mathematical economy, game theory, etc., see, for example [1,3,11,13,19,22] and the references therein.

One of the most interesting and important problems in the theory of variational inequality is the development of an efficient iterative algorithm to compute approximate solutions. Under Hilbert space setting, one of the most efficient numerical technique is the project method and its variant forms, see [4,5,9,12–16,22]. Since the standard projection method strictly depend on the inner product property of Hilbert spaces, it can no longer be applied for general mixed type variational inequalities in Banach spaces. The fact motivate us to develop alterative methods to study existence and iterative algorithm of solutions for generalized mixed variational inequalities in Banach spaces. Recently, [6–8] extended the auxiliary principle technique to study the existence of solutions and to suggest the iterative algorithms for solving various mixed type variational inequalities in Banach spaces. Some related works, we refer to [2,17,18] and the references therein.

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* Corresponding author.

E-mail address: nanjinghuang@hotmail.com (N.-J. Huang).

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Motivated and inspired by the research work going on this field, in this paper, we first introduce a new concept of \( \eta \)-proximal mapping for a proper subdifferentiable functional on a Banach space. We prove an existence theorem and Lipschitz continuity of the \( \eta \)-proximal mapping. Using the properties of the \( \eta \)-proximal mapping, we consider a new class of general mixed variational inequalities in a Banach space. We prove an existence theorem of solutions for this problem under some suitable conditions and suggest a new iterative algorithm to compute the approximate solutions. Finally, we suggest the convergence criteria of the iterative sequence generated by the new algorithm. Our results are new and include some known results of [5–7,9,14] as special cases.

2. Preliminaries

Let \( B \) be a Banach space with the topological dual space of \( B^* \) and \( \langle u, v \rangle \) be the pairing between \( u \in B^* \) and \( v \in B \). Let \( 2^{B^*} \) and \( CB(B^*) \) denote the family of all subsets of \( B^* \) and the family of all nonempty closed bounded subset of \( B^* \), respectively. Let \( T, A : B \rightarrow B^* \) and \( g : B \rightarrow B \) be single-valued mappings, and \( \phi : B \rightarrow (-\infty, +\infty] \) be a proper lower semicontinuous and subdifferentiable functional. We consider the following general mixed variational inequality problem (for short, GMVIP): find \( u \in D \) such that

\[
\langle Tu - Au, v - g(u) \rangle \geq \phi(g(u)) - \phi(v), \quad \forall v \in B.
\]  

Some special cases of problem (2.1):

(I) If \( A \equiv 0 \), \( g \) is an identity mapping on \( B \), and \( \phi \) is a proper convex lower semicontinuous functional, then GMVIP (2.1) reduces to the general mixed variational inequality problem considered in [5].

(II) If \( B = H \) is a Hilbert space and \( \phi \) is a proper convex lower semicontinuous functional on \( H \), then GMVIP (2.1) was studied by many authors (see, for example, [2,3,7,8]).

We first recall the following definitions and some known results.

**Definition 2.1.** Let \( T : B \rightarrow 2^{B^*} \) be a set-valued mapping, \( A : B \rightarrow B^* \) and \( g : B \rightarrow B \) be two single-valued mappings. We say that

1. \( A \) is \( \alpha \)-strongly monotone with constant \( \alpha > 0 \) if, for any \( x, y \in B \),
   \[
   (Ax - Ay, x - y) \geq \alpha \|x - y\|^2;
   \]
2. \( T \) is \( \lambda \)-strongly monotone if, for any \( x, y \in B, u \in Tx, \) and \( v \in Ty \),
   \[
   (u - v, x - y) \geq \lambda \|x - y\|^2;
   \]
3. \( T \) is \( \beta \)-Lipschitz continuous with constant \( \beta \geq 0 \) if, for all \( x, y \in B \),
   \[
   H(Tx, Ty) \leq \beta \|x - y\|,
   \]
   where \( H(\cdot, \cdot) \) is the Hausdorff metric on \( CB(B^*) \).
4. \( g \) is \( k \)-strongly accretive (where \( k \in (0, 1) \)) if, for any \( x, y \in B \), there exists \( j(x - y) \in J(x - y) \) such that
   \[
   \langle j(x - y), g(x) - g(y) \rangle \geq k \|x - y\|^2,
   \]
   where \( J : B \rightarrow 2^{B^*} \) is the normalized duality mapping defined by
   \[
   J(x) = \{ f \in B^* : \langle f, x \rangle = \|f\| \cdot \|x\|, \|f\| = \|x\| \}, \quad \forall x \in B.
   \]

