MATHEMATICS

COMPACTNESS CONDITIONS FOR INTEGRAL OPERATORS IN BANACH FUNCTION SPACES

BY

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§ 0. *Introduction*

It is well known that integral operators of finite double-norm in a Lebesgue space L_p $(1<p<\infty)$ are compact. Furthermore, it is known that, although an operator of finite double-norm in L_1 need not be compact, its square is always compact. We present a theorem from which these facts follow, thereby generalizing a theorem by W. A. J. LUXEMBURG and A. C. ZAANEN [3]. The theorem (theorem 4.2) is presented in the setting of a Banach function space (also called a normed Köthe space), which is a generalization of the Lebesgue spaces L_p $(1\leqslant p\leqslant\infty)$ and of the Orlicz spaces L_Φ.

§ 1. *Banach function spaces*

Let X be a non-empty point set and let μ be a countably additive and non-negative measure in X such that the triple (X, Λ, μ) is a σ-finite measure space (Λ denoting the collection of all measurable subsets of X). By M we denote the set of all μ-measurable complex functions on X, and by M^+ the subset of all $f \in M$ such that $f(x)\geqslant 0$ holds μ-almost everywhere (μ-a.e.) on X. The notation $\int d\mu$ will denote integration over the whole set X. If it is necessary to show that integration is performed with respect to a certain variable x, we shall use the notation $\int du(x)$. Finally, we denote the characteristic function of the set $E \subset X$ by $\chi_E(x)$.

Let ϱ be a function norm defined on M, and suppose that ϱ satisfies the following conditions.

(a) ϱ has the weak Fatou property, i.e., from

$$0\leqslant f_1\leqslant f_2\leqslant f_3\leqslant \dots \uparrow f \text{ with all } f_n \in M^+ \qquad (n=1, 2, \dots)$$

and $\lim \varrho(f_n)<\infty$ it follows that $\varrho(f)<\infty$.

(b) ϱ is saturated, i.e., if E is any subset of X such that $\mu(E)>0$ and $\varrho(\chi_E)=\infty$, then E contains at least one subset F such that $\mu(F)>0$ and $\varrho(\chi_F)<\infty$.

It is well known (see for example [4]) that these hypotheses on ϱ imply that the space L_ϱ, consisting of all $f \in M$ such that $\varrho(f)<\infty$, is a complete normed space, i.e., a Banach space. We shall also assume that $\mu(X) > 0$, and so it follows from (b) that there exists an L_ϱ-admissable sequence $(\pi): X_n \uparrow X$ of measurable subsets of X (see [3]).

We recall that the associate norm ϱ' of ϱ is defined by

$$\varrho'(g) = \sup \{ \int |fg| d\mu : \varrho(f) \leqslant 1 \},$$

holding for every $g \in M$. We denote $L_{\varrho'}$ also by L_ϱ', and recall that $L_\varrho' \subset L_\varrho^*$. A functional $G \in L_\varrho^*$ belongs to L_ϱ' if and only if there exists a function $g \in L_\varrho'$ such that $\langle f, G \rangle = G(f) = \int fg d\mu$ for all $f \in L_\varrho$.

A function $f \in L_\varrho$ is said to be of absolutely continuous norm whenever $\varrho(\chi_{E_n} f) \downarrow 0$ for every sequence E_n $(n = 1, 2, \ldots)$ of measurable subsets of X such that $E_n \downarrow \emptyset$. The set $L_\varrho{}^a$ of all functions $f \in L_\varrho$ which are of absolutely continuous norm is a closed linear subspace of L_ϱ. We observe that, for $1 \leqslant p < \infty$, every $f \in L_p$ is of absolutely continuous norm. If μ has no atoms, then the only function of absolutely continuous norm in L_∞ is the null function.

Let $(\pi): X_n \uparrow X$ be an L_ϱ-admissable sequence. The closure of the set of all $f \in L_\varrho$, bounded on some subset $F_f \subset X_n$ for some n and vanishing outside F_f, is also a closed linear subspace of L_ϱ, which we denote by L_ϱ^π (see [3]).

