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A Characterization of Rings with Krull Dimension

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1. INTRODUCTION

Throughout this paper rings will mean associative rings with identity and all modules are assumed to be unitary.

As is well known, cyclic modules play an important role in ring theory. Many nice properties of rings can be characterized by their cyclic modules, see for example [1, 2, 9–11]. One of the most important results in this direction is the result of Osofsky [8, Theorem] which says that a ring R is semisimple (i.e., R is right Artinian with zero Jacobson radical) if and only if every cyclic right R-module is injective. Starting from this and in connection with a result of Vamos [11] the following result has been recently obtained in [2, Theorem 1.1]:

A ring R is right Artinian if and only if every cyclic right R-module is a direct sum of an injective module and a finitely cogenerated module.

In the present paper we follow this investigation and aim to show a similar result for rings with Krull dimension. We consider the following condition about a ring R:

(P) Every cyclic right R-module is a direct sum of an injective module and a module containing an essential submodule with Krull dimension at most α.

We shall prove that a ring satisfying (P) has right Krull dimension at most α (Theorem 7).

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2. Preliminaries

Let R be a ring. For a module M, M_R means that M is a right R-module. To say that M has finite uniform dimension means that M does not contain an infinite direct sum of non-zero submodules. In this case there exists a smallest positive integer k such that every direct sum of non-zero submodules of M in M has at most k terms. This k is called the uniform dimension of M. A submodule K of M is called essential in M if for each non-zero submodule H of M, $K \cap H \neq 0$. M is called uniform if every non-zero submodule of M is essential in M. For a ring R and $x \in R$ we set $r(x) = \{r \in R, xr = 0\}$. The set $Z(R) = \{x \in R, r(x) \text{ is essential in } R_R\}$ is an ideal of R which is called the right singular ideal of R. In case $Z(R) = 0$, R is called right non-singular.

For a module M, the Krull dimension of M is defined in [6]. In case M has Krull dimension we denote it by $K\dim M$. Let α be an ordinal and M be a non-zero module. Then M is called α-critical if $K\dim M = \alpha$ and for each proper homomorphic image M' of M, $K\dim M' < \alpha$. A critical module is a module which is α-critical for some α.

3. Results

We start our investigation by giving the following easy but useful lemma.

Lemma 1. *Let R be a ring with the property* (P) *and K be an ideal of R. Then the factor ring R/K also has the property* (P).

Lemma 2. *Let R be a von Neumann regular, right self-injective ring such that every cyclic right R-module is a direct sum of an injective module and a module with finite uniform dimension. Then R is semisimple.*

Proof. We use the argument in the proof of [2, Theorem 1.1] which showed that the ring B considered there is semisimple.

Lemma 3. *Let R be a semiprime, right non-singular ring and $A \neq 0$ be a right ideal of R such that A_R is indecomposable and injective. Then A is a minimal right ideal of R.*

Proof. Let T be a maximal right quotient ring of R. Since R is right non-singular, T has the following properties (see for example [3] or [5, Chap. 2]):

(a) If X is a submodule of T_R then X_R is essential in $(XT)_R$.

(b) T is a von Neumann regular, right self-injective ring.

(c) T_R is injective.

By (a), A_R is essential in $(AT)_R$. Since A_R is injective by assumption $A = AT$, therefore A is a right ideal of T. Moreover it is easy to see that A is a minimal right ideal of T. Hence $A = eT = eR$ for some idempotent e in A. It follows that $eTe = eRe$. But eTe is a skew field, so we get that A is a minimal right ideal of R by the semiprimeness of R, completing the proof.

Lemma 3 has the following immediate consequence.

COROLLARY 4. *For a right ideal A of a semiprime right Goldie ring R the following conditions are equivalent:*

(a) *A_R is semisimple (i.e., a direct sum of simple modules).*

(b) *A_R is quasi-injective.*

(c) *A_R is injective.*

LEMMA 5. *Let R be a semiprime ring, A and B be right ideals of R with $B \subseteq A$ such that B_R is essential in A_R. If B_R has Krull dimension so also has A_R and K dim $A_R = K$ dim B_R.*

Proof. Our proof is similar to the argument proving [6, Theorem 3.3]. Let B_R be essential in A_R and have Krull dimension. We will show that there is a c in B such that $r(c) \cap A = 0$. Then $A_R \cong (cA)_R \subseteq B_R$ and it follows that $K \dim A_R$ exists with $K \dim A_R = K \dim B_R$.