**Definition 2.2.** Let \( B \) be a Banach space with the dual space \( B^* \), \( \eta : E \rightarrow E^* \), \( \phi : B \rightarrow \mathbb{R} \cup {+\infty} \) be a proper subdifferentiable functional (may not be convex). If for any given \( x^* \in B^* \) and any constant \( \rho > 0 \), there is a unique \( x \in B \) satisfying

\[
\langle \eta x - x^*, y - x \rangle + \rho \phi(y) - \rho \phi(x) \geq 0, \quad \forall y \in B,
\]

then the mapping \( x^* \mapsto x \), denoted by \( x = J^\phi_\rho(x^*) \), is said to be an \( \eta \)-proximal mapping of \( \phi \).
By the definitions of the subdifferential and (2.2), we know that $x^* - \eta x \in \rho \partial \phi(x)$ and so
\[ x = J^\phi_\rho (x^*) = (\eta + \rho \partial \phi)^{-1}(x^*). \]

**Remark 2.1.** If $B = H$ is a Hilbert space, $\eta : H \to H$ is an identity mapping on $H$ and $\phi$ is a proper convex lower semicontinuous functional on $H$, then the $J$-proximal mapping of $\phi$ reduces to the resolvent operator of $\phi$ on $H$.

**Lemma 2.1** (10). Let $D$ be a nonempty convex subset of a topological vector space and let $f : D \times D \to [-\infty, +\infty]$ be such that

(i) for each $x \in D$, $y \mapsto f(x, y)$ is lower semicontinuous on each nonempty compact subset of $D$;

(ii) for each nonempty finite set $\{x_1, \ldots, x_m\} \subset D$ and for each $y = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$,
\[ \min_{1 \leq i \leq m} f(x_i, y) \leq 0; \]

(iii) there exist a nonempty compact convex subset $D_0$ of $D$ and a nonempty compact subset $K$ of $D$ such that for each $y \in D \setminus K$, there is an $x \in co(D_0 \cup \{y\})$ with $f(x, y) > 0$.

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in D$.

Now we give some sufficient conditions which guarantee the existence and Lipschitz continuity of $J$-proximal mappings for a lower semicontinuous subdifferentiable proper functional $\phi$ on a reflexive Banach space $B$.

**Theorem 2.1.** Let $B$ be a reflexive Banach space with the dual space $B^*$, $\phi : B \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional, and $\eta : B \to B^*$ be an $\alpha$-strongly monotone and continuous mapping. Then for any given $x^* \in B^*$ and any $\rho > 0$, there exists a unique $x \in B$ such that
\[ \langle \eta x - x^*, y - x \rangle + \rho \phi(y) - \rho \phi(x) \geq 0, \quad \forall y \in B, \tag{2.3} \]
that is, $x = J^\phi_\rho (x^*)$ and the $\eta$-proximal mapping of $\phi$ is well defined.

**Proof.** For any given $x^* \in B$ and $\rho > 0$, define a functional $f : B \times B \to \mathbb{R} \cup \{+\infty\}$ as follows:
\[ f(y, x) = \langle x^* - \eta x, y - x \rangle + \rho \phi(x) - \rho \phi(y), \quad \forall x, y \in B. \]

By the continuity of $\eta$ and the lower semicontinuity of $\phi$, the function $x \mapsto f(y, x)$ is lower semicontinuous on $B$ for each fixed $y \in B$.

We now claim that $f(y, x)$ satisfies condition (ii) of Lemma 2.1. If it is false, then there exist a finite set $\{y_1, \ldots, y_m\} \subset B$ and $x_0 = \sum_{i=1}^m \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$ such that
\[ \langle x^* - \eta x_0, y_i - x_0 \rangle + \rho \phi(x_0) - \rho \phi(y_i) > 0, \quad \forall i = 1, \ldots, m. \]

Since $\phi$ is subdifferentiable at $x_0$, there exists a point $f_0 \in B^*$ such that
\[ \rho \phi(y_i) - \rho \phi(x_0) \geq \rho(f^*, y_i - x_0), \quad \forall i = 1, \ldots, m. \]

It follows that
\[ \langle x^* - \eta x_0 - \rho f^*, y_i - x_0 \rangle > 0, \quad \forall i = 1, \ldots, m. \]

Thus,
\[ 0 < \sum_{i=1}^m \lambda_i \langle x^* - \eta x_0 - \rho f^*, y_i - x_0 \rangle = \langle x^* - \eta x_0 - \rho f^*, x_0 - x_0 \rangle = 0, \]
which is a contradiction. Therefore, $f(y, x)$ satisfies condition (ii) of Lemma 2.1.
For any given point \( \hat{y} \in \text{dom}\phi \), since \( \phi \) is subdifferentiable at \( \hat{y} \), there exists a point \( f^* \in B^* \) such that
\[
\phi(x) - \phi(\hat{y}) \geq \langle f^*, x - \hat{y} \rangle, \quad \forall x \in B.
\]

