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Variational iteration method for Sturm–Liouville differential equations

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ABSTRACT

In this article, He's variational iteration method is applied to linear Sturm–Liouville eigenvalue and boundary value problems, including the harmonic oscillator. In this method, solutions of the problems are approximated by a set of functions that may include possible constants to be determined from the boundary conditions. By computing variations, the Lagrange multipliers are derived and the generalised expressions of variational iterations are constructed. Numerical results show that the method is simple, however powerful and effective.

1. Introduction

It is not in general easy to find a closed-form solution of differential equations [1,2]. Apart from numerical integration methods, several analytical ones, such as the variational iteration and the Adomian decomposition methods are proposed to find approximate, and if possible in closed form, solutions of differential equations.

This paper focuses on the variational iteration method (VIM), which was introduced by He [3] to solve approximately the differential equations. The proposed method can be applied successfully to different types of systems. For example, He applied the method to autonomous ordinary differential equations and approximated solutions of some nonlinear problems in [2,4] and, recently, in [5,6]. In [7], an application of the method to the differential equations with delay was shown. Momani et al. [8] used VIM for solving nonlinear boundary value problems. Furthermore, in [9], He applied the method to the initial–boundary value problems of order eight; in [10], Noor and Mohyud-Din used He's polynomials to approximate solutions of higher-order boundary value problems. In [11], Momani and Abuasad implemented the scheme to solve the Helmholtz equation, and Abulwafa et al. successfully applied the method to the nonlinear Boltzmann equation in [12]. Recently, Rafei et al., in [13], applied the technique efficiently to approximate the solutions of the epidemic and the predator and prey models.

In order to illustrate the basic concepts of He's variational iteration method, let us consider the system

$$Tu(x) = g(x), \quad x \in I,$$

where T is a differential operator that acts on sufficiently smooth functions u defined on some interval $I \subseteq \mathbb{R}$. The function g is given, and defined for all $x \in I$. The VIM is based on splitting T into linear and nonlinear operators as follows:

$$Lu(x) + Nu(x) = g(x), \tag{1}$$

where L is a linear and N is a nonlinear differential operator.

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A correction functional of Eq. (1) is the variational iteration method described iteratively, for each $k = 0, 1, 2, \ldots$, by

$$u_{k+1}(x) = u_k(x) + \int_0^x \mu \Big\{ Lu_k(s) + N\tilde{u}_k(s) - g(s) \Big\} \mathrm{d}s, \tag{2}$$

given the initial approximation $u_0(x)$, where μ is the so-called general Lagrange multiplier [14]. The Lagrange multiplier can be identified via the variational theory [15,16]. The iterates u_k denote the kth-order approximate solutions, and the $\tilde{u}_k$ are the restricted variations so that their variations are zero, $\delta\tilde{u}_k = 0$. Employing the restricted variation in Eq. (2) makes it easy to compute the Lagrange multiplier, see for instance [2–4,7,16–19]. Under reasonable choice of u_0, the fixed point of the correction functional (2) is considered as an approximate solution of (1), see [20] for the convergence of He's VIM. In case of linear equations, exact solutions can be obtained by only a single iteration. For nonlinear operators, reliable approaches are introduced in [21], recently.

Sturm–Liouville differential equations arise in physical and engineering problems. In fact, in the framework of operator splitting methods, there are some other studies on the eigenvalue problems: Attili [22] and Somali, Gokmen [23] applied the Adomian decomposition for computing eigenvalues and eigenfunctions; further in [24] Attili used the homotopy perturbation method to eigen-pairs (eigenelements) of two-point boundary value problems.

In what follows, we consider using the variational iteration method to find the eigen-pair of the linear Sturm–Liouville problem. A linear Sturm–Liouville operator has the form,

$$Ty(x) := Ly(x) = \lambda r(x) y(x), \tag{3}$$

where

$$L = -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{d}{\mathrm{d}x} \right] + q(x), \quad x \in I := [a, b].$$

Associated with the differential equation (3) are the separated homogeneous boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = 0,$$
$$\alpha_2 y(b) + \beta_2 y'(b) = 0,$$

where α_1, α_2, β_1 and β_2 are arbitrary constants provided that $\alpha_i^2 + \beta_i^2 \neq 0$ for $i = 1, 2$. The values of λ for which the boundary value problem has a nontrivial solution are called eigenvalues of L. A nontrivial solution corresponding to an eigenvalue is called an eigenfunction. We will assume that $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous and $p(x) > 0$, $r(x) > 0$ for all $x \in I$ (we refer to [25,26]), for simplicity.

