Journal of Algebra 356 (2012) 115–132

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the classification of regular holonomic $\mathcal{D}$-modules on skew-symmetric matrices

Philibert Nang

École Normale Supérieure, Laboratoire de Recherche en Mathématiques (LAREMA), BP 8637, Libreville, Gabon

ARTICLE INFO

Article history:
Received 18 April 2011
Available online 8 February 2012
Communicated by V. Srinivas

In honor of Professor Louis Boutet de Monvel

MSC:
primary 32C38
secondary 32S25, 32S60

Keywords:
$\mathcal{D}$-modules
Holonomic $\mathcal{D}$-modules
Invariant differential operators
Capelli identity
Multiplicity free spaces
Linear representations
Invariant sections
Homogeneous sections
Skew-symmetric matrices

ABSTRACT

We give a classification of regular holonomic $\mathcal{D}$-modules on $2m \times 2m$-skew-symmetric matrices related to the action of the general linear group $GL(2m, \mathbb{C})$. Actually we establish one more case of the conjecture by T. Levasseur (Levasseur, 2009 [12, p. 508, Conjecture 5.17]).

1. Introduction

One knows by the Riemann–Hilbert correspondence [6] that there is a general equivalence between the category consisting of regular holonomic $\mathcal{D}_V$-modules with characteristic variety Σ and the category consisting of perverse sheaves on V (where V is a complex manifold) with microsupport Σ. This gives a classification of regular holonomic $\mathcal{D}_V$-modules theoretically, but in practice the classification of perverse sheaves is not always much simpler. A more accessible problem is as follows:

E-mail addresses: nangphilibert@yahoo.fr, pnang@ictp.it.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2012.01.021

given a complex manifold V on which a Lie group acts with finitely many orbits $(V_j)_{j\in J}$; the problem is to classify regular holonomic $\mathcal{D}_V$-modules whose characteristic variety is contained in the union of conormal bundles ($\Sigma := \overline{\bigcup_{j\in J} T^*_{V_j}V}$) to these orbits. These modules form a full category which we denote by $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}_V)$.

In this paper we consider the action of $GL(N,\mathbb{C})$ on skew-symmetric tensors. This last induces a natural action on $V := \Lambda^2(\mathbb{C}^N)$, which we will think of as the space of complex skew-symmetric matrices of size N. There are $(\lfloor \frac{N}{2} \rfloor + 1)$ orbits $V_{2k} := \{X \in V, \mathrm{rank}(X) = 2k\}$ the set of rank $2k$-skew-symmetric matrices in V $(0 \leqslant k \leqslant \lfloor \frac{N}{2} \rfloor$ where $\lfloor \frac{N}{2} \rfloor$ is the integer part of $\frac{N}{2}$). This study is done here for $N = 2m$ even which is the most interesting case (see [17]).

The main ingredient to get the classification is the extension of the action of $GL(2m,\mathbb{C})$ on V to the action of its universal covering $SL(2m,\mathbb{C}) \times \mathbb{C}$ on regular holonomic $\mathcal{D}_V$-modules in $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}_V)$ (see Remark 5). In particular we take a closer look on the action of $SL(2m,\mathbb{C})$ on these objects. It turns out that such $\mathcal{D}_V$-modules are generated by finitely many global sections which are invariant by $SL(2m,\mathbb{C})$ (see Theorem 9).

Let us point out that here there is a natural $\mathbb{C}$-algebra associated to this situation: the graded algebra $\mathcal{A}$ of (polynomial coefficients) invariant differential operators acting on polynomials of the pfaffian. It is precisely the quotient of $\overline{\mathcal{A}} := \Gamma(V, \mathcal{D}_V)^{SL(2m,\mathbb{C})}$, the Weyl algebra of $SL(2m,\mathbb{C})$-invariant differential operators on V, by an ideal described in Section 3 (see Proposition 6 and Corollary 8).

The main result of this paper is Theorem 18 saying that there is an equivalence of categories between the category $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}_V)$ consisting of regular holonomic $\mathcal{D}_V$-modules as above and the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ consisting of graded $\mathcal{A}$-modules of finite type for the Euler vector field on V. Actually the image by this equivalence of a regular holonomic $\mathcal{D}_V$-module is its set of global homogeneous sections (i.e. global sections of finite type for the Euler vector field on V) which are invariant under the action of $SL(2m,\mathbb{C})$. Note that we establish here one more case of the conjecture by T. Levasseur (see [12, p. 508, Conjecture 5.17]).

The $\mathbb{C}$-algebra $\mathcal{A}$ is described simply by generators and relations (see Corollary 8) thanks to skew-Capelli identities constructed by R. Howe and T. Umeda (see [5, p. 592, Corollary (11.3.19)]) and explicitly calculated by K. Kinoshita and M. Wakayama [11]. This leads to the description of the latter category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ as an "elementary" category consisting of diagrams of finite-dimensional complex vector spaces and linear maps between them satisfying certain relations (quiver category, see Section 6).

We should note that, even before our study, several authors, notably L. Boutet de Monvel [1] gave, very elegantly, a description of regular holonomic $\mathcal{D}$-modules in one variable by using pairs of finite-dimensional $\mathbb{C}$-vector spaces and certain linear maps. Galligo, Granger and Maisonobe [3] obtained, using the Riemann–Hilbert correspondence, a classification of regular holonomic $\mathcal{D}_{\mathbb{C}^n}$-modules with singularities along the hypersurface $x_1 \cdots x_n = 0$ by 2^n-tuples of $\mathbb{C}$-vector spaces with a set of linear maps. L. Narvaez Macarro [18] treated the case $y^2 = x^p$ using the method of Beilinson and Verdier and generalized this study to the case of reducible plane curves. R. Macpherson and K. Vilonen [13] treated the case with singularities along the curve $y^n = x^m$ etc. Finally let us mention that the author has obtained similar results for holonomic $\mathcal{D}$-modules on $M_n(\mathbb{C})$ the space of complex square matrices associated to the action of $GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ (see [14,16]) and on $\mathbb{C}^n$ associated to the action of the orthogonal group (see [15]).

Throughout the paper we assume that the reader is familiar with all basic notions of $\mathcal{D}$-modules theory (see [2,7–9]).

2. Preliminary results

As in the Introduction V denotes the complex vector space of $2m \times 2m$-skew-symmetric matrices. We denote a typical element of the space by X, with entries x_{ij} $(1 \leqslant i, j \leqslant 2m)$ with the understanding that $x_{ji} = -x_{ij}$ and $x_{ii} = 0$. The action of $GL(2m,\mathbb{C})$ on skew-symmetric matrices X is given by $g \cdot X := gXg^t$ for $g \in GL(2m,\mathbb{C})$. The orbits for this action are the set of skew-symmetric matrices X in V of rank exactly $2k$ $(0 \leqslant k \leqslant m)$ which we will denote by $V_{2k} := \{X \in V, \mathrm{rank}(X) = 2k\}$. We have the following proposition:

Proposition 1. *For $k < m$ the $GL(2m,\mathbb{C})$-orbits V_{2k} are simply connected.*

Proof. Note that for $k < m$ the $SL(2m,\mathbb{C})$ and $GL(2m,\mathbb{C})$-orbits are the same. If the skew-symmetric matrices are thought of as the skew-symmetric bilinear forms, then the $SL(2m,\mathbb{C})$-orbits V_{2k} are represented by the forms $\omega_{2k} := e^1 \wedge e^2 + \cdots + e^{2k-1} \wedge e^{2k}$ (where $\{e^1,\ldots,e^{2m}\}$ is a basis of $\mathbb{C}^{2m}$).

Denote by $H_{w_{2k}} \subset SL(2m,\mathbb{C})$ the stabilizer of the form ω_{2k}, then V_{2k} can be identified with the space of cosets $SL(2m,\mathbb{C})/H_{w_{2k}}$ under the correspondence $g\omega_{2k} \longrightarrow gH_{w_{2k}}$. Now, since $SL(2m,\mathbb{C})$ is simply connected, the fundamental group of the homogeneous space $SL(2m,\mathbb{C})/H_{w_{2k}}$ is $\pi_0(H_{w_{2k}})$ the component group of the stabilizer $H_{w_{2k}}$:

$$\pi_1(V_{2k}) \simeq \pi_1\big(SL(2m,\mathbb{C})/H_{w_{2k}}\big) \xrightarrow{\sim} \pi_0(H_{w_{2k}}). \tag{1}$$

It remains to determine the stabilizer for the $SL(2m,\mathbb{C})$-action. Here

$$H_{w_{2k}} = Sp(2k,\mathbb{C}) \times SL(2m-2k,\mathbb{C}) \times \exp \operatorname{Hom}\big(\mathbb{C}^{2m-2k},\mathbb{C}^{2k}\big) \tag{2}$$

is a connected group i.e. $\pi_0(H_{w_{2k}}) = \{1\}$. Hence, from (1) we get

$$\pi_1(V_{2k}) \simeq \pi_0(H_{w_{2k}}) = \{1\}, \tag{3}$$

that is, for $k < m$ the orbits V_{2k} are simply connected. $\square$

As usual $\mathcal{D}_V$ denotes the sheaf of rings of differential operators on V with holomorphic coefficients. We denote by $\theta := \sum_{1\leqslant i<j\leqslant 2m} x_{ij}\frac{\partial}{\partial x_{ij}}$ the Euler vector field on V. We note the following definition:

Definition 2. Let $\mathcal{M}$ be a $\mathcal{D}_V$-module.