It follows that
\[
f(\hat{y}, x) = \langle x - \eta x, \hat{y} - x \rangle + \rho \phi(x) - \rho \phi(\hat{y})
\>= \langle \eta \hat{y} - \eta x, \hat{y} - x \rangle + \langle x - \eta \hat{y}, \hat{y} - x \rangle + \rho(f^*, \hat{y} - x)
\geq \|\hat{y} - x\|^2 - (\|x\| + \|\eta \hat{y}\| + \rho \|f^*\|)\|\hat{y} - x\|
\>= \|\hat{y} - x\| \{\|x\| - (\|x\| + \|\eta \hat{y}\| + \rho \|f^*\|)\}.
\]

Let
\[
\|\hat{y} - x\| = \frac{1}{\alpha} (\|x\| + \|\eta \hat{y}\| + \rho \|f^*\|), \quad K = \{z \in B : \|\hat{y} - z\| \leq r\}.
\]

Then \( D_0 = \{\hat{y}\} \) and \( K \) are both weakly compact convex subsets of \( B \). For each \( x \in B \setminus K \), there exists a point \( \hat{y} \in \text{co}(D_0 \cup \{\hat{y}\}) \) such that \( f(\hat{y}, x) > 0 \) and so all conditions of Lemma 2.1 are satisfied. By Lemma 2.1, there exists a point \( \hat{x} \in B \) such that \( f(y, \hat{x}) \leq 0 \) for all \( y \in B \), that is,
\[
\langle \eta x - \hat{x}, y - \hat{x} \rangle + \rho \phi(y) - \rho \phi(\hat{x}) \geq 0, \quad \forall y \in B.
\]

Now we show that \( \hat{x} \) is a unique solution of auxiliary variational inequality (2.3). Suppose that \( x_1, x_2 \in B \) are arbitrary two solutions of auxiliary variational inequality (2.3). Then,
\[
\langle \eta x_1 - x_1, y - x_1 \rangle + \rho \phi(y) - \rho \phi(x_1) \geq 0, \quad \forall y \in B \quad (2.4)
\]
and
\[
\langle \eta x_2 - x_2, y - x_2 \rangle + \rho \phi(y) - \rho \phi(x_2) \geq 0, \quad \forall y \in B \quad (2.5)
\]
Taking \( y = x_2 \) in (2.4) and \( y = x_1 \) in (2.5) and adding these inequalities, we have
\[
\langle \eta x_1 - \eta x_2, x_1 - x_2 \rangle \leq 0.
\]

The \( \alpha \)-strongly monotonicity of \( \eta \) implies that
\[
\alpha \|x_1 - x_2\|^2 \leq \langle \eta x_1 - \eta x_2, x_1 - x_2 \rangle \leq 0,
\]
and so \( x_1 = x_2 \). This completes the proof. \( \square \)

**Remark 2.2.** Theorem 2.1 shows that for any strongly monotone and continuous mapping \( \eta : B \to B^* \), the \( \eta \)-proximal mapping \( J^\phi_\rho : B^* \to B \) for the lower semicontinuous subdifferentiable proper functional \( \phi : B \to \mathbb{R} \cup \{+\infty\} \) is well defined and for each \( x^* \in B^* \),
\[
J^\phi_\rho(x^*) = (\eta + \rho \partial \phi)^{-1}(x^*)
\]
is the unique solution of auxiliary variational inequality (2.3).

**Theorem 2.2.** Let \( B \) be a reflexive Banach space with the dual space \( B^* \), \( \eta : B \to B^* \) be \( \alpha \)-strongly monotone and continuous mapping, and \( \phi : B \to \mathbb{R} \cup \{+\infty\} \) be a lower semicontinuous subdifferentiable proper functional. Then the \( \eta \)-proximal mapping \( J^\phi_\rho(x^*) = (\eta + \rho \partial \phi)^{-1} \) is \( \frac{1}{2} \)-Lipschitz continuous. Furthermore, if the subdifferential \( \partial \phi : B \to 2^{B^*} \) for \( \phi \) is \( \xi \)-strongly monotone, then the \( \eta \)-proximal mapping \( J^\phi_\rho = (\eta + \rho \partial \phi)^{-1} \) is \( \frac{1}{\alpha + \rho \xi} \)-Lipschitz continuous.
Proof. For any \( x^*, y^* \in B^* \), let \( x = J^\phi_\rho(x^*) \) and \( y = J^\phi_\rho(y^*) \). Then \( x^* - \eta x \in \rho \partial \phi(x) \) and \( y^* - \eta y \in \rho \partial \phi(y^*) \). By the definition of the subdifferential, we have

\[
\rho \phi(u) - \rho \phi(x) \geq \langle x^* - \eta x, u - x \rangle, \quad \forall u \in B
\]

and

\[
\rho \phi(u) - \rho \phi(y) \geq \langle y^* - \eta y, u - y \rangle, \quad \forall u \in B.
\]