§ 2. *Weak sequential compactness*

As usual we denote the weak topology generated in L_ϱ by the subset $M \subset L_\varrho^*$ by $\sigma(L_\varrho, M)$. The subset $S \subset L_\varrho$ is said to be $\sigma(L_\varrho, M)$ sequentially compact whenever every sequence in S contains a subsequence which is $\sigma(L_\varrho, M)$ convergent to an element of S. If it is only required that the subsequence converges to an element of L_ϱ, then S is said to be conditionally $\sigma(L_\varrho, M)$ sequentially compact.

The following theorem was proved by W. A. J. LUXEMBURG and A. C. ZAANEN in [3].

Theorem 2.1. *Let $(\pi): X_n \uparrow X$ be an L_ϱ-admissable as well as L_ϱ'-admissable sequence and let M be a closed linear subspace of L_ϱ' such that $M \supset L_{\varrho'}^\pi$ and such that $g \in M$ implies $g\chi_E \in M$ for any μ-measurable set E. Then the subset $S \subset L_\varrho$ is conditionally $\sigma(L_\varrho, M)$ sequentially compact if and only if*

(i) *$N(g) = \sup \{ \int |fg| d\mu : f \in S\}$ is finite for every $g \in M$,*

(ii) *$N(\chi_{E_n} g) \downarrow 0$ for every $g \in M$ and for every sequence $E_n \downarrow \emptyset$ of measurable subsets of X.*

We shall make use of the following facts which can be deduced from the above theorem.

Theorem 2.2. (*For a proof see* [3]). *Every norm-bounded subset $S \subset L_\varrho$ is conditionally $\sigma(L_\varrho, L_{\varrho'}{}^a)$ sequentially compact.*

The subset $S \subset L_\varrho^a$ is said to be of uniformly absolutely continuous norm whenever, given $\varepsilon > 0$ and any sequence of measurable sets $E_n \downarrow \emptyset$, there exists an index N such that $\varrho(\chi_{E_n} f) < \varepsilon$ for all $n \geqslant N$ and for all $f \in S$ simultaneously.

Theorem 2.3. *Let the norm-bounded subset $S \subset L_\varrho$ be of uniformly absolutely continuous norm. Then S is conditionally $\sigma(L_\varrho, L_\varrho')$ sequentially compact.*

Proof. Obviously L_ϱ' satisfies the conditions required for M in theorem 2.1. Since S is norm bounded, there exists a number C such that $\varrho(f) \leqslant C$ for all $f \in S$. Hence,

$$N(g) = \sup \{ \int |fg| d\mu : f \in S \} \leqslant \sup \{\varrho(f)\varrho'(g) : f \in S\} \leqslant C\varrho'(g) < \infty$$

for every $g \in L_\varrho'$. Furthermore, if $E_n \downarrow \emptyset$ is a sequence of measurable sets, then

$$N(\chi_{E_n} g) = \sup \{ \int |f\chi_{E_n} g| d\mu : f \in S\} \leqslant \sup \{\varrho(\chi_{E_n} f)\varrho'(g) : f \in S\} \downarrow 0$$

for every $g \in L_\varrho'$, since S is of uniformly absolutely continuous norm. By theorem 2.1 we conclude that S is conditionally $\sigma(L_\varrho, L_\varrho')$ sequentially compact.

§ 3. *Integral Operators*

We shall restrict our attention to a certain class of integral operators mapping the Banach function space L_ϱ into itself, i.e., the class of operators having L_ϱ-kernels.

Definition 3.1. *The $(\mu \times \mu)$-measurable function $T(x, y)$ defined on $X \times X$ is called an L_ϱ-kernel whenever*

$$\int |T(x, y) f(y)| d\mu(y) \in L_\varrho \text{ for every } f \in L_\varrho.$$

Let $T(x, y)$ be an L_ϱ-kernel. Then the following properties are well known ([1], [3]).

(a) $Tf = \int T(x, y) f(y) d\mu(y)$ defines a bounded linear operator on L_ϱ.