The main objective of this study is to implement the variational iteration method for computing, if possible, in closed form, the eigenvalues and the eigenfunctions of the Sturm–Liouville problems. In Section 2, we present some examples that we can compute exactly, as well as approximately, the eigenvalues and corresponding eigenfunctions. Section 3 concludes the paper with discussions of results.

2. Numerical applications

Each subsection below consists of a specific example of eigenvalue problems that are frequently studied in the context of Sturm–Liouville operators.

2.1. Dirichlet boundary conditions on $[-\ell, \ell]$

Consider the Sturm–Liouville system

$$-y''(x) - \lambda y(x) = 0, \qquad x \in I = (-\ell, \ell), \quad \ell > 0, \tag{4}$$

with homogeneous Dirichlet boundary conditions

$$y(-\ell) = 0, \qquad y(\ell) = 0. \tag{5}$$

The correction functional of differential equation (4) has the following form:

$$y_{k+1}(x, \lambda) = y_k(x, \lambda) + \int_0^x \mu(s; x, \lambda) \Big\{ -y_k''(s, \lambda) - \lambda y_k(s, \lambda) \Big\} \mathrm{d}s, \tag{6}$$

where $\mu = \mu(s; x, \lambda)$ is the Lagrange multiplier [4,14]. Then, we have

$$\delta y_{k+1}(x, \lambda) = \delta y_k(x, \lambda) + \delta \int_0^x \mu(s; x, \lambda) \Big\{ -y_k''(s, \lambda) - \lambda y_k(s, \lambda) \Big\} \mathrm{d}s.$$

Note that $\delta y_k(0, \lambda) = 0$. Calculus of variations and integration by parts give the stationary conditions

$$\mu''(s; x, \lambda) + \lambda\mu(s; x, \lambda) = 0,$$
$$1 + \mu'(s; x, \lambda)|_{s=x} = 0,$$
$$\mu(s; x, \lambda)|_{s=x} = 0,$$

for which the Lagrange multiplier μ should satisfy. Solving this system of equations for μ yields

$$\mu(s; x, \lambda) = -\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(s - x)).$$

Thus, by inserting the Lagrange multiplier into (6) we obtain the recursion for the variational iterates as

$$y_{k+1}(x, \lambda) = y_k(x, \lambda) + \int_0^x -\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(s - x)) \left\{ -y_k''(s, \lambda) - \lambda y_k(s, \lambda) \right\} \mathrm{d}s. \tag{7}$$

As an initial approximating solution, let us choose

$$y_0(x, \lambda) = A + Bx,$$

where A and B are constants that are going to be defined by imposing the boundary conditions (5) on the iterates and normalisation of the corresponding eigenfunctions. Substituting the proposed initial iterate $y_0(x, \lambda)$ in (7) gives

$$y_1(x, \lambda) = A \cos(\sqrt{\lambda}x) + \frac{B}{\sqrt{\lambda}} \sin(\sqrt{\lambda}x)$$

as the first-order approximation. If the function $y_1(x, \lambda)$ is now forced to satisfy the boundary conditions at $x = -\ell$ and $x = \ell$, then we get a system of algebraic equations:

$$y_1(-\ell, \lambda) = A \cos(\sqrt{\lambda}\ell) - \frac{B}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\ell) = 0,$$
$$y_1(\ell, \lambda) = A \cos(\sqrt{\lambda}\ell) + \frac{B}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\ell) = 0.$$