Taking \( u = y \) in (2.6) and \( u = x \) in (2.7) and adding these inequalities, we obtain

\[
\langle y^* - \eta x, y - x \rangle \leq \langle y - x, y^* - x^* \rangle.
\]

Since \( \eta \) is \( \alpha \)-strongly monotone,

\[
\alpha \| y - x \|^2 \leq \langle y - x, y^* - x^* \rangle \leq \| y - x \| \| y^* - x^* \|,
\]

which implies that the \( \eta \)-proximal mapping \( J^\phi_\rho \) is \( \frac{1}{\alpha} \)-Lipschitz continuous.

Now we suppose that the subdifferential \( \partial \phi : B \to 2^{B^*} \) is \( \zeta \)-strongly monotone. Then

\[
\langle x^* - \eta x - (y^* - \eta y), x - y \rangle \geq \rho \zeta \| x - y \|^2.
\]

Since \( \eta \) is \( \alpha \)-strongly monotone,

\[
\langle x^* - y^*, x - y \rangle - \langle \eta x - \eta y, x - y \rangle \geq \rho \zeta \| x - y \|^2
\]

and so

\[
\langle x^* - y^*, x - y \rangle \geq (\alpha + \rho \zeta) \| x - y \|^2.
\]

It follows that

\[
(\alpha + \rho \zeta) \| y - x \|^2 \leq \| y - x \| \| y^* - x^* \|
\]

and this implies

\[
\| J^\phi_\rho(y^*) - J^\phi_\rho(x^*) \| \leq \frac{1}{\alpha + \rho \zeta} \| y^* - x^* \|.
\]

Thus, \( J^\phi_\rho \) is \( \frac{1}{\alpha + \rho \zeta} \)-Lipschitz continuous. This completes the proof. \( \square \)

Remark 2.3. If \( B = H \) is a Hilbert space and \( \eta = I \) is an identity mapping on \( H \), then the \( \eta \)-proximal mapping of \( \phi \) reduces the resolvent operator of \( \phi \). Therefore, Theorem 2.2 generalizes Lemma 3 in [20].

3. Existence and algorithm

We first transfer GMVIP (2.1) into a fixed point problem.

**Theorem 3.1.** \( q \) is a solution of the GMVIP (2.1) if and only if \( q \) satisfies the following relation:

\[
g(q) = J^\phi_\rho [\eta(g(q)) - \rho(Tq - Aq)],
\]

where \( J^\phi_\rho = (\eta + \rho \partial \phi)^{-1} \) is the \( \eta \)-proximal mapping of \( \phi \) and \( \rho > 0 \) is a constant.

**Proof.** Assume that \( q \) satisfies relation (3.1). Noting \( J^\phi_\rho = (\eta + \rho \partial \phi)^{-1} \), relation (3.1) holds if and only if

\[
\eta(g(q)) - \rho(Tq - Aq) \in \eta(g(q)) + \rho \partial \phi(q).
\]
Remark 3.1. Relation (3.1) can be written as
\[ q \in R(T - Aq, v - g(q)) + \phi(v) - \phi(g(q)) \geq 0, \quad \forall v \in B. \]
i.e.,
\[ \langle Tq - Aq, v - g(q) \rangle + \phi(v) - \phi(g(q)) \geq 0, \quad \forall v \in B. \]
Thus, \( q \) is a solution of GMVIP (2.1) if and only if \( q \) satisfied (3.1). This completes the proof. \( \square \)

**Remark 3.2.** By Theorem 2.1, we can choose a strongly monotone and Lipschitz continuous mapping \( \eta : B \to B^* \)
such that it is easy to compute the values of the \( \eta \)-proximal mapping \( J_\rho^\phi \) of \( \phi \) on \( B^* \). Theorem 3.1 shows that, by using the \( \eta \)-proximal mapping, GMVIP (2.1) can be transfer into a fixed point problem (3.2). Based on these observations, we can suggest the following new and general iterative algorithms for computing the approximate solutions of GMVIP (2.1) in reflexive Banach spaces.

**Lemma 3.1** ([21]). Let \( B \) be a real Banach space and \( J : B \to 2^{B^*} \) be the normalized duality mapping. Then for any \( x, y \in B \), the following inequality holds:
\[ \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y). \]
We now use Theorem 3.1 to construct the following algorithm for solving the general mixed variational inequality (2.1) in Banach spaces.