(i) $\mathcal{M}$ is said to be homogeneous if it has a good filtration stable under the action of the Euler vector field θ.

(ii) A section u in $\mathcal{M}$ is homogeneous if $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty$. u is homogeneous of degree $\lambda \in \mathbb{C}$, if there exists $j \in \mathbb{N}$ such that $(\theta - \lambda)^j u = 0$.

We recall the following useful theorem (see [15, Theorem 1.3]):

Theorem 3. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_V$-module equipped with a good filtration $(\mathcal{M}_k)_{k\in\mathbb{Z}}$ stable under the action of θ. Then*

i) *$\mathcal{M}$ is generated over $\mathcal{D}_V$ by finitely many homogeneous global sections.*

ii) *For any $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$, the vector space $\Gamma(V,\mathcal{M}_k) \cap [\bigcup_{p\in\mathbb{N}} \ker(\theta-\lambda)^p]$ of homogeneous global sections in $\mathcal{M}_k$ of degree λ is finite-dimensional.*

Remark 4. We will describe a holomorphic classification of regular holonomic $\mathcal{D}_V$-modules in $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}_V)$ but Theorem 3 permits to reduce these objects to algebraic (homogeneous) $\mathcal{D}_V$-modules.

Now recall that the universal covering of $GL(2m,\mathbb{C})$ is $SL(2m,\mathbb{C}) \times \mathbb{C}$: the morphism being described by $SL(2m,\mathbb{C}) \times \mathbb{C} \longrightarrow GL(2m,\mathbb{C})$, $(X,t) \longmapsto e^t X$. We denote $G := SL(2m,\mathbb{C}) \times \mathbb{C}$ and $G_0 := SL(2m,\mathbb{C})$. We have the following useful remark:

Remark 5. From [15, Proposition 1.6.] we see that the infinitesimal action of $GL(2m,\mathbb{C})$ on $\mathcal{M}$ lifts to an action of its universal covering G on $\mathcal{M}$. In particular G_0 acts on $\mathcal{M}$.

3. Invariant differential operators on alternating matrices

Recall that the connected (reductive) Lie group $GL(2m,\mathbb{C})$ acts on the vector space of skew-symmetric matrices V by $g\cdot X := gXg^t$ where $X=(x_{ij})$ is a $2m\times 2m$-skew-symmetric matrix in V and $g\in GL(2m,\mathbb{C})$. This defines a finite-dimensional linear representation $(GL(2m,\mathbb{C}),V)$. This action extends to the algebra $\mathbb{C}[V]$ of polynomial functions on V and to the algebra $\Gamma(V,\mathcal{D}_V)^{\mathrm{pol}}$ of differential operators with coefficients in $\mathbb{C}[V]$ by $g\cdot D := (g^t)^{-1}Dg^{-1}$. We thus obtain algebras of invariant $\mathbb{C}[V]^{GL(2m,\mathbb{C})}$ and $\Gamma(V,\mathcal{D}_V)^{GL(2m,\mathbb{C})}$. Recall that $G_0 := SL(2m,\mathbb{C})$ is the derived subgroup of $GL(2m,\mathbb{C})$ (i.e. the subgroup of commutators $SL(2m,\mathbb{C}) = [GL(2m,\mathbb{C}),GL(2m,\mathbb{C})]$). In this section we describe $\overline{\mathcal{A}} := \Gamma(V,\mathcal{D}_V)^{G_0}$ the $\mathbb{C}$-algebra of G_0-invariant differential operators with polynomial coefficients on V (see formula (15)) and its quotients by some ideals (see Proposition 6 and Corollary 8).

Let $X=(x_{ij})$ be a $2m\times 2m$-skew-symmetric matrix in V, the pfaffian $pf(X)$ for X is defined by

$$pf(X) := \frac{1}{2^m m!}\sum_{\sigma\in S_{2m}} \mathrm{sign}(\sigma)x_{\sigma(1)\sigma(2)}x_{\sigma(3)\sigma(4)}\cdots x_{\sigma(2m-1)\sigma(2m)} \tag{4}$$

(S_{2m} is the symmetric group and $\mathrm{sign}(\sigma)$ is the signature of σ) and satisfies $\det(X)=pf(X)^2$.

The action of $GL(2m,\mathbb{C})$ on the pfaffian is $g\cdot pf(X) = pf(gXg^t) = \det(g)pf(X)$ where $X\in V$ and $g\in GL(2m,\mathbb{C})$; in particular, if $g\in G_0$ we have $pf(gXg^t)=pf(X)$. Then the pfaffian is a relative invariant of the representation $(GL(2m,\mathbb{C}),V)$ (i.e. there exists a character $\chi: GL(2m,\mathbb{C})\longrightarrow\mathbb{C}$ such that $g\cdot pf(X)=\chi(g)pf(X)$ for all $g\in GL(2m,\mathbb{C})$) and an invariant for G_0.

Note that the algebra of G_0-invariant polynomial functions is generated by the pfaffian $pf(X)$ that is

$$\mathbb{C}[V]^{G_0} = \mathbb{C}\big[pf(X)\big] \quad \text{such that} \quad pf(X)\notin \mathbb{C}[V]^{GL(2m,\mathbb{C})}. \tag{5}$$

In that case the representation $(GL(2m,\mathbb{C}),V)$ is said to be with one-dimensional quotient (see T. Levasseur [12]).

We know from R. Howe and T. Umeda [5, p. 589, (11.3.4)] that the canonical generators for the algebra $\Gamma(V,\mathcal{D}_V)^{GL(2m,\mathbb{C})}$ of $GL(2m,\mathbb{C})$-invariant differential operators on V are the following skew Capelli operators defined with the pfaffian:

$$\Gamma_k := \sum_{|I|=2k} pf(X_I)pf(D_I) \tag{6}$$

($1\leqslant k\leqslant m$). Here X_I and D_I indicate the submatrices $X_I=(x_{ij})_{i,j\in I}$ and $D_I=(\frac{\partial}{\partial x_{ij}})_{i,j\in I}$ for $I\subseteq\{1,2,\ldots,2m\}$ respectively. This last algebra is known to be commutative (see [5, p. 581, (10.3) (the abstract Capelli problem) and p. 612, Table (15.1)] or [23]):

$$\Gamma(V,\mathcal{D}_V)^{GL(2m,\mathbb{C})} = \mathbb{C}[\Gamma_1,\ldots,\Gamma_m]. \tag{7}$$

Note that here $\Gamma_1=\theta := \sum_{1\leqslant i<j\leqslant 2m} x_{ij}\frac{\partial}{\partial x_{ij}}$ is the Euler vector field on V.

Let us recall (see T. Levasseur [12, p. 508, Appendix, (3)] or Sato and Kimura [22, p. 145, (3)]) that $(GL(2m,\mathbb{C}),V)$ is an irreducible finite-dimensional linear representation which is "multiplicity free" that is the associated representation of $GL(2m,\mathbb{C})$ on polynomial functions $\mathbb{C}[V]$ decomposes without multiplicities. More precisely each irreducible representation of $GL(2m,\mathbb{C})$ occurs at most once in $\mathbb{C}[V]$ (see T. Levasseur [12, p. 484, Definition 4.1] or Howe and Umeda [5] for details). Moreover the representation $(GL(2m,\mathbb{C}),V)$ has an open dense orbit and is called a prehomogneous vector space (see T. Kimura [10, Chap. 2]). Note that the complement S of this dense orbit is a hypersurface defined by the pfaffian $S: X\in V,\ pf(X)=0$. In that case the representation $(GL(2m,\mathbb{C}),V)$ is called regular (see T. Kimura [10, Theorem 2.28, p. 43]).