In order to have a nontrivial solution $y_1(x, \lambda)$, the system yields two infinite sequences of eigenvalues λ_n:

$$\lambda_n = \left(\frac{(2n + 1)\pi}{2\ell} \right)^2, \quad n = 0, 1, 2, \ldots,$$
$$\lambda_n = \left(\frac{n\pi}{\ell} \right)^2, \quad n = 1, 2, \ldots.$$

The corresponding linearly independent eigenfunctions u_n and v_n are therefore

$$u_n(x) = A \cos\left(\frac{(2n + 1)\pi}{2\ell} x \right), \quad n = 0, 1, 2, \ldots,$$
$$v_n(x) = B \frac{\ell}{n\pi} \sin\left(\frac{n\pi}{\ell} x \right), \quad n = 1, 2, \ldots.$$

Here, the u_n and v_n are of class $\mathcal{C}(I, \mathbb{R})$; that is, $u_n(x)$ and $v_n(x)$ are continuous real-valued functions of $x \in I = (-\ell, \ell)$. The standard inner product on $\mathcal{C}(I, \mathbb{R})$ is defined by

$$\langle u_n, v_n \rangle = \int_{-\ell}^{\ell} u_n(s) v_n(s) \mathrm{d}s, \quad u_n, v_n \in \mathcal{C}(I, \mathbb{R}),$$

and the norm induced by the inner product is

$$\|u_n\|_2 = \langle u_n, u_n \rangle^{\frac{1}{2}} = \left(\int_{-\ell}^{\ell} |u_n(s)|^2 \mathrm{d}s \right)^{\frac{1}{2}}.$$

Consequently, we obtain the normalisation constants as

$$A = \frac{1}{\sqrt{\ell}}, \qquad B = \frac{n\pi}{\ell^{\frac{3}{2}}}.$$

Hence, the normalised eigenfunctions $t_n(x) = \frac{u_n(x)}{\|u_n\|_2}$ and $z_n(x) = \frac{v_n(x)}{\|v_n\|_2}$ together with their associated eigenvalues have the forms

$$t_n(x) = \frac{1}{\sqrt{\ell}} \cos\left(\sqrt{\lambda_n} x \right), \qquad \lambda_n = \left(\frac{(2n + 1)\pi}{2\ell} \right)^2, \quad n = 0, 1, 2, \ldots,$$
$$z_n(x) = \frac{1}{\sqrt{\ell}} \sin\left(\sqrt{\lambda_n} x \right), \qquad \lambda_n = \left(\frac{n\pi}{\ell} \right)^2, \quad n = 1, 2, \ldots.$$

These eigenvalues and eigenfunctions are the exact eigen-pairs of the Sturm–Liouville system in (4).

2.2. Neumann boundary conditions on $[-\ell, \ell]$

As another example, consider the eigenvalue problem

$$-y''(x) - \lambda y(x) = 0, \qquad x \in I = (-\ell, \ell), \quad \ell > 0, \tag{8}$$

with Neumann boundary conditions,

$$y'(-\ell) = 0, \qquad y'(\ell) = 0. \tag{9}$$

Using the variational iteration method, we can construct the correction functional of (8) as follows:

$$y_{k+1}(x, \lambda) = y_k(x, \lambda) + \int_0^x \mu(s; x, \lambda)\left\{-y_k''(s, \lambda) - \lambda y_k(s, \lambda)\right\}\mathrm{d}s, \tag{10}$$

where $\mu = \mu(s; x, \lambda)$ is the Lagrange multiplier. Note that $\delta y_k(0, \lambda) = 0$. In this case, the Lagrange multiplier can be identified as

$$\mu(s; x, \lambda) = -\frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}(s - x))$$

so that the variational iterations in (10) take the form

$$y_{k+1}(x, \lambda) = y_k(x, \lambda) + \int_0^x -\frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}(s - x))\left\{-y_k''(s, \lambda) - \lambda y_k(s, \lambda)\right\}\mathrm{d}s. \tag{11}$$

We begin with an initial approximation $y_0(x, \lambda) = A + Bx$, where A and B are constants to be determined as before. By the correction functional (11), we obtain the first-order approximation

$$y_1(x, \lambda) = A\cos(\sqrt{\lambda}x) + \frac{B}{\sqrt{\lambda}}\sin(\sqrt{\lambda}x). \tag{12}$$