**Algorithm 3.1.** Let \( T, A : B \to B^* \) be two single-valued mappings, \( g : B \to B \) be a single-valued mapping with \( g(B) = B \), \( \eta : B \to B^* \) be a \( \alpha \)-strongly monotone and \( \theta \)-Lipschitz continuous mapping, and \( \phi : B \to R \cup \{+\infty\} \) be a lower semicontinuous subdifferentiable proper functional. For any given \( x_0 \in B \), an iterative sequence \( \{x_n\} \) is defined by
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[x_n - g(x_n) + j_\rho^\phi(\eta(g(x_n)) - \rho(Tx_n - Ax_n))], \quad n = 0, 1, \ldots, \]
where \( \rho > 0 \) and \( \alpha_n \in [0, 1] \) for all \( n \geq 0 \) with \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Algorithm 3.1 is called Mann-type iterative algorithm.

**Algorithm 3.2.** Let \( T, A, g, \eta, \) and \( \phi \) be the same as in Algorithm 3.1. For any given \( x_0 \in B \), the iterative sequences \( \{x_n\} \) and \( \{y_n\} \) are defined by
\[
\begin{align*}
\left\{ \begin{array}{l}
\quad y_n = (1 - \beta_n)x_n + \beta_n[x_n - g(x_n) + j_\rho^\phi(\eta(g(x_n)) - \rho(Tx_n - Ax_n))], \\
\quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[y_n - g(y_n) + j_\rho^\phi(\eta(g(y_n)) - \rho(Ty_n - Ay_n))],
\end{array} \right.
\end{align*}
\]
where \( \alpha_n, \beta_n \in [0, 1] \) for all \( n \geq 0 \) with \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Algorithm 3.2 is called the Ishikawa-type iterative algorithm.

**Remark 3.3.** If \( \beta_n = 0 \) for all \( n \geq 0 \), then Algorithm 3.2 reduces to Algorithm 3.1.

Now we prove an existence theorem of solution for GMVIP (2.1).

**Theorem 3.2.** Let \( B \) be a reflexive Banach space with the dual space \( B^* \), \( T, A : B \to B^* \) be two continuous mappings and \( g : B \to B \) be a continuous mapping. Let \( \eta : B \to B^* \) be \( \alpha \)-strongly monotone and continuous, and \( \phi : B \to R \cup \{+\infty\} \) be a lower semicontinuous and subdifferentiable proper functional. If the ranges \( R(I - g), R(\eta g) \) and \( R(T - A) \) are bounded, then there exists \( q \in B \) which is a solution of GMVIP (2.1).
Proof. Define $F : B \rightarrow B$ by

$$F(x) = x - g(x) + J_\rho^\phi [\eta g(x) - \rho(Tx - Ax)], \quad \forall x \in B.$$ 

By Theorem 2.2, the mapping $J_\rho^\phi$ is Lipschitz continuous. Since the ranges $R(I - g)$, $R(\eta g)$ and $R(T - A)$ are bounded, we know that the range $R(F)$ is also bounded in $B$, i.e., $F(B)$ is bounded in $B$. Thus, $F(B)$ is weakly compact subset of $B$. Since $T, A, g, \eta$, and $J_\rho^\phi$ are continuous, so does $F : B \rightarrow B$. By Schauder fixed point Theorem, $F : B \rightarrow B$ has a fixed point $q \in B$. It follows from Theorem 3.1 that $q$ is a solution of GMVIP (2.1). This completes the proof. □

Now we give some sufficient conditions which guarantee the convergence of the iterative sequences generated by Algorithm 3.2.

**Theorem 3.3.** Let $B$ be a reflexive Banach space with the dual space $B^*$, $T : B \rightarrow B^*$ be $\delta$-Lipschitz continuous, $A : B \rightarrow B^*$ be $\gamma$-Lipschitz continuous, and $g : B \rightarrow B$ be $k$-strongly accretive and $\varepsilon$-Lipschitz continuous. Suppose that $\eta : B \rightarrow B^*$ is $\alpha$-strongly monotone and $\theta$-Lipschitz continuous, $\phi : B \rightarrow R \cup \{+\infty\}$ is a lower semicontinuous subdifferentiable proper functional, and the subdifferential $\partial \phi : B \rightarrow 2^{B^*}$ of $\phi$ is $\zeta$-strongly monotone. Let $\{x_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ with $x_n \rightarrow 0$, $\beta_n \rightarrow 0$, and $\sum_{n=0}^\infty x_n = \infty$. If the ranges $R(I - g)$, $R(\eta g)$, and $R(T - A)$ are bounded and

$$k\zeta > \delta + \gamma; \quad \theta \varepsilon > k\zeta,$$

then for any given $x_0 \in B$, the iterative sequence $\{x_n\}$ defined by Algorithm 3.2 converges strongly to the solution $q$ of GMVIP (2.1).