Now, following H. Rubenthaler [19, p. 1346; 3] or [20, p. 5, 6; 2.2], we need more information on the irreducible regular prehomogeneous vector space $(GL(2m,\mathbb{C}), V)$ to get the algebra of G_0-invariant differential operators $\overline{\mathcal{A}}$. Indeed, let us choose the matrix of the quadratic form on $\mathbb{C}^{2m}$ to be $J = \left[\begin{smallmatrix} 0 & I_{2m} \\ I_{2m} & 0 \end{smallmatrix}\right]$ with I_{2m} the $2m$ by $2m$ identity matrix. Identify $\mathrm{End}(\mathbb{C}^{2m})$ to the space $\mathrm{M}_{2m}(\mathbb{C})$ of $2m \times 2m$-matrices and let $\mathfrak{g}$ be the orthogonal Lie algebra $\mathfrak{so}(\mathbb{C}^{2m}, J)$ i.e.

$$\mathfrak{g} = \left\{M \in \mathrm{End}\left(\mathbb{C}^{2m}\right)\colon\ {}^tMJ + JM = 0\right\}. \tag{8}$$

Let $M \in \mathfrak{g}$ and write $M = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \eta \end{smallmatrix}\right]$ with $\alpha, \beta, \gamma, \eta \in \mathrm{M}_{2m}(\mathbb{C})$. Then we have

$${}^t\eta = -\alpha, \qquad {}^t\beta = -\beta, \qquad {}^t\gamma = -\gamma. \tag{9}$$

Thus

$$M = B^- + A + B^+, \tag{10}$$

where

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & -{}^t\alpha \end{bmatrix}, \qquad B^+ = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}, \qquad B^- = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}. \tag{11}$$

With an obvious notation, it follows that $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}^+ \tag{12}$$

where $\mathfrak{l} \cong \mathfrak{gl}(2m,\mathbb{C})$ and $\mathfrak{n}^\pm$ are commutative. Moreover, the adjoint action of $\mathfrak{l}$ on $\mathfrak{n}^+$ identifies with the natural action of $\mathfrak{gl}(2m,\mathbb{C})$ on the space of $2m \times 2m$-skew-symmetric matrices V. Going to the associated Lie group L associated to $\mathfrak{l}$ gives the representation

$$\left(L, \mathfrak{n}^+\right) \cong \left(GL(2m,\mathbb{C}), V\right). \tag{13}$$

For H. Rubenthaler [19, p. 1346, 3], such a (multiplicity free) irreducible regular prehomogeneous vector space with one-dimensional quotient $(GL(2m,\mathbb{C}), V)$ satisfying (12)–(13) is said to be of "commutative parabolic type".

In that case, according to H. Rubenthaler [19, Proposition 3.1, 3), p. 1346] or [20, Theorem 5.3.3, p. 24] by adding the following pfaffian operators

$$\delta := pf(X) \quad \text{and} \quad \Delta := pf(D) \tag{14}$$

to the list of the Capelli operators (6): $\Gamma_1, \ldots, \Gamma_{m-1}$ above, we obtain the generators for the noncommutative algebra of G_0-invariant differential operators $\overline{\mathcal{A}}$ that is:

$$\overline{\mathcal{A}} = \mathbb{C}\langle \delta, \Delta, \Gamma_1, \Gamma_2, \ldots, \Gamma_{m-1}\rangle.^1 \tag{15}$$

More precisely, denote by

$$\mathcal{J} := \left\{Q \in \overline{\mathcal{A}}/Qf = 0\ \forall f \in \mathbb{C}[\delta]\right\} \subset \overline{\mathcal{A}}$$

the annihilator of G_0-invariant polynomials $\mathbb{C}[\delta]$. In the following proposition, we give relations that hold in the quotient algebra $\overline{\mathcal{A}}/\mathcal{J}$.

[1] In (15) the brackets $\langle\ \rangle$ indicate the noncommutativity of $\overline{\mathcal{A}}$.

Proposition 6. *The following relations hold in the quotient algebra $\overline{\mathcal{A}}/\mathcal{J}$:*

$$\text{(a)}\quad [\theta,\delta]=m\cdot\delta,\qquad [\theta,\Delta]=-m\cdot\Delta,$$

$$\text{(b)}\quad [\Gamma_k,\Gamma_l]=0\quad \forall k,l=1,\dots,m-1,$$

$$\text{(c)}\quad \delta\Delta=\prod_{j=0}^{m-1}\left(\frac{\theta}{m}+2j\right),$$

$$\text{(d)}\quad \Delta\delta=\prod_{j=0}^{m-1}\left(\frac{\theta}{m}+2j+1\right),$$

$$\text{(e)}\quad \Gamma_k=\binom{m}{k}\prod_{j=0}^{k-1}\left(\frac{\theta}{m}+2j\right),$$

$$\text{(f)}\quad [\Gamma_k,\delta]=\binom{m}{k}\delta\left\{\prod_{j=0}^{k-1}\left(\frac{\theta}{m}+2j+1\right)-\prod_{j=0}^{k-1}\left(\frac{\theta}{m}+2j\right)\right\},$$

$$\text{(g)}\quad [\Gamma_k,\Delta]=\binom{m}{k}\left\{\prod_{j=0}^{k-1}\left(\frac{\theta}{m}+2j+1\right)-\prod_{j=0}^{k-1}\left(\frac{\theta}{m}+2j\right)\right\}\Delta.$$

Remark 7. Let $\mathcal{B}$ denote the quotient of the free associative algebra on $\delta,\theta=\Gamma_1,\Gamma_2,\dots,\Gamma_{m-1},\Delta$ by the ideal generated by the relations above. The above proposition says there is a natural surjective map $\mathcal{B}\longrightarrow\overline{\mathcal{A}}/\mathcal{J}$, but it is not claimed that the map is an isomorphism.

Before starting the proof of Proposition 6, we recall the explicit eigenvalues of the Capelli operators Γ_l $(1\leqslant l\leqslant m)$ obtained by K. Kinoshita and M. Wakayama [11]. From [11, p. 463, formula (3.10)] we deduce that, for $1\leqslant l\leqslant m$, $k\in\mathbb{Z}$,

$$\Gamma_l\delta^k=\binom{m}{l}\left(\prod_{j=0}^{l-1}(k+2j)\right)\delta^k. \tag{16}$$

In particular for $l=m$ $(\Gamma_m=\delta\Delta)$ we get the (simplest) Cayley-type formula or more generally the Bernstein–Sato polynomial (b-function) attached to V:

$$\Delta\delta^k=\left(\prod_{j=0}^{m-1}(k+2j)\right)\delta^{k-1}\quad\bigl(\text{see [11, p. 463, Corollary 3.13]}\bigr). \tag{17}$$

Proof of Proposition 6. The proof is devoted to demonstrating that the equations (a)–(g) hold in $\overline{\mathcal{A}}/\mathcal{J}$. The formula (a) are the well-known homogeneity relations since the pfaffian δ (resp. Δ) is a homogeneous polynomial of degree m (resp. $-m$). (b) holds because the $GL(2m,\mathbb{C})$-invariant operators commute (see (7)). Now, we should note that the algebra $\overline{\mathcal{A}}$ acts on the ring $\mathbb{C}[\delta]$ of polynomials of the pfaffian. In particular $\Delta\delta$ (resp. $\delta\Delta$), homogeneous of degree 0 (i.e. $[\theta,\Delta\delta]=0$), acts on $\mathbb{C}[\delta]$. This implies that the differential operator $\Delta\delta$ (resp. $\delta\Delta$) is a polynomial of $\delta,\frac{\partial}{\partial\delta}$, that is, $\Delta\delta\in\mathbb{C}[\delta,\frac{\partial}{\partial\delta}]$

with $\delta\frac{\partial}{\partial\delta} = \frac{1}{m}\theta$. Thus $\Delta\delta$ (resp. $\delta\Delta$) is a polynomial in θ. Letting the polynomial δ act from the left of Δ in the first member and in the second member of the Bernstein–Sato formula (17) yields

$$\delta\Delta\delta^k = \left(\prod_{j=0}^{m-1}(k+2j)\right)\delta^k \quad \forall k \in \mathbb{Z}. \tag{18}$$

Since the pfaffian polynomial function δ (resp. δ^k) is a homogeneous function of degree m (resp. mk) the second member of (18) can be written as follows:

$$\prod_{j=0}^{m-1}\left(\frac{\theta}{m}+2j\right)\delta^k = \left(\prod_{j=0}^{m-1}(k+2j)\right)\delta^k. \tag{19}$$

Then, from (18) and (19), we deduce

$$\left(\delta\Delta - \prod_{j=0}^{m-1}\left(\frac{\theta}{m}+2j\right)\right)\delta^k = 0. \tag{20}$$

This last means that the operator $(\delta\Delta - \prod_{j=0}^{m-1}(\frac{\theta}{m}+2j))$ annihilates any polynomial function in $\mathbb{C}[\delta]$. Thus $(\delta\Delta - \prod_{j=0}^{m-1}(\frac{\theta}{m}+2j))$ belongs to $\mathcal{J}$ that is we get the relation (c): $\delta\Delta = \prod_{j=0}^{m-1}(\frac{\theta}{m}+2j) \bmod \mathcal{J}$. In the same way, by the Bernstein–Sato polynomial (17) we have

$$\Delta\delta^{k+1} = \left(\prod_{j=0}^{m-1}(k+2j+1)\right)\delta^k, \tag{21}$$

$$\Delta\delta\delta^k = \left(\prod_{j=0}^{m-1}(k+2j+1)\right)\delta^k, \tag{22}$$

and the homogeneity of the pfaffian polynomial gives

$$\prod_{j=0}^{m-1}\left(\frac{\theta}{m}+2j+1\right)\delta^k = \left(\prod_{j=0}^{m-1}(k+2j+1)\right)\delta^k. \tag{23}$$

So from (22) and (23), we have

$$\Delta\delta\delta^k = \prod_{j=0}^{m-1}\left(\frac{\theta}{m}+2j+1\right)\delta^k \tag{24}$$

and the relation (d) is deduced. Next, we get the relation (e) from the formula (16) and the following equality obtained thanks to the homogeneity of the polynomial function δ:

$$\Gamma_l\delta^k = \binom{m}{l}\left(\prod_{j=0}^{l-1}\left(\frac{\theta}{m}+2j\right)\right)\delta^k. \tag{25}$$

Therefore (f) and (g) are obtained from (e). $\square$

Now, denote by $\overline{\mathcal{J}}$ the preimage in $\overline{\mathcal{A}}$ of the ideal in $\overline{\mathcal{A}}/\mathcal{J}$ defined by the relations (a), (c), (d) of Proposition 6. Put $\mathcal{A} := \overline{\mathcal{A}}/\overline{\mathcal{J}}$ the quotient algebra of $\overline{\mathcal{A}}$ by $\overline{\mathcal{J}}$.