Imposing the boundary conditions in (9) yields

$$y_1'(-\ell, \lambda) = A\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) + B\cos(\sqrt{\lambda}\ell) = 0,$$
$$y_1'(\ell, \lambda) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) + B\cos(\sqrt{\lambda}\ell) = 0.$$

Thus, the eigenvalues of this system have the form

$$\lambda_n = \left(\frac{(2n+1)\pi}{2\ell}\right)^2, \quad n = 0, 1, 2, \ldots,$$
$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 1, 2, \ldots.$$

The corresponding linearly independent nontrivial solutions are

$$u_n(x) = B\frac{2\ell}{(2n+1)\pi}\sin\left(\sqrt{\lambda_n}x\right), \qquad \lambda_n = \left(\frac{(2n+1)\pi}{2\ell}\right)^2, \quad n = 0, 1, 2, \ldots,$$
$$v_n(x) = A\cos\left(\sqrt{\lambda_n}x\right), \qquad \lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 1, 2, \ldots.$$

Hence, the normalisation constants are

$$A = \frac{1}{\sqrt{\ell}}, \qquad B = \frac{(2n+1)\pi}{2\ell^{\frac{3}{2}}}.$$

Denoting by $t_n(x) = \frac{u_n(x)}{\|u_n\|_2}$ and $z_n(x) = \frac{v_n(x)}{\|v_n\|_2}$, the normalised eigenfunctions, and plugging in the constants A and B, we have

$$t_n(x) = \frac{1}{\sqrt{\ell}}\sin\left(\frac{(2n+1)\pi}{2\ell}x\right), \quad n = 0, 1, 2, \ldots,$$
$$z_n(x) = \frac{1}{\sqrt{\ell}}\cos\left(\frac{n\pi}{\ell}x\right), \quad n = 1, 2, \ldots.$$

2.3. Dirichlet boundary conditions on $[0, \ell]$

Consider the following equation

$$-y''(x) - \lambda y(x) = 0, \quad x \in I = (0, \ell), \tag{13}$$

with homogeneous Dirichlet boundary conditions, but at $x = 0$ and $x = \ell$:

$$y(0) = 0, \qquad y(\ell) = 0.$$

As before, we have the iteration formula

$$y_{k+1}(x, \lambda) = y_k(x, \lambda) + \int_0^x -\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(s - x)) \left\{ -y_k''(s, \lambda) - \lambda y_k(s, \lambda) \right\} \mathrm{d}s.$$

The zeroth approximation, $y_0(x, \lambda) = A + Bx$, is again used, where A and B are to be determined. Then,

$$y_1(x, \lambda) = A \cos(\sqrt{\lambda}x) + \frac{B}{\sqrt{\lambda}} \sin(\sqrt{\lambda}x).$$

If the boundary conditions are imposed, eigenvalues of Eq. (13) can simply be identified as

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 0, 1, 2, \ldots,$$

and the corresponding eigenfunctions for the eigenvalues are of the form

$$u_n(x) = B \frac{\ell}{n\pi} \sin\left(\frac{n\pi}{\ell} x\right), \quad n = 0, 1, 2, \ldots.$$

By normalising $u_n(x)$, we consequently obtain

$$B = \frac{\sqrt{2} n\pi}{\ell^{\frac{3}{2}}}.$$

Notice that the eigenvalues and eigenfunctions obtained by the variational iteration method are the same as the exact ones.

2.4. Harmonic oscillator

Now we consider the following equation

$$-y''(x) + (x^2 - \lambda) y(x) = 0, \qquad x \in I = (-\infty, \infty), \tag{14}$$

which describes a quantum mechanical harmonic oscillator. It is well-known that the eigen-pairs of the system (14) are given as follows:

$$y_n^{\infty} = A_n \mathrm{e}^{-x^2/2} H_n(x), \qquad \lambda_n^{\infty} = 2n + 1, \quad n = 0, 1, \ldots,$$

where $H_n(x)$ denotes the Hermite polynomials and A_n are normalisation constants.