By [6, Theorem 2.1], B_R contains a critical submodule C_1. Since R is semiprime, [6, Corollary 2.5, Lemma 3.2] shows the existence of a c_1 in C_1 such that $r(c_1) \cap C_1 = 0$. If $r(c_1) \cap A = 0$, we are done. If not, there is a critical submodule C_2 in $r(c_1) \cap B$ and a c_2 in C_2 such that $r(c_2) \cap C_2 = 0$. Clearly $C_1 \cap C_2 = 0$ and $r(c_1 \oplus c_2) \cap (C_1 \oplus C_2) = 0$. In this way we get for example after i steps a direct sum of i critical submodules $C_1, ..., C_i$ of B_R. But since B_R has finite uniform dimension, there exists a positive integer m such that $c = c_1 + \cdots + c_m \in B$ and $r(c) \cap A = 0$.

The next lemma is implicit in [6, pp. 16, 33] but we give a proof for completeness. We use a technique of [7].

LEMMA 6. *Let R be a ring, M be a right R-module, and α be an ordinal. Then M has Krull dimension at most α if and only if every homomorphic image of M has an essential submodule with Krull dimension at most α.*

Proof. The necessity is clear. Conversely, suppose that every homomorphic image of M contains an essential submodule with Krull dimension at most α. It is clear that there exists an ascending chain of submodules

$$0 = B_0 \subset B_1 \subset \cdots \subset B_\sigma \subset B_{\sigma+1} \subset \cdots \subset B_\rho = M$$

such that for each ordinal $\sigma \geqslant 0$, $B_{\sigma+1}/B_\sigma$ is an essential submodule of M/B_σ with $K\dim(B_{\sigma+1}/B_\sigma) \leqslant \alpha$, and

$$B_\sigma = \bigcup_{0 \leqslant \tau < \sigma} B_\tau \tag{1}$$

if σ is a limit ordinal. Suppose that $K\dim M \nleqslant \alpha$. Let σ denote the least ordinal such that $K\dim B_\sigma \nleqslant \alpha$. If σ is not a limit ordinal, then $K\dim B_{\sigma-1} \leqslant \alpha$ and hence $K\dim B_\sigma \leqslant \alpha$ by [6, Lemma 1.1], a contradiction. Thus σ is a limit ordinal and hence (1) holds. Because $K\dim B_\sigma \nleqslant \alpha$, there exists an infinite descending chain

$$B_\sigma = M_0 \supset M_1 \supset \cdots \tag{2}$$

of submodules M_i of B_σ such that $K\dim(M_i/M_{i+1}) \nleqslant \alpha - 1$ for all i. By (1) there exists $0 \leqslant \mu < \sigma$ such that $B_\mu \nsubseteq M_1$. Let $A_0 = 0$, $A_1 = B_\mu$ and note that

$$M_0 \cap A_1 = B_\mu \nsubseteq M_1 = A_0 + M_1.$$

Consider the descending chain

$$A_1 \cap M_0 \supseteq A_1 \cap M_1 \supseteq A_1 \cap M_2 \supseteq \cdots.$$

Because $K\dim A_1 \leqslant \alpha$, there exists $k \geqslant 1$ such that $K\dim[(A_1 \cap M_k)/(A_1 \cap M_{k+1})] \leqslant \alpha - 1$. Suppose that $M_k \subseteq A_1 + M_{k+1}$. Then $M_k = (A_1 \cap M_k) + M_{k+1}$, and hence $M_k/M_{k+1} \cong (A_1 \cap M_k)/(A_1 \cap M_{k+1})$, which gives the contradiction $K\dim(M_k/M_{k+1}) \leqslant \alpha - 1$. Thus $M_k \nsubseteq A_1 + M_{k+1}$. It follows that $M_1 \nsubseteq A_1 + M_{k+1}$, so that without loss of generality we can assume that $k = 1$. By (1) there exists $0 \leqslant \nu < \sigma$ such that

$$M_1 \cap B_\nu \nsubseteq A_1 + M_2. \tag{3}$$

Let $A_2 = B_\nu$. Note that if $A_2 \subseteq A_1$, then (3) is contradicted. Thus $A_1 \subset A_2$. Repeating this argument we obtain an ascending chain of submodules of B_σ,

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots, \tag{4}$$

such that $M_i \cap A_{i+1} \nsubseteq A_i + M_{i+1}$ for all i. Now we use (2) and (4) to show that there is a submodule N of B_σ such that B_σ/N does not have finite uniform dimension.