Corollary 8. *The quotient algebra $\mathcal{A}$ is generated by δ, θ, Δ satisfying the relations*

$$[\theta, \delta] = m \cdot \delta,$$

$$[\theta, \Delta] = -m \cdot \Delta,$$

$$\delta\Delta = \prod_{j=0}^{m-1}\left(\frac{\theta}{m} + 2j\right),$$

$$\Delta\delta = \prod_{j=0}^{m-1}\left(\frac{\theta}{m} + 2j + 1\right).$$

Actually, this corollary is a particular case of T. Levasseur's result in [12, Theorem 3.9, p. 483] or H. Rubenthaler [19, Theorem 2.8, p. 1345], [20, Theorem 7.3.2, p. 37].

Proof of Corollary 8. Let P be an operator in $\overline{\mathcal{A}}$, we decompose it into homogeneous components ($P = \sum_{j \in \mathbb{Z}} P_j$) P_j of degree jm (i.e. $[\theta, P_j] = jmP_j$) so that if $j = 0$ then $P_0 = \varphi(\theta)$ is a polynomial in θ. Indeed, P_0 acts on $\mathbb{C}[\delta]$ then $P_0 \in \mathbb{C}[\delta, \frac{\partial}{\partial\delta}]$ with $\delta\frac{\partial}{\partial\delta} = \frac{1}{m}\theta$. If $j > 0$ then $\Delta^j P_j = \psi(\theta)$ is a polynomial in θ because $\Delta^j P_j$ is homogeneous of degree 0. Likewise if $j < 0$ then $\delta^{-j} P_j = \phi(\theta)$ is a polynomial in θ. Thus for any P_j homogeneous of degree jm, its class modulo $\overline{\mathcal{J}}$ is of the form

$$P_j \mod \overline{\mathcal{J}} = \begin{cases} \delta^j \phi_j(\theta) & \text{if } j \geqslant 0, \\ \delta^{-j}\psi_j(\theta) & \text{if } j \leqslant 0 \end{cases} \tag{26}$$

where $\phi_j(\theta)$, $\psi_j(\theta)$ are (polynomials) homogeneous of degree 0. □

4. Invariant sections of $\mathcal{D}_V$-modules

This section is devoted to the main general argument of the paper. The idea is to show that the $\mathcal{D}_V$-modules studied here are generated by their invariant global sections under the action of G_0. The proof makes use of $\mathcal{D}_V$-modules with support in the closure of the orbits V_{2k} ($0 \leqslant k \leqslant m$).

Theorem 9. *A $\mathcal{D}_V$-module $\mathcal{M}$ in $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}_V)$ is generated by its G_0-invariant global sections.*

To prove this key Theorem 9, we need some preliminary results.

4.1. *A description of $\mathcal{D}_V$-modules supported by the closure of the $GL(2m, \mathbb{C})$-orbits V_{2k} ($0 \leqslant k \leqslant m$)*

Let us denote by $\overline{V}_{2k} := \bigcup_{j \leqslant k} V_{2j}$ the closure of the $GL(2m, \mathbb{C})$-orbit V_{2k}, that is, $\overline{V}_{2k} := \{X \in V := \Lambda^2\mathbb{C}^{2m} / \operatorname{rank}(X) \leqslant 2k\}$ the set of $2m \times 2m$-skew-symmetric matrices of rank $2k$ or less for $0 \leqslant k \leqslant m$. Then $\overline{V}_{2m-2}$ is the hypersurface defined by the equation $pf(X) = 0$ where $pf : V \longrightarrow \mathbb{C}$, $X \longmapsto pf(X)$ is the pfaffian mapping. Here we study $\mathcal{D}_V$-modules with support on $\overline{V}_{2k}$. These modules will be used in the sequel to prove the central Theorem 9.

4.1.1. Invariant sections of $\mathcal{O}(\frac{1}{\delta})$

Recall that we have denoted by $\delta := pf(X)$ and its dual $\Delta := pf(D)$ the pfaffians for $X \in V$ and $D = (\frac{\partial}{\partial x_{ij}}) \in \mathcal{D}_V$ respectively.

In this subsection we describe the subquotient modules for $F := \mathcal{O}_V(\frac{1}{\delta})$. Put $e_k := \delta^{-k}$ $(k \geqslant 0)$. Actually F is generated by its G_0-invariant homogeneous sections e_k satisfying the following relations deduced from formulas (17)–(16):

$$\delta.e_k = e_{k-1}, \qquad \theta e_k = -mke_k, \tag{27}$$

$$\Delta e_k = \left(\prod_{j=0}^{m-1}(2j-k)\right)e_{k+1}, \tag{28}$$

$$\Gamma_l e_k = \binom{m}{l}\left(\prod_{j=0}^{l-1}(2j-k)\right)e_k \tag{29}$$

for $1 \leqslant l \leqslant m$.

4.1.2. Characterization of quotient modules for $\mathcal{O}(\frac{1}{\delta})$

We consider the submodules $F_k := \mathcal{D}_V\delta^{-k}$ of F generated respectively by $e_k := \delta^{-k}$ $(0 \leqslant k \leqslant 2m)$ in $\mathcal{O}_V(\frac{1}{\delta})$:

$$F_0 := \mathcal{O}_V \subset F_1 := \mathcal{D}_V\delta^{-1} \subset F_2 := \mathcal{D}_V\delta^{-2} \subset \cdots \subset F_{2m} := \mathcal{D}_V\delta^{-2m}. \tag{30}$$

Remark 10. The Bernstein–Sato equation (17)

$$\Delta\delta^k = \left(\prod_{j=0}^{m-1}(k+2j)\right)\delta^{k-1}$$

guarantees the following equalities

$$F_{2k+1} = F_{2k+2} \quad (0 \leqslant k \leqslant m-1). \tag{31}$$

Then we will use in the sequel the modules F_{2k} and its quotient F_{2k}/F_{2k-2} $(1 \leqslant k \leqslant m)$.

Denote by

$$F^{2k} := F_{2k}/F_{2k-2} = \mathcal{D}_V\delta^{-2k}/\mathcal{D}_V\delta^{-(2k-2)} \tag{32}$$

the quotient module associated with F_{2k} $(1 \leqslant k \leqslant m)$.

Proposition 11. *The quotient module F^{2k} is a simple holonomic $\mathcal{D}_V$-module of multiplicity* 1 *which is supported by $\overline{V}_{2m-2k}$ for $1 \leqslant k \leqslant m$.*

Indeed the relation (31) of the preceding Remark 10 implies the following equalities

$$F^0 = \mathcal{O}_V, \qquad F^2 := \mathcal{D}_V\delta^{-2}/\mathcal{O}_V = \mathcal{D}_V\delta^{-1}/\mathcal{O}_V, \tag{33}$$

$$F^{2k} := \mathcal{D}_V\delta^{-2k}/\mathcal{D}_V\delta^{-(2k-2)} = \mathcal{D}_V\delta^{-(2k-1)}/\mathcal{D}_V\delta^{-(2k-2)}, \tag{34}$$

and the following description for the quotient module $F^{2k} := F_{2k}/F_{2k-2}$ $(1 \leqslant k \leqslant m)$:

$$F^{2k} := \begin{cases} \text{one generator } \bar{e}_{2k} := e_{2k} \bmod \delta^{-(2k-2)}, \\ \theta \bar{e}_{2k} = -2mk\bar{e}_{2k}, \\ pf(X_I)\bar{e}_{2k} = 0 \text{ for } |I| = (2m-2k)\ +2. \end{cases}$$

Then the quotient modules F^{2k} are generated by invariant sections $\bar{e}_{2k}$. They are simple $\mathcal{D}_V$-modules supported by $\overline{V}_{2m-2k}$.