**Proof.** By Theorem 2.2 and the assumptions in Theorem 3.3, we know that the solution set of GMVIP (2.1) is nonempty. Let $q$ be a solution of GMVIP (2.1). Since $k\zeta > \delta + \gamma$ and $\theta \varepsilon > k\zeta$, we can choose a constant $\rho$ such that

$$\rho > \frac{\theta \varepsilon - k\zeta}{k\zeta - (\delta + \gamma)}. \quad (3.3)$$

By Algorithm 3.2, we have

$$y_n = (1 - \beta_n)x_n + \beta_n[x_n - g(x_n) + J_\rho^\phi(\eta g(x_n)) - \rho(Tx_n - Ax_n))],$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[y_n - g(y_n) + J_\rho^\phi(\eta g(y_n)) - \rho(Ty_n - Ay_n)].$$

Let

$$p_n = y_n - g(y_n) + J_\rho^\phi(\eta g(y_n)) - \rho(Ty_n - Ay_n),$$

$$r_n = x_n - g(x_n) + J_\rho^\phi(\eta g(x_n)) - \rho(Tx_n - Ax_n).$$

Then

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n p_n, \quad y_n = (1 - \beta_n)x_n + \beta_n r_n.$$

It follows from Theorem 2.2 that $J_\rho^\phi$ is Lipschitz continuous. Since the ranges $R(I - g)$, $R(\eta g)$, and $R(T - A)$ are bounded, we know that $x - g(x) + J_\rho^\phi(\eta g(x)) - \rho(Tx - Ax))$ is bounded. Let

$$M = \sup\{\|w - q\| : w \in (x - g(x) + J_\rho^\phi(\eta g(x)) - \rho(Tx - Ax)), x \in B\} + \|x_0 - q\| < +\infty.$$

This implies that

$$\|p_n - q\| \leq M, \quad \|r_n - q\| \leq M, \quad \forall n \geq 0. \quad (3.4)$$
Since \( \|x_0 - q\| \leq M \),

\[
\|y_0 - q\| = \| (1 - \beta_0)(x_0 - q) + \beta_0(r_0 - q) \|
\leq (1 - \beta_0)\|x_0 - q\| + \beta_0\|r_0 - q\|
\leq M.
\]

It follows that

\[
\|x_1 - q\| \leq (1 - \alpha_0)\|x_0 - q\| + \alpha_0\|p_0 - q\| \leq M,
\]

\[
\|y_1 - q\| \leq (1 - \beta_1)\|x_1 - q\| + \beta_1\|r_1 - q\| \leq M.
\]

By induction we can prove that

\[
\|x_n - q\| \leq M, \quad \|y_n - q\| \leq M, \quad \forall n \geq 0. \tag{3.5}
\]

On the other hand, by Lemma 3.1,

\[
\|x_{n+1} - q\|^2 = \| (1 - \alpha_n)(x_n - q) + \alpha_n(p_n - q) \|^2
\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle p_n - q, J(x_{n+1} - q) \rangle
\]

\[
= (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle p_n - q, J(y_n - q) \rangle
\]

\[
+ 2\alpha_n\langle p_n - q, J(x_{n+1} - q) - J(y_n - q) \rangle. \tag{3.6}
\]

Now we consider the third term on the right side of (3.6). By (3.4) and (3.5), we know that

\[
\|(x_{n+1} - q) - (y_n - q)\| = \|x_{n+1} - y_n\|
\leq (1 - \alpha_n)\beta_n\|x_n - r_n\| + \alpha_n\|p_n - q\| + \|y_n - q\|
\leq (1 - \alpha_n)\beta_n\|x_n - q\| + \alpha_n\|p_n - q\| + \|y_n - q\|
\leq 2((1 - \alpha_n)\beta_n + \alpha_n)M \to 0 \quad (n \to \infty).
\]

By the uniform continuity of the normalized duality mapping \( J : B \to 2^{B^*} \), we have

\[
J(x_{n+1} - q) - J(y_n - q) \to 0 \quad (n \to \infty).
\]

Let

\[
\delta_n = |\langle p_n - q, J(x_{n+1} - q) - J(y_n - q) \rangle|.
\]

Since \( \{p_n - q\} \) is bounded,

\[
\delta_n = |\langle p_n - q, J(x_{n+1} - q) - J(y_n - q) \rangle| \to 0 \quad (n \to \infty). \tag{3.7}
\]