Let $N = \sum_{i \geqslant 1} (A_i \cap M_i)$ and let $\varphi\colon B_\sigma \to B_\sigma/N$ denote the canonical epimorphism. Then B_σ/N has no finite uniform dimension. To prove this we note first that $\varphi(A_i \cap M_{i-1}) \neq 0$ for each $i \geqslant 1$. Suppose on the contrary that $\varphi(A_i \cap M_{i-1}) = 0$ for some $i \geqslant 1$. There exists a $j \geqslant 1$ such that

$A_i \cap M_{i-1} \subseteq (A_1 \cap M_i) + \cdots + (A_j \cap M_j)$. Let $x \in A_i \cap M_{i-1}$. Then $x = y_1 + \cdots + y_t$ for some $y_t \in A_t \cap M_t$ $(t = 1, ..., j)$. Suppose $j > i$. Then

$$y_j = x - y_1 - \cdots - y_{j-1} \in A_{j-1} \cap M_j \subseteq A_{j-1} \cap M_{j-1}.$$

Thus without loss of generality we can suppose that $j = i$. It follows

$$x = y_1 + \cdots + y_i = (y_1 + \cdots + y_{i-1}) + y_i \in A_{i-1} + M_i.$$

Thus $\varphi(A_i \cap M_{i-1}) = 0$ implies $A_i \cap M_{i-1} \subseteq A_{i-1} + M_i$, a contradiction. Hence we have $\varphi(A_i \cap M_{i-1}) \neq 0$ for each $i \geqslant 1$. Second we have to check that $\varphi(A_1) + \varphi(A_2 \cap M_1) + \varphi(A_3 \cap M_2) + \cdots$ is a direct sum in B_σ/N. Suppose $a_i \in A_i \cap M_{i-1}$ $(i = 1, ..., k)$ and $\varphi(a_1) + \cdots + \varphi(a_k) = 0$ for some positive integer k. Then there exist a positive integer n and elements $b_i \in A_i \cap M_i$ $(i = 1, ..., n)$ such that

$$a_1 + \cdots + a_k = b_1 + \cdots + b_n.$$

By the above argument we can suppose without loss of generality that $n = k$. In this case $a_k - b_k = b_1 + \cdots + b_{k-1} - a_1 - \cdots - a_{k-1} \in A_{k-1}$, and hence $a_k - b_k \in A_{k-1} \cap M_{k-1}$. Thus $\varphi(a_k) = 0$. By induction on k, $\varphi(a_i) = 0$ for $i = 1, ..., k-1$. Thus the sum $\varphi(A_1) + \varphi(A_2 \cap M_1) + \cdots$ is direct, so the factor module B_σ/M has no finite uniform dimension. By hypothesis there exists a submodule K of M containing N such that K/N is an essential submodule of M/N and K/N has Krull dimension. It follows that M/N, and hence B_σ/N, has finite uniform dimension. This contradiction proves that M has Krull dimension at most α.

Note that Lemma 6 generalizes a result of Vamos, who proved it in the case $\alpha = 0$ (see [11, Proposition 2*]).

Now we are in a position to prove the main result of the paper.

THEOREM 7. *Every ring satisfying* (P) *has right Krull dimension at most* α.

Proof. We consider first the case that R is semiprime. In this case we first show that R is right non-singular. Assume on the contrary that the right singular ideal Z of R contains a non-zero element x. By (P), $xR = I \oplus K$, where I_R is injective and K_R contains an essential submodule with Krull dimension. Since Z cannot contain non-zero idempotents, it follows that $I = 0$ and $xR = K$. Hence xR contains a critical submodule C. We have $C^2 \neq 0$ since R is semiprime. Therefore C contains a d with $dC \neq 0$. Since $r(d) \neq 0$, dC is a proper homomorphic image of C_R. Hence $K\dim dC < K\dim C$ by the definition of critical modules. On the other hand, since dC is a non-zero submodule of C, we have $K\dim dC = K\dim C$

by [6, Proposition 2.3]. This is a contradiction. Hence $Z=0$, i.e., R is right non-singular.