Now we are interested in the following quotient modules for $\mathcal{O}_V(\frac{1}{\delta})$ by the F_{2k-2} which will be used in the sequel:

$$R_{2k} := \mathcal{O}_V\left(\frac{1}{\delta}\right)/F_{2k-2} = \mathcal{O}_V\left(\frac{1}{\delta}\right)/\mathcal{D}\delta^{-(2k-2)}. \tag{35}$$

They are generated by finitely many global homogeneous invariant sections and there exists the following Jordan–Hölder sequence:

$$\mathcal{O}_V\left(\frac{1}{\delta}\right)/F_{2k-2},\ F_{2k-2}/F_{2k-4},\ F_{2k-4}/F_{2k-6}, \ldots, F_2/F_0,\ F_0$$

supported respectively by

$$\overline{V}_{2m-2k},\ \overline{V}_{2m-(2k-2)},\ \overline{V}_{2m-(2k-4)}, \ldots, \overline{V}_{2m-2},\ \overline{V}_{2m}.$$

Then we can see that the R_{2k} are supported by the closure $\overline{V}_{2m-2k}$ for $1 \leqslant k \leqslant m$.

Lemma 12. *The $\mathcal{D}_V$-modules $R_{2k} := \mathcal{O}(\frac{1}{\delta})/F_{2k-2}$ are generated by finitely many global invariant sections and they are supported by $\overline{V}_{2m-2k}$ $(1 \leqslant k \leqslant m)$ the closure of $GL(2m, \mathbb{C})$-orbits.*

4.1.3. Sections of R_{2k} extended

In this subsection we show that any section u of the $\mathcal{D}_V$-module R_{2k} in the complement of $\overline{V}_{2m-2k-2}$ extends to the whole V.

Proposition 13. *A section $u \in \Gamma(V \backslash \overline{V}_{2m-2k-2}, R_{2k})$ of the $\mathcal{D}_V$-module R_{2k} in the complement of $\overline{V}_{2m-2k-2}$ extends to the whole V $(k = 1, \ldots, m-1)$.*

Proof. First, note that the hypersurface $\overline{V}_{2m-2}$ is smooth out of $\overline{V}_{2m-4}$ and it is a normal variety along V_{2m-4} (smooth). Likewise the variety $\overline{V}_{2m-2k}$ is smooth out of $\overline{V}_{2m-2k-2}$ and normal along $V_{2m-2k-2}$ for $k = 1, \ldots, m-1$.

Next, the $\mathcal{D}_V$-module R_{2k} is the union of modules $\mathcal{O}_V \bar{e}_{2j}$ $(k \leqslant j \leqslant m)$ such that the associated graded modules $\operatorname{gr}(R_{2k})$ is the sum of modules $\mathcal{O}_{T^*_{\overline{V}_{2m-2k-2}}V}\bar{e}_{2j}$ $(k \leqslant j \leqslant m)$. In this case the property of extension here is true for functions because $\overline{V}_{2m-2k}$ is normal along $\overline{V}_{2m-2k-2}$ $(k = 1, \ldots, m-1)$. $\square$

Actually for the proof of Theorem 9, we need more information about the $\mathcal{D}_V$-module R_2 that is a more precise statement as that the restriction of R_2 to $V - \overline{V}_{2m-4}$ is isomorphic to $\mathcal{O}_{V-\overline{V}_{2m-4}}(1/\delta)/\mathcal{O}_{V-\overline{V}_{2m-4}}$.

Lemma 14. *The restriction of R_2 to $V - \overline{V}_{2m-4}$ is isomorphic to the quotient $\mathcal{O}_{V-\overline{V}_{2m-4}}(1/\delta)/\mathcal{O}_{V-\overline{V}_{2m-4}}$.*

Proof. Let $j : V - \overline{V}_{2m-4} \longrightarrow V$ be an open embedding and recall that we have denoted $R_2 := \mathcal{O}_V(1/\delta)/\mathcal{O}_V$. We have the following exact sequence $0 \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V(1/\delta) \longrightarrow R_2 \longrightarrow 0$. Since the inverse image j^+ is an exact functor we obtain the following exact sequence:

$$0 \longrightarrow j^+(\mathcal{O}_V) \longrightarrow j^+\big(\mathcal{O}_V(1/\delta)\big) \longrightarrow j^+(R_2) \longrightarrow 0.$$

Thus we have the isomorphism

$$j^+(R_2) \simeq j^+\big(\mathcal{O}_V(1/\delta)\big)/j^+(\mathcal{O}_V) \simeq \mathcal{O}_{V-\overline{V}_{2m-4}}(1/\delta)/\mathcal{O}_{V-\overline{V}_{2m-4}}. \qquad \square$$

Next, we will also need the following remark, which will be used in the algebraic context, in the proof of Theorem 9.

Remark 15. (See [4,21].) Here V is an algebraic variety under the action of $G_0 = SL(2m, \mathbb{C})$.

i) For $\mathcal{F}$ a quasi-coherent sheaf on the algebraic variety V, U the complement of the closed set defined by an equation $f = 0$ and $s \in \Gamma(U, \mathcal{F})$ a section of $\mathcal{F}$ on U, for all large enough $n \in \mathbb{N}$, the multiplication by the n-th power of f that is $s^* f^n$[2] extends from the open set U to the whole V (i.e. $s^* f^n \in \Gamma(V, \mathcal{F})$);

ii) If s', $s'' \in \Gamma(V, \mathcal{F})$ are two extensions of the above section $s \in \Gamma(U, \mathcal{F})$, then for all large enough $k \in \mathbb{N}$, we have the equality

$$s'^* f^k = s''^* f^k. \tag{36}$$

This, on $V \times G_0$, gives the G_0-invariance of the extension.

Now we embark on the proof of Theorem 9.

4.2. Proof of Theorem 9

Recall $F := \mathcal{O}_V(\frac{1}{\delta})$ and $R_{2k} := \mathcal{O}(\frac{1}{\delta})/\mathcal{D}\delta^{-(2k-2)}$ for $1 \leqslant k \leqslant m$ (see Section 4.1.2). The $\mathcal{D}$-module F is generated by its G_0-invariant homogeneous sections $e_k = \delta^{-k}$ where $k \geqslant 0$ subject to the relations (27)–(29). In particular, F has subquotient modules R_{2k} with support in $\overline{V}_{2m-2k}$ for $1 \leqslant k \leqslant m$ (see Lemma 12).

Denote by $\widetilde{\mathcal{M}} \subset \mathcal{M}$ the submodule generated over $\mathcal{D}_V$ by G_0-invariant homogeneous global sections i.e.

$$\widetilde{\mathcal{M}} := \mathcal{D}_V\big\{u \in \Gamma(V, \mathcal{M})^{G_0},\ \dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty\big\}. \tag{37}$$

We will see successively that the quotient module $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by the closure of the $GL(2m, \mathbb{C})$-orbits $\overline{V}_{2m-2k}$ $(1 \leqslant k \leqslant m)$, and the monodromy is trivial since $\overline{V}_{2m-2k} \backslash \overline{V}_{2m-2k-2} = V_{2m-2k}$ is simply connected (see Proposition 1).

To begin with, $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by $\overline{V}_{2m-2}$: this will be proved in the algebraic context since the result in the algebraic context implies all in the $\mathbb{C}$-analytic context.

Denote by $U := V \backslash \overline{V}_{2m-2}$ (the complement of the pfaffian hypersurface) the algebraic variety, and by $U(\mathbb{C})$ the set of its complex points with its usual topology. On the Zariski open set U, the given $\mathcal{M}$ is equivalent to that of a local system $\mathcal{F}$ on U (i.e. on $U(\mathbb{C})$). Since G_0 is simply connected, such a local system $\mathcal{F}$ is (only) an inverse image by the pfaffian map δ of a local system $\mathcal{L}$ on $\mathbb{G}_m := \mathbb{P}^1 \backslash \{0, \infty\}$ (i.e. on $\mathbb{C}^*$) (one has a fiber bundle with connected and simply connected fibers):

$$\mathcal{F} = \delta^{-1}\mathcal{L} \quad \text{with}\ \delta : U \longrightarrow \mathbb{G}_m. \tag{38}$$

[2] Here $*$ is the multiplication of sections of the structural sheaf $\mathcal{O}$.

The corresponding D-module $\mathcal{N}$ on $\mathbb{G}_m$ is generated by its sections $\sigma_1, \ldots, \sigma_p \in \Gamma(\mathbb{G}_m, \mathcal{N})$ ($\mathbb{G}_m$ is affine):

$$\mathcal{N} = D_{\mathbb{G}_m}\langle \sigma_1, \ldots, \sigma_p \rangle. \tag{39}$$

The inverse images on U of these sections are G_0-invariant and on U they generate $\mathcal{M}$:

$$\delta^{-1}(\sigma_1), \ldots, \delta^{-1}(\sigma_p) \in \Gamma(U, \mathcal{M})^{G_0} \tag{40}$$

and

$$\mathcal{M}_{|U} = D_U \langle \delta^{-1}(\sigma_1), \ldots, \delta^{-1}(\sigma_p) \rangle \tag{41}$$

(because the action of G_0 on the inverse image comes from the action of G_0 on U, which is compatible with the projection $\delta : U \longrightarrow \mathbb{G}_m$ (i.e. $\delta : U \longrightarrow \mathbb{C}^*$) and compatible with the trivial action on $\mathbb{G}_m$).