Next we consider the second term on the right side of (3.6). Since \( q \) is a solution of GMVIP (2.1), by Theorem 2.1, we have

\[
q = q - g(q) + J_\rho^\phi[y(g(q)) - \rho(T(q) - A(q))].
\]
It follows that
\[
2\alpha_n \langle p_n - q, J(y_n - q) \rangle \\
= 2\alpha_n (\langle y_n - g(y_n) + J_\rho^\phi (\eta g(y_n) - \rho(T(y_n) - A(y_n))) - (q-g(q) \\
\quad + J_\rho^\phi (\eta g(q) - \rho(T(q) - A(q)))), J(y_n - q) \rangle \\
= 2\alpha_n (y_n - q, J(y_n - q)) - 2\alpha_n (g(y_n) - g(q), J(y_n - q)) \\
\quad + 2\alpha_n (J_\rho^\phi (\eta g(y_n)) - \rho(T(y_n) - A(y_n))) - J_\rho^\phi (\eta g(q)) - \rho(T(q) - A(q)))], J(y_n - q) \\
\leq 2\alpha_n \|y_n - q\|^2 - k\|y_n - q\|^2 + \lambda\|y_n - q\|^2 \\
= 2\alpha_n (1 - k + \lambda)\|y_n - q\|^2.
\]
where
\[
\lambda = \frac{1}{\alpha + \rho\xi}(\theta + \rho\delta + \rho\gamma).
\]
Substituting (3.7) and (3.8) into (3.6), we have
\[
\|x_{n+1} - q\|^2 \leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n (1 - k + \lambda)\|y_n - q\|^2 + 2\alpha_n \delta_n.
\]
Next we make an estimation for \(\|y_n - q\|^2\). In fact,
\[
\|y_n - q\|^2 = (1 - \beta_n)(x_n - q) + \beta_n (r_n - q) \|^2 \\
\leq (1 - \beta_n)^2 \|x_n - q\|^2 + 2\beta_n \langle r_n - q, J(y_n - q) \rangle \\
\leq (1 - \beta_n)^2 \|x_n - q\|^2 + 2\beta_n \|r_n - q\| \|y_n - q\| \\
\leq \|x_n - q\|^2 + 2\beta_n M^2.
\]
Substituting (3.10) into (3.9) and simplifying, we have
\[
\|x_{n+1} - q\|^2 \leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n (1 - k + \lambda)\{|\|x_n - q\|^2 + 2\beta_n M^2\| + 2\alpha_n \delta_n \\
= \|x_n - q\|^2 - 2\alpha_n (k - \lambda)\|x_n - q\|^2 + \alpha_n^2 \|x_n - q\|^2 \\
\quad + 4\alpha_n \beta_n (1 - k + \lambda) M^2 + 2\alpha_n \delta_n.
\]
Let
\[
\sigma = \inf_{n \geq 0} \|x_n - q\|.
\]
Next we prove that \(\sigma = 0\). Suppose that \(\sigma > 0\). Then we have \(\|x_n - q\| \geq \sigma > 0\) for all \(n \geq 0\). By condition (3.3), we have \(k > \lambda\). It follows from (3.11) that
\[
\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - 2\alpha_n (k + \lambda)\sigma^2 + \alpha_n^2 M^2 + 4\alpha_n \beta_n (1 - k + \lambda) M^2 + 2\alpha_n \delta_n \\
= \|x_n - q\|^2 - \alpha_n (k - \lambda)\sigma^2 - \alpha_n [(k - \lambda)\sigma^2 - \alpha_n M^2 \\
\quad - 4\beta_n (1 - k + \lambda) M^2 - 2\delta_n].
\]
Since \(\beta_n \to 0, \alpha_n \to 0, \) and \(\delta_n \to 0, \) there exists \(n_0\) such that, for \(n \geq n_0, \)
\[
(k - \lambda)\sigma^2 - \alpha_n M^2 - 4\beta_n (1 - k + \lambda) M^2 - 2\delta_n > 0.
\]
Therefore, it follows from (3.12) that
\[
\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n (k - \lambda)\sigma^2, \ \forall n \geq n_0,
\]

i.e.,
\[ x_n (k - \lambda) s^2 \leq \| x_n - q \|^2 - \| x_{n+1} - q \|^2, \quad \forall n \geq n_0. \]

Thus, for any \( m \geq n_0 \), we have
\[
\sum_{n=n_0}^{m} x_n (k - \lambda) s^2 \leq \| x_{n_0} - q \|^2 - \| x_{m+1} - q \|^2 \\
\leq \| x_{n_0} - q \|^2.
\]

Letting \( m \to \infty \), we have
\[
\infty = (k - \lambda) s^2 \sum_{n=n_0}^{\infty} x_n \leq \| x_{n_0} - q \|^2,
\]
which is a contradiction. Therefore, \( \sigma = 0 \) and so there exists a subsequence \( \{ x_{n_j} \} \subset \{ x_n \} \) such that \( x_{n_j} \to q \), i.e., for any given \( \varepsilon > 0 \), there exists \( j_0 \in N \) such that \( \| x_{n_j} - q \| < \varepsilon \) as \( n_j \geq j_0 \). Since \( \lim_{j \to +\infty} x_{n_j} = 0 \), there exists \( N_1 \in N \) such that, for \( n_j \geq N_1 \),
\[
\| x_{n_j} \| \leq \frac{\varepsilon}{2M}. \tag{3.13}
\]