Now, we have by (P)

$$R_R = I \oplus K, \tag{1}$$

where I_R is injective and K_R contains an essential submodule with Krull dimension at most α. If K_R is injective, R is right self-injective. Moreover, by the above, R is right non-singular. Hence by [3, Corollary 19.28] R is von Neumann regular. By Lemma 2, R is semisimple.

We consider now the case that K_R is not injective. By Lemma 5, K_R has Krull dimension at most α. Since the uniform dimension of K_R is finite, by (1) we can, without loss of generality, assume that K_R does not contain non-zero injective submodules. Let $U \neq 0$ be a uniform submodule in K_R and for this U let $B(U)$ be the sum of all such uniform submodules V of R_R which are R-isomorphic to U_R. By [4, Lemma 5.1] for a uniform submodule V of R_R and an $x \in R$, if $r(x) \cap V = 0$, then $xV \cong V_R$, and if $r(x) \cap V \neq 0$, $xV = 0$. From this $B(U)$ is a (two-sided) ideal of R. We shall show that $B(U) \subseteq K$.

By (1) we have

$$B(U) = (I \cap B(U)) \oplus (K \cap B(U)). \tag{2}$$

Suppose that $I \cap B(U) \neq 0$. Then there are finitely many uniform submodules $V_1, ..., V_t$ of R_R such that $V_i \cong U_R$ and

$$V = (V_1 + \cdots + V_t) \cap I \neq 0.$$

Then it is clear that V_R has Krull dimension. Therefore V contains a uniform submodule $W \neq 0$. Denote by $E(W)$ the injective hull of W in I_R. Then Lemma 3 shows that $E(W)$ is a minimal right ideal of R. Hence $E(W) = W$, so W_R is a simple injective right R-module. Now let $0 \neq w \in W$. As we have mentioned above, if $wU \neq 0$, then $wU \cong U_R$, implying the injectivity of U_R, a contradiction to the assumption about U. Hence $wU = 0$ for each $w \in W$, i.e., $WU = 0$. By the semiprimeness of R we have also $UW = 0$. Now let X be a right ideal of R such that there is an R-isomorphism α of X_R onto U_R. Then $\alpha(XW) = \alpha(X)\, W = UW = 0$; therefore $XW \subseteq \ker \alpha = 0$. From this it follows that $B(U)\, W = 0$, in particular $W^2 = 0$, a contradiction to the semiprimeness of R. Hence $I \cap B(U) = 0$, so by (2), $B(U)$ is actually contained in K. In conclusion, if K_R is not injective, then K_R contains a non-zero ideal of R.

By Zorn's Lemma there is an ideal F of R which is maximal with respect to the condition $F \subseteq K$. The factor ring $\bar{R} = R/F$ satisfies (P) by Lemma 1. Let A be an ideal of R such that $A^k \subseteq F$ for some positive integer k. Then by (1), $(I \cap A)^k = 0$, therefore $I \cap A = 0$, implying that $A \subseteq K$. Hence $A \subseteq F$

by the maximality of F in K. This shows that $\bar{R}$ is semiprime, so $\bar{R}$ is right non-singular. From (1) we have

$$\bar{R} = \bar{I} \oplus \bar{K},$$

where $\bar{I} = (I + F)/F$, $\bar{K} = K/F$. It is not difficult to show that $\bar{I}_{\bar{R}}$ is injective and $\bar{K}_{\bar{R}}$ has Krull dimension.

Now, if $\bar{K}_{\bar{R}}$ is not injective, then by the above, $\bar{K}_{\bar{R}}$ contains a non-zero ideal of $\bar{R}$, a contradiction to the maximality of F in K. Hence $\bar{K}_{\bar{R}}$ is injective, i.e., $\bar{R}$ is a right self-injective, semiprime ring satisfying (P). As in the beginning of the proof we can conclude that $\bar{R}$ is semisimple. Since F_R has Krull dimension at most α, it follows that R has right Krull dimension at most α.