Note that each of these invariant sections $\delta^{-1}(\sigma_1), \ldots, \delta^{-1}(\sigma_p)$ extends from U to the whole space V after a multiplication by a large enough power of the pfaffian δ (see Remark 15 i)):

$$\delta^{-1}(\sigma_1), \ldots, \delta^{-1}(\sigma_p) \in \Gamma(V, \mathcal{M}). \tag{42}$$

Moreover, after a multiplication by another power of δ, the extension is G_0-invariant (see Remark 15 ii)):

$$\delta^{-1}(\sigma_1), \ldots, \delta^{-1}(\sigma_p) \in \Gamma(V, \mathcal{M})^{G_0}. \tag{43}$$

Now taking the quotient of $\mathcal{M}$ by $\widetilde{\mathcal{M}}$ the module generated by G_0-invariant sections, we then deduce from (41) and (43) that

$$\mathcal{M}/\widetilde{\mathcal{M}} = 0 \quad \text{on } U. \tag{44}$$

This means that $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by $\overline{V}_{2m-2}$.

Next, if $\mathcal{M}$ is supported by $\overline{V}_{2m-2}$, it is isomorphic out of $\overline{V}_{2m-4}$ to a direct sum of a certain number of copies of R_2 (see Lemma 14), then there is a morphism $v : \mathcal{M} \longrightarrow R_2^q$ whose sections extend from $V \backslash \overline{V}_{2m-4}$ to the whole V (see Proposition 13). Here, the image $v(\mathcal{M})$ is a submodule of R_2 then it is generated by its invariant homogeneous section (see Lemma 12) so that $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by $\overline{V}_{2m-4}$.

In the same way by induction on k, if $\mathcal{M}$ is with support on $\overline{V}_{2m-2k}$ ($1 \leqslant k \leqslant m$) then there is a morphism $\mathcal{M} \longrightarrow R_{2k}^q$ which is an isomorphism out of $\overline{V}_{2m-2k-2}$, such that $\mathcal{M}/\widetilde{\mathcal{M}}$ is with support on $\overline{V}_{2m-2k-2}$ because the submodules of R_{2k} are also generated by their invariant homogeneous sections. Finally, if $\mathcal{M}$ is supported by V_0 (the Dirac module with support at the origin) then the result is obvious. This completes the proof of Theorem 9.

5. Equivalence of categories

In this section we establish the main result of this paper: Theorem 18.

Recall that $\overline{\mathcal{A}} := \mathbb{C}\langle \delta, \Delta, \theta, \Gamma_2, \ldots, \Gamma_{m-1} \rangle$ is the algebra of G_0-invariant differential operators. Since the Euler vector field θ belongs to $\overline{\mathcal{A}}$, we can decompose the algebra $\overline{\mathcal{A}}$ under the adjoint action of θ:

$$\overline{\mathcal{A}} = \bigoplus_{k \in \mathbb{N}} \overline{\mathcal{A}}[k], \quad \overline{\mathcal{A}}[k] = \{ P \in \overline{\mathcal{A}} \colon [\theta, P] = kP \} \tag{45}$$

and we can check that

$$\forall k, l \in \mathbb{N}, \quad \overline{\mathcal{A}}[k] \cdot \overline{\mathcal{A}}[l] \subset \overline{\mathcal{A}}[k+l] \tag{46}$$

so $\overline{\mathcal{A}}$ is a graded algebra.

Recall also that $\mathcal{J} \subset \overline{\mathcal{A}}$ is the annihilator of $\mathbb{C}[\delta]$. We have denoted $\overline{\mathcal{J}}$ the preimage in $\overline{\mathcal{A}}$ of the ideal in $\overline{\mathcal{A}}/\mathcal{J}$ defined by the relations (a), (c), (d) of Proposition 6:

$$\begin{aligned}
[\theta, \delta] &= m \cdot \delta, \\
[\theta, \Delta] &= -m \cdot \Delta, \\
\delta\Delta &= \prod_{j=0}^{m-1}\left(\frac{\theta}{m} + 2j\right), \\
\Delta\delta &= \prod_{j=0}^{m-1}\left(\frac{\theta}{m} + 2j + 1\right).
\end{aligned}$$

We put $\mathcal{A}$ the quotient of $\overline{\mathcal{A}}$ by $\overline{\mathcal{J}}$: $\mathcal{A} := \overline{\mathcal{A}}/\overline{\mathcal{J}}$ (see Corollary 8).

Now, since $\overline{\mathcal{J}}$ is an ideal of $\overline{\mathcal{A}}$ it decomposes also under the adjoint action of θ:

$$\overline{\mathcal{J}} = \bigoplus_{k \in \mathbb{N}} \overline{\mathcal{J}}[k], \quad \overline{\mathcal{J}}[k] = \overline{\mathcal{J}} \cap \overline{\mathcal{A}}[k]. \tag{47}$$

Note that $\overline{\mathcal{J}}$ is a homogeneous ideal of the graded algebra $\overline{\mathcal{A}}$, thus the quotient algebra $\mathcal{A} = \overline{\mathcal{A}}/\overline{\mathcal{J}}$ is naturally graded by

$$\mathcal{A}[k] := (\overline{\mathcal{A}}/\overline{\mathcal{J}})[k] = \overline{\mathcal{A}}[k]/\overline{\mathcal{J}}[k]. \tag{48}$$

As in the Introduction we denote by $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the category consisting of graded $\mathcal{A}$-modules T of finite type such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty$ for any u in T. In other words, T is a direct sum of finite-dimensional $\mathbb{C}$-vector spaces:

$$T = \bigoplus_{\lambda \in \mathbb{C}} T_\lambda, \quad T_\lambda := \bigcup_{p \in \mathbb{N}} \ker(\theta - \lambda)^p \ (\text{with } \dim_{\mathbb{C}} T_\lambda < \infty) \tag{49}$$

equipped with the endomorphisms δ, θ, Δ of degree m, 0, $-m$, respectively and satisfying the relations (a), (c), (d) with $(\theta - \lambda)$ being a nilpotent operator on each T_λ.

Recall that $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}_V)$ stands for the category consisting of regular holonomic $\mathcal{D}_V$-modules whose characteristic variety is contained in Σ the union of conormal bundles to the orbits for the action of $GL(2m, \mathbb{C})$ on skew-symmetric matrices.

Let $\mathcal{M}$ be an object in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}_V)$, denote by $\Psi(\mathcal{M})$ the submodule of $\Gamma(V, \mathcal{M})$ consisting of G_0-invariant homogeneous global sections u in $\mathcal{M}$ such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty$:

$$\Psi(\mathcal{M}) := \left\{u \in \Gamma(V, \mathcal{M})^{G_0},\ \dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty\right\}. \tag{50}$$

We are going to show that $\Psi(\mathcal{M})$ is an object in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$.

Let $(\sigma_1, \dots, \sigma_p) \in \Gamma(V, \mathcal{M})^{G_0}$ be a finite family of homogeneous invariant global sections generating the $\mathcal{D}_V$-module $\Psi(\mathcal{M})$ (see Theorem 9):

$$\Psi(\mathcal{M}) := \mathcal{D}_V \langle \sigma_1, \dots, \sigma_p \rangle. \tag{51}$$

We are going to see that the family $(\sigma_1, \dots, \sigma_p)$ generates also $\Psi(\mathcal{M})$ as an $\mathcal{A}$-module: indeed, an invariant section $\sigma \in \Psi(\mathcal{M})$ can be written as

$$\sigma = \sum_{j=1}^{p} q_j(X, D)\sigma_j \quad \text{where } q_j \in \mathcal{D}_V. \tag{52}$$

Denote by $\widetilde{q}_j := \int_{SU(2m)} g \cdot q_j \, dg$ the average of q_j over $SU(2m)$ (compact maximal subgroup of G_0). Then the average $\widetilde{q}_j$ belongs to the algebra $\overline{\mathcal{A}}$ (i.e. $\widetilde{q}_j \in \overline{\mathcal{A}}$). Now denote by f_j the class of $\widetilde{q}_j$ modulo $\overline{\mathcal{J}}$:

$$f_j := \widetilde{q}_j \mod \overline{\mathcal{J}} \quad \text{that is } f_j \in \mathcal{A}. \tag{53}$$

Therefore we also have

$$\sigma = \sum_{j=1}^{p} \widetilde{q}_j \sigma_j = \sum_{j=1}^{p} f_j \sigma_j \quad \text{with } f_j \in \mathcal{A}. \tag{54}$$