Since \( \beta_{n_j} \to 0 \), \( x_{n_j} \to 0 \), \( \delta_{n_j} \to 0 \), and \( k > \lambda \), there exists \( N_2 \in N \) such that, for \( n_j \geq N_2 \),
\[
x_{n_j} M^2 + 4 \beta_{n_j} (1 - k + \lambda) M^2 + 2 \delta_{n_j} \leq \frac{1}{4} (k - \lambda) \varepsilon^2. \tag{3.14}
\]

Now we prove that for all \( n_j \geq \max \{ n_{j_0}, N_1, N_2 \} \), \( \| x_{n_j+1} - q \| < \varepsilon \). Assume that \( \| x_{n_j+1} - q \| \geq \varepsilon \) as \( n_j \geq \max \{ n_{j_0}, N_1, N_2 \} \), then we have
\[
e \leq \| x_{n_j+1} - q \| = \| (1 - x_{n_j}) x_{n_j} + x_{n_j} p_{n_j} - q \| \\
\leq (1 - x_{n_j}) \| x_{n_j} - q \| + x_{n_j} \| p_{n_j} - q \| \\
\leq \| x_{n_j} - q \| + x_{n_j} M \quad \text{(from (3.4) and } x_{n_j} \in (0, 1)) \\
\leq \| x_{n_j} - q \| + \frac{\varepsilon}{2} \quad \text{(from (3.13))}
\]

That is, \( \| x_{n_j} - q \| > \frac{\varepsilon}{2} \) as \( n_j \geq \max \{ n_{j_0}, N_1, N_2 \} \). It follows from (3.11), (3.5), and (3.14) that, for \( n_j \geq \max \{ n_{j_0}, N_1, N_2 \} \),
\[
\| x_{n_j+1} - q \|^2 \leq (1 - x_{n_j})^2 \| x_{n_j} - q \|^2 + 2 x_{n_j} (1 - k + \lambda) \| x_{n_j} - q \|^2 + 2 \beta_{n_j} \| x_{n_j} \|^2 \\
= \| x_{n_j} - q \|^2 - 2 x_{n_j} (k - \lambda) \| x_{n_j} - q \|^2 + x_{n_j}^2 \| x_{n_j} - q \|^2 \\
+ 4 x_{n_j} \beta_{n_j} (1 - k + \lambda) M^2 + 2 x_{n_j} \delta_{n_j} \\
\leq \| x_{n_j} - q \|^2 - \frac{1}{2} x_{n_j} (k - \lambda) \varepsilon^2 + x_{n_j}^2 \| x_{n_j} - q \|^2 + 4 x_{n_j} \beta_{n_j} (1 - k + \lambda) M^2 + 2 x_{n_j} \delta_{n_j} \\
\leq \| x_{n_j} - q \|^2 - \frac{1}{4} x_{n_j} (k - \lambda) \varepsilon^2 \\
- x_{n_j} \left( \frac{1}{4} (k - \lambda) \varepsilon^2 - x_{n_j} M^2 - 4 \beta_{n_j} (1 - k + \lambda) M^2 - 2 \delta_{n_j} \right) \\
\leq \| x_{n_j} - q \|^2 - \frac{1}{4} x_{n_j} (k - \lambda) \varepsilon^2 \\
\leq \| x_{n_j} - q \|^2,
\]
i.e., $\|x_{n_j} - q\| \geq \varepsilon$, which is contradiction. This implies that $\|x_{n_j+1} - q\| < \varepsilon$ as $n_j \geq \max\{n_0, N_1, N_2\}$. By induction, we can prove that $\|x_{n_j+i} - q\| < \varepsilon$ for all $i \in N$. This implies that for any given $\varepsilon > 0$,

$$\|x_n - q\| = \|x_{n_j+n-n_j} - q\| < \varepsilon,$$

for all $n > \max\{n_0, N_1, N_2\}$. Therefore, the sequence $\{x_n\}$ defined by Algorithm 3.2 converges strongly to the solution $q$ of GMVIP (2.1). This completes the proof. \Box

**Remark 3.2.** We would like to point out that, in Theorem 3.3, the functional $\phi$ may not be convex, the mappings $T$ and $A$ may not have any monotonicity and their domains and ranges are reflexive Banach space $B$ and the dual space $B^*$ of $B$, respectively. Hence Theorem 3.3 improves and generalizes some known results in [5–7,9,14]. Furthermore, the argument methods presented in this paper are quite different from those in [5,6,9,12,14,16,18].

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**References**