Now we consider the general case. Let R be a ring satisfying (P) and N be the prime radical of R. By Lemma 1 and the result above, R/N has right Krull dimension at most α. Since N is a nil ideal of R, it follows that $R_R = e_1R \oplus \cdots \oplus e_mR$, where $\{e_i\}_{i=1}^m$ is a system of orthogonal primitive idempotents of R. By (P) each e_iR is injective or contains an essential submodule with Krull dimension at most α. Suppose that for some e_i, e_iR is injective. If $e_iR \cap N = 0$, e_iR has Krull dimension at most α. If $e_iR \cap N \neq 0$, let x be a non-zero element in $e_iR \cap N$. Clearly xR does not contain non-zero injective submodules, therefore xR contains an essential submodule with Krull dimension at most α. Thus R contains an essential right ideal with Krull dimension at most α. This together with Lemma 1 shows that any homomorphic image of R contains an essential right ideal with Krull dimension at most α. Now using [6, Corollary 5.10] we easily see that the prime radical N is nilpotent.

Let k be a positive integer such that $N^k = 0$ and $N^{k-1} \neq 0$. By induction on k we can assume that R/N^{k-1} has right Krull dimension at most α. Let M be a cyclic injective right R-module and $L = \{x \in M, xN = 0\}$. Since $N^k = 0$, it follows that L is an essential submodule of M. Moreover, M/L is a cyclic right R/N^{k-1}-module, hence $K\dim(M/L) \leqslant \alpha$. By [6, Proposition 1.4], M/L has finite uniform dimension, n say. We show that L has finite uniform dimension. First we see that L is an injective right R/N-module. Suppose on the contrary that L contains an infinite direct sum K of non-zero submodules K_1, K_2, ... Without loss of generality we can assume that K is essential in L. One can easily see that there exist submodules Q_j of K ($j = 1, ..., n+1$) such that

$$K = Q_1 \oplus \cdots \oplus Q_{n+1}$$

and each Q_j is an infinite direct sum of some K_i ($i = 1, 2, ...$). Denote by E_j the R/N-injective hull in L of Q_j. Then

$$L = E_1 \oplus \cdots \oplus E_{n+1}. \tag{3}$$

Let F_j denote the R-injective hull of E_j in M. Then

$$M = F_1 \oplus \cdots \oplus F_{n+1}. \tag{4}$$

If for any j, $E_j = F_j$, then E_j is a direct summand of M. It follows that E_j is a cyclic right R/N-module; therefore E_j has Krull dimension at most α. In particular, E_j has finite uniform dimension by [6, Proposition 1.4], a contradiction. Hence for each j, $E_j \neq F_j$. But by (3) and (4) we have

$$M/L \cong (F_1/E_1) \oplus \cdots \oplus (F_{n+1}/E_{n+1}),$$

which shows that the uniform dimension of M/L is at least $n+1$, a contradiction. Thus L has finite uniform dimension. There exist a positive integer t and elements x_i $(i = 1, ..., t)$ in L such that $x_1 R \oplus \cdots \oplus x_t R$ is essential in L. But note that for each i, $K \dim x_i R \leqslant \alpha$ since $LN = 0$. Thus every cyclic injective right R-module contains an essential submodule with Krull dimension at most α. By (P) it follows that every cyclic right R-module has an essential submodule with Krull dimension at most α. Now Lemma 6 shows that R has right Krull dimension at most α.

The proof of Theorem 7 is complete.

Note that for $\alpha = 0$ we obtain the main result of [2] mentioned in the Introduction.

In [1, Theorem 4.1] Chatters proves that if a ring R has the property that there exists an ordinal α such that every cyclic right R-module is a direct sum of a projective module and a module with Krull dimension at most α then R has right Krull dimension at most $\alpha + 1$. The ring of integers is an example for the fact that in this result of Chatters, $\alpha + 1$ cannot be replaced by α. Also in [1, Theorem 3.1] Chatters characterized right Noetherian rings as rings whose cyclic right modules are a direct sum of a projective module and a Noetherian module. A question arises naturally whether or not the same statement holds also if "projective" is replaced by "injective." Using Theorem 7 and Corollary 4 we can answer this question positively in case of semiprime rings.

PROPOSITION 8. *A semiprime ring R is right Noetherian if every cyclic right R-module is a direct sum of an injective module and a Noetherian module.*

Proof. Since every Noetherian module has Krull dimension, we can use Theorem 7 to see that R has right Krull dimension (at most α for some ordinal α). Then, as is well known, R is right Goldie. By assumption, $R_R = I \oplus M$, where I_R is injective and M_R is Noetherian. By Corollary 4, I_R is semisimple. Thus R is right Noetherian, completing the proof.

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