This last means that

$$\Psi(\mathcal{M}) := \mathcal{A}\langle \sigma_1, \dots, \sigma_p \rangle, \tag{55}$$

and $\Psi(\mathcal{M})$ is an $\mathcal{A}$-module. Moreover, according to Theorem 3 ii), we have

$$\Psi(\mathcal{M}) = \bigoplus_{\lambda \in \mathbb{C}} \Psi(\mathcal{M})_\lambda \tag{56}$$

where

$$\Psi(\mathcal{M})_\lambda := \left[\Psi(\mathcal{M})\right] \cap \left[\bigcup_{p \in \mathbb{N}} \ker(\theta - \lambda)^p\right] \quad \left(\text{with } \dim_{\mathbb{C}} \Psi(\mathcal{M})_\lambda < \infty\right) \tag{57}$$

is the finite-dimensional $\mathbb{C}$-vector space of homogeneous global sections of degree $\lambda \in \mathbb{C}$ in $\Psi(\mathcal{M})$. Finally we can check that

$$\mathcal{A}[k]\Psi(\mathcal{M})_\lambda \subset \Psi(\mathcal{M})_{\lambda+k} \quad \text{for all } k \in \mathbb{N},\ \lambda \in \mathbb{C}. \tag{58}$$

So $\Psi(\mathcal{M})$ is a graded $\mathcal{A}$-module of finite type for the Euler vector field θ thanks to (55)–(58). This means that $\Psi(\mathcal{M})$ is an object in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$.

Conversely, let T be an object in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$, one associates to it the $\mathcal{D}_V$-module

$$\Phi(T) := \mathcal{M}_0 \bigotimes_{\mathcal{A}} T \tag{59}$$

where $\mathcal{M}_0 := \mathcal{D}_V / \overline{\mathcal{J}}$. Then $\Phi(T)$ is an object in the category $\mathrm{Mod}_{\Sigma}^{\mathrm{rh}}(\mathcal{D}_V)$.

Thus, we have defined two functors

$$\Psi : \mathrm{Mod}_{\Sigma}^{\mathrm{rh}}(\mathcal{D}_V) \longrightarrow \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}), \qquad \Phi : \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}) \longrightarrow \mathrm{Mod}_{\Sigma}^{\mathrm{rh}}(\mathcal{D}_V). \tag{60}$$

We need the two following lemmas:

Lemma 16. *The canonical morphism*

$$T \longrightarrow \Psi\big(\Phi(T)\big), \quad t \longmapsto 1 \otimes t \tag{61}$$

is an isomorphism, and defines an isomorphism of functors $\mathrm{Id}_{\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})} \longrightarrow \Psi \circ \Phi$.

Proof. As above $\mathcal{M}_0 := \mathcal{D}_V/\overline{\mathcal{J}}$. Denote by ε (the class of $1_{\mathcal{D}}$ modulo $\overline{\mathcal{J}}$) the canonical generator of $\mathcal{M}_0$. Let $h \in \mathcal{D}_V$, denote by $\tilde{h} \in \overline{\mathcal{A}}$ its average on $SU(2m)(\mathbb{C})$ and by φ the class of $\tilde{h}$ modulo $\overline{\mathcal{J}}$, that is, $\varphi \in \mathcal{A}$.

Since ε is G_0-invariant, we get $\widetilde{h\varepsilon} = \tilde{h}\varepsilon = \varepsilon\varphi$. Moreover, we have $\tilde{h}\varphi = 0$ if and only if $\tilde{h} \in \overline{\mathcal{J}}$, in other words $\varphi = 0$. Therefore the average operator (over $SU(2m)$) $\mathcal{D}_V \longrightarrow \overline{\mathcal{A}}$, $h \longmapsto \tilde{h}$ induces a surjective morphism of $\mathcal{A}$-modules $v : \mathcal{M}_0 \longrightarrow \mathcal{A}$. More generally, for any $\mathcal{A}$-module T in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the morphism $v \otimes 1_T$ is surjective

$$v_T : \mathcal{M}_0 \bigotimes_{\mathcal{A}} T \longrightarrow \mathcal{A} \bigotimes_{\mathcal{A}} T = T \tag{62}$$

which is the left inverse of the morphism

$$u_T : T \longrightarrow \mathcal{M}_0 \bigotimes_{\mathcal{A}} T, \quad t \longmapsto \varepsilon \otimes t, \tag{63}$$

that is, $(v \otimes 1_T) \circ (\varepsilon \otimes 1_T) = v(\varepsilon) = 1_T$. This means that the morphism u_T is injective. Next, the image of u_T is exactly the set of invariant sections of $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T = \Phi(T)$, that is, $\Psi(\Phi(T))$: indeed if $\sigma = \sum_{i=1}^{p} h_i \otimes t_i$ is an invariant section in $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T$, we may replace each h_i by its average $\widetilde{h_i} \in \mathcal{A}$, then we get

$$\sigma = \sum_{i=1}^{p} \widetilde{h_i} \otimes t_i = \varepsilon \otimes \sum_{i=1}^{p} \widetilde{h_i} t_i \in \varepsilon \otimes T, \tag{64}$$

that is, $\sum_{i=1}^{p} \widetilde{h_i} t_i \in T$. Therefore the morphism u_T is an isomorphism from T to $\Psi(\Phi(T))$ and defines an isomorphism of functors. $\square$

Next we note the following

Lemma 17. *The canonical morphism*

$$w : \Phi\big(\Psi(\mathcal{M})\big) \longrightarrow \mathcal{M} \tag{65}$$

is an isomorphism and defines an isomorphism of functors $\Phi \circ \Psi \longrightarrow \mathrm{Id}_{\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}_V)}$.

Proof. As in Theorem 9 the $\mathcal{D}_V$-module $\mathcal{M}$ is generated by a finite family of invariant sections $(\sigma_i)_{i=1,\dots,p} \in \Psi(\mathcal{M})$ so that the morphism w is surjective. Now consider $\mathcal{Q}$ the kernel of the morphism $w : \Phi(\Psi(\mathcal{M})) \longrightarrow \mathcal{M}$. It is also generated over $\mathcal{D}_V$ by its invariant sections, that is, by $\Psi(\mathcal{Q})$. Then we get

$$\Psi(\mathcal{Q}) \subset \Psi\big[\Phi\big(\Psi(\mathcal{M})\big)\big] = \Psi(\mathcal{M}) \tag{66}$$

where we used $\Psi \circ \Phi = Id_{\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})}$ (see the preceding Lemma 16). Since the morphism $\Psi(\mathcal{M}) \longrightarrow \mathcal{M}$ is injective ($\Psi(\mathcal{M}) \subset \Gamma(V, \mathcal{M})$) we obtain $\Psi(\mathcal{Q}) = 0$. Therefore $\mathcal{Q} = 0$ (because $\Psi(\mathcal{Q})$ generates $\mathcal{Q}$). $\square$

This section ends by Theorem 18 established by means of the preceding lemmas.

Theorem 18. *The functors Φ and Ψ induce equivalence of categories*

$$\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D}_V) \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}). \tag{67}$$

6. Description of $\mathcal{A}$-modules by certain kinds of quivers

This section consists in the description of objects in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ by certain kinds of quivers. Note that a graded $\mathcal{A}$-module T in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ defines an infinite diagram consisting of finite-dimensional vector spaces T_λ (with $(\theta-\lambda)$ being a nilpotent operator on each T_λ, $\lambda \in \mathbb{C}$) and linear maps between them deduced from δ, θ, Δ:

$$\cdots \rightleftarrows T_\lambda \underset{\Delta}{\overset{\delta}{\rightleftarrows}} T_{\lambda+m} \rightleftarrows \cdots \tag{68}$$

satisfying the following $(\theta-\lambda)T_\lambda \subset T_\lambda$,

$$\delta\Delta = \prod_{j=0}^{m-1}\left(\frac{\theta}{m}+2j\right), \tag{69}$$

$$\Delta\delta = \prod_{j=0}^{m-1}\left(\frac{\theta}{m}+2j+1\right). \tag{70}$$

These diagrams are determined by finite subsets of objects and arrows:

a) For $\sigma \in \mathbb{C}/m\mathbb{Z}$, denote by $T^\sigma \subset T$ the submodule $T^\sigma = \bigoplus_{\lambda=\sigma \bmod m\mathbb{Z}} T_\lambda$. Then T is generated by the finite direct sum of T^σ's

$$T = \bigoplus_{\sigma\in\mathbb{C}/m\mathbf{Z}} T^\sigma = \bigoplus_{\sigma\in\mathbb{C}/m\mathbf{Z}}\left(\bigoplus_{\lambda=\sigma \bmod m\mathbb{Z}} T_\lambda\right). \tag{71}$$

b) If $\sigma \neq 0 \bmod m\mathbb{Z}$ then the linear maps δ and Δ are bijective. Therefore T^σ is determined by one T_λ with the nilpotent action of $(\theta-\lambda)$.