Let us consider system (14) on a truncated domain, $-\ell \le x \le \ell$, for all $\ell > 0$, as proposed in [26]: the new system now consists of the differential equation

$$-y''(x) + (x^2 - \lambda) y(x) = 0, \tag{15}$$

for $x \in (-\ell, \ell)$ and the Dirichlet boundary conditions

$$y(-\ell) = 0, \qquad y(\ell) = 0.$$

To find the approximate eigen-pair of the truncated system by means of the variational method, the following correction functional can be constructed:

$$y_{k+1}(x, \lambda) = y_k(x, \lambda) + \int_0^x \mu(s; x, \lambda) \left\{ -y_k''(s, \lambda) + s^2 \tilde{y}_k(s, \lambda) - \lambda y_k(s, \lambda) \right\} \mathrm{d}s,$$

where $\mu = \mu(s; x, \lambda)$ is the Lagrange multiplier and $\tilde{y}_k$ denotes restricted variation, that is, $\delta \tilde{y}_k = 0$. Following the discussions presented in the previous examples, we obtain the iteration formula

$$y_{k+1}(x, \lambda) = y_k(x, \lambda) + \int_0^x -\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(s - x)) \left\{ -y_k''(s, \lambda) + s^2 y_k(s, \lambda) - \lambda y_k(s, \lambda) \right\} \mathrm{d}s.$$

Let us start again with an initial approximation of the form, $y_0(x, \lambda) = A + Bx$, where A and B are constant to be determined so as the solution is nontrivial. Imposing the boundary conditions to the kth-order iterates, for instance, the first-order one,

$$y_1(x, \lambda) = A + Bx + \int_0^x -\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(s - x)) \left\{ s^2 (A + Bs) - \lambda (A + Bs) \right\} \mathrm{d}s,$$

we approximate the eigenvalues and eigenfunctions. In Table 1, we present the ground and the first excited state approximate eigenvalues of system (15) for different values of ℓ. Obtained eigenvalues at the kth iterate are compared to the exact eigenvalues of the harmonic oscillator: $\lambda_n^{\infty} = 2n + 1$ for $n = 0$ and $n = 1$.

Table 1
Eigenvalues of Eq. (15) obtained by VIM.

n	ℓ	k	λ	$\lvert\lambda - \lambda_n^{\infty}\rvert$	λ_n^{∞}
0	2	3	1.08231643774281	0.0823164377428101	1
		6	1.07492263142208	0.0749226314220801	
		7	1.07490823973019	0.0749082397301899	
	π	3	1.39521365408429	0.39521365408429	
		6	1.00304940330591	0.00304940330591008	
		7	1.00064596085758	0.000645960857579952	
	4.5	3	NaN		
		6	1.27710535363838	0.27710535363838	
		7	1.10065201458686	0.10065201458686	
1	2	3	1.08231643774291	1.91768356225709	3
		6	1.07492261826316	1.92507738173684	
		7	1.0748655886969	1.9251344113031	
	π	3	3.16650693566247	0.16650693566247	
		6	1.0030495485462	1.9969504514538	
		7	1.00304948090116	1.99695051909884	
	4.5	3	NaN		
		6	3.29162140783956	0.29162140783956	
		7	4.07642824294159	1.07642824294159	

3. Conclusion

In this study, the applicability of He's variational iteration method for obtaining solutions of boundary value problems and linear Sturm–Liouville problems is demonstrated with some patterns and with the most common eigenvalue problems in applied mathematics. Sections 2.1–2.3 show that the use of the variational iteration method may result in exact solutions, the eigenvalues and the corresponding eigenfunctions, by just a single iteration.

However, this has not been the case for the harmonic oscillator described in Section 2.4. It is possible to overcome some of the shortcomings of the classical variational iteration method. In this respect, the use of the multistage VIM [1,27] may be an alternative modified technique of He's in order to improve the numerical efficiency of the calculations, especially in cases where the restricted variations are necessary.

It can be concluded that the VIM is a very powerful and relatively easy tool for solving different types of systems, even for finding eigen-pairs of Sturm–Liouville operators.

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