c) If $\sigma = 0 \bmod m\mathbb{Z}$ then T^σ is determined by one diagram of $2m$ elements

$$T_{-(2m-1)m} \underset{\Delta}{\overset{\delta}{\rightleftarrows}} T_{-(2m-2)m} \cdots \rightleftarrows T_{-m} \underset{\Delta}{\overset{\delta}{\rightleftarrows}} T_0. \tag{72}$$

In the other degrees δ or Δ are bijective. Indeed, we have

$$T_0 \simeq \delta^k T_0 \simeq T_{mk} \tag{73}$$

and

$$T_{-(2m-1)m} \simeq \Delta^k T_{-(2m-1)m} \simeq T_{-(2m-1+k)m} \quad (k \in \mathbb{N}) \tag{74}$$

thanks to relations (69)–(70). The operator $\delta\Delta$ (resp. $\Delta\delta$) on T_λ has only one eigenvalue $\frac{\lambda}{m}(\frac{\lambda}{m}+2)(\frac{\lambda}{n}+4)\times\cdots\times(\frac{\lambda}{m}+2m-2)$ (resp. $(\frac{\lambda}{m}+1)(\frac{\lambda}{m}+3)\times\cdots\times(\frac{\lambda}{m}+2m-1)$) so that Eqs. (69)–(70) have each of them a unique solution θ of eigenvalue λ if λ is not a critical value. Here $\lambda = 0, -2m, -4m, \ldots, -(2m-2)m$ or $\lambda = -m, \ldots, -(2m-1)m$ thus it is always the case.

6.1. Examples of diagrams

Example 19. The irreducible $\mathcal{D}_V$-module $\mathcal{O}_V$ is generated by $e_0 := 1_V$ a homogeneous section of degree 0 such that $\theta e_0 = 0$ and $\Delta e_0 = 0$. This yields a graded $\mathcal{A}$-module of finite type in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ with a basis (e_q) where $q = mk$ ($k \in \mathbb{N}$) such that $\Delta e_0 = 0$ and satisfying the following system:

$$S_0 = \begin{cases} \theta e_q = q e_q \ (q = mk,\ k \in \mathbb{N}), \\ \delta e_q = e_{q+m}, \\ \Delta e_q = \prod_{j=0}^{m-1} (\frac{q}{m} + 2j) e_{q-m}. \end{cases} \tag{75}$$

Since $\Delta e_0 = 0$ (i.e. $\Delta T_0 = 0$), the arrows on the left of T_0 in the diagram vanish, that is,

$$0 \longrightarrow T_0 \underset{\Delta}{\overset{\delta}{\rightleftarrows}} T_m \rightleftarrows \cdots. \tag{76}$$

Example 20. The Dirac module supported by $\{0\}$: $\mathcal{B}_{\{0\}|V}$. It is the Fourier transform of $\mathcal{O}_V$ and is generated by $e_{-(2m-1)m}$ a homogeneous section of degree $-(2m-1)m$ satisfying the equations: $\theta e_{-(2m-1)m} = -(2m-1)m e_{-(2m-1)m}$ and $\delta e_{-(2m-1)m} = 0$.

This yields a graded $\mathcal{A}$-module with basis (e_q) where $q = -(2m-1)m - mk$ ($k \in \mathbb{N}$) such that $\delta e_{-(2m-1)m} = 0$ satisfying the system:

$$S_1 = \begin{cases} \theta e_q = q e_q \ (q = -(2m-1)m - mk,\ k \in \mathbb{N}), \\ \delta e_q = \prod_{j=0}^{m-1} (\frac{q}{m} + 2j + 1) e_{q+m}. \end{cases} \tag{77}$$

Since $\delta e_{-(2m-1)m} = 0$ (i.e. $T_{-(2m-1)m} = 0$), the arrows on the right of $T_{-(2m-1)m}$ in the diagram vanish, that is,

$$\cdots \rightleftarrows T_{-2m^2} \underset{\Delta}{\overset{\delta}{\rightleftarrows}} T_{-(2m-1)m} \longrightarrow 0. \tag{78}$$

Acknowledgments

We acknowledge the generous help we received from Professor Pierre Deligne on the proof of Theorem 9. It is a great pleasure to thank Professor Louis Boutet de Monvel for many helpful discussions, criticisms, corrections and necessary encouragements during the preparation of this paper. Thanks are also due to Professor Thierry Levasseur, Professor Hubert Rubenthaler for kindly pointed out his paper [20] and specially to Professor Gerald Schwarz for valuable suggestions. We are also thankful for the referee's details comments on the manuscript helping us to improve its readability in several points. Finally but, not least, we are grateful to the Max-Planck Institute for Mathematics (MPIM) in Bonn for financial support, hospitality and the exceptional scientific atmosphere that made this project possible.

References

[1] L. Boutet de Monvel, $\mathcal{D}$-modules holonômes réguliers en une variable, in: Mathématiques et Physique, Séminaire de L'ENS, in: Progr. Math., vol. 37, Birkhäuser, Boston, MA, 1972, 1982, pp. 313–321.
[2] L. Boutet de Monvel, Revue sur la théorie des $\mathcal{D}$-modules et modèles d'opérateurs pseudodifférentiels, in: Math. Phys. Stud., vol. 12, Kluwer Acad. Publ., 1991, pp. 1–31.
[3] A. Galligo, M. Granger, P. Maisonobe, $\mathcal{D}$-modules et faisceaux pervers dont le support singulier est un croisement normal, I, Ann. Inst. Fourier 35 (1) (1985) 1–48; II, Astérisque 130 (1985) 240–259.
[4] R. Hartshorne, Residues and Duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Havard 1963/1964. With an appendix by P. Deligne, Lecture Notes in Math., vol. 20, Springer Verlag, Berlin, New York, 1966.

[5] R. Howe, T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity free actions, Math. Ann. 290 (1991) 565–619.

[6] M. Kashiwara, The Riemann–Hilbert problem for holonomic systems, Publ. Res. Inst. Math. Sci. 20 (1984) 319–365.

[7] M. Kashiwara, $\mathcal{D}$-Modules and Microlocal Calculus, Iwanami Ser. Mod. Math., Transl. Math. Monogr., vol. 217, Amer. Math. Soc., 2003.

[8] M. Kashiwara, Algebraic study of systems of partial differential equations, Mem. Soc. Math. Fr. 63 (123 fascicule 4) (1995).

[9] M. Kashiwara, T. Kawai, On the characteristic variety of a holonomic system with regular singularities, Adv. Math. 34 (2) (1979) 163–184.

[10] T. Kimura, Introduction to Prehomogeneous Vector Spaces, Transl. Math. Monogr., vol. 215, Amer. Math. Soc., Providence, RI, 2003.

[11] K. Kinoshita, M. Wakayama, The explicit Capelli identities for skew symmetric matrices, Proc. Edinb. Math. Soc. 45 (2002) 449–465.

[12] T. Levasseur, Radial components, prehomogeneous vector spaces, and rational Cherednik algebras, Int. Math. Res. Not. IMRN 3 (2009) 462–511.

[13] R. Macpherson, K. Vilonen, Perverse sheaves with regular singularities along the curve $y^n = x^m$, Comment. Math. Helv. 63 (1988) 89–102.

[14] P. Nang, On a class of holonomic $\mathcal{D}$-modules on $M_n(\mathbb{C})$ related to the action of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$, Adv. Math. 218 (3) (2008) 635–648.

[15] P. Nang, $\mathcal{D}$-modules associated to the group of similitudes, Publ. Res. Inst. Math. Sci. 35 (2) (1999) 223–247.

[16] P. Nang, $\mathcal{D}$-modules associated to determinantal singularities, Proc. Japan Acad. Ser. A Math. Sci. 80 (5) (2004) 139–144.

[17] P. Nang, Regular holonomic $\mathcal{D}$-modules on skew symmetric matrices, Preprint Max-Planck-Institut für Mathematik MPIM2007-42, 2007.

[18] L. Narvaez Macarro, Cycles évanescents et faisceaux pervers I: cas des courbes planes irréductibles, Compos. Math. 65 (3) (1988) 321–347; II: cas des courbes planes réductibles, London Math. Soc. Lecture Notes Ser. 201 (1994) 285–323.

[19] H. Rubenthaler, Algebras of invariant differential operators on a class of multiplicity free spaces, C. R. Math. Acad. Sci. Paris 347 (2009) 1343–1346.

[20] H. Rubenthaler, Invariant differential operators and infinite dimensional Howe-type correspondence, preprint, arXiv: 0802.0440v1 [math.RT], 4 February 2008.

[21] J.P. Serre, Faisceaux algébriques cohérents, Ann. of Math. (2) 61 (1955) 197–278.

[22] M. Sato, T. Kimura, A classification of prehomogeneous vector spaces and their relative invariants, Nagoya Math. J. 65 (1977) 1–155.

[23] T. Umeda, The Capelli Identities, a Century After, in: Selected Papers on Harmonic Analysis, Groups, and Invariants, in: Amer. Math. Soc. Transl. Ser. 2, vol. 183, 1998, pp. 51–78.