(b) $T = O$ (the null operator) if and only if $T(x, y) = 0$ holds $(\mu \times \mu)$-a.e. on $X \times X$.

(c) $T(x, y)$ is an L_ϱ-kernel if and only if

$$\int |T(x, y) g(x)| d\mu(x) \in L_\varrho' \text{ for every } g \in L_\varrho'.$$

In other words, $T(x, y)$ is an L_ϱ-kernel if and only if $T^\sim(x, y) = T(y, x)$ is an L_ϱ'-kernel.

(d) The operator $T^\sim$ with kernel $T^\sim(x, y)$ is the restriction of the adjoint operator T^* to the subspace L_ϱ' of L_ϱ^*.

(e) If T_1 and T_2 are integral operators with corresponding L_ϱ-kernels $T_1(x, y)$ and $T_2(x, y)$ respectively, then $T_3 = T_1 T_2$ is an integral operator with L_ϱ-kernel $T_3(x, y)$, where

$$T_3(x, y) = \int T_1(x, z) T_2(z, y) d\mu(z).$$

§ 4. *Compactness of Integral Operators*

We recall that the subset S of a Banach space W is called conditionally sequentially compact whenever every sequence in S contains a subsequence which converges to some element of W. The linear operator T, mapping a Banach space V into a Banach space W is called compact whenever T maps the closed unit ball of V into a conditionally sequentially compact subset of W.

In this section we shall give a necessary and sufficient condition for an integral operator T with L_ϱ-kernel $T(x, y)$ to be compact. We shall, however, make one restrictive assumption concerning the operator T, namely, that the range $R(T)$ of T is a subset of the space $L_\varrho{}^a$.

The following lemma is essentially due to W. A. J. Luxemburg [2].

Lemma 4.1. *Let $f_n (n=1, 2, \ldots)$ be a sequence in L_ϱ which converges pointwise μ-a.e. to a function $f_0 \in L_\varrho$ such that the set $\{f_n : n=1, 2, \ldots\}$ is of uniformly absolutely continuous norm. Then f_n converges in norm to f_0, i.e., $\varrho(f_n - f_0) \to 0$ as $n \to \infty$.*

Proof. Let $(\pi): X_k \uparrow X$ be any L_ϱ-admissable sequence. Then f_n converges pointwise μ-a.e. on X_k for every k. Determine for $k=1, 2, \ldots$ subsets Z_k of X_k such that $\mu(X_k - Z_k) < 1/k$ and such that f_n converges uniformly on Z_k. This is possible by Egoroff's theorem. We may assume that the sequence Z_k is ascending and hence $Z_k \uparrow X$ is an L_ϱ-admissable sequence. By the uniformly absolute continuity of the f_n, we have for some index N that $\varrho(\chi_{X-Z_N} f_n) < \varepsilon/4$ for all n. Hence,

$$\varrho\{\chi_{X-Z_N}(f_n - f_m)\} < \varepsilon/2 \text{ for all } m \text{ and } n.$$

On Z_N, the sequence f_n converges uniformly and we therefore have that

$$\varrho\{\chi_{Z_N}(f_n - f_m)\} < \varepsilon/2 \text{ for } m, n \geq N_1.$$

Hence, for all $m, n \geq N_1$, we have

$$\varrho(f_n - f_m) \leq \varrho\{\chi_{X-Z_N}(f_n - f_m)\} + \varrho\{\chi_{Z_N}(f_n - f_m)\} < \varepsilon.$$

This holds for arbitrarily given $\varepsilon > 0$, so the sequence f_n converges in norm to some $g \in L_\varrho{}^a$. But then some subsequence of f_n converges pointwise μ-a.e. to g. By hypothesis f_n converges pointwise μ-a.e. to f_0 and hence $g = f_0$ holds μ-a.e. on X. This shows that $\varrho(f_0 - f_n) \to 0$ as $n \to \infty$, which ends the proof.

A direct proof without making use of Egoroff's theorem is possible [1].

We now come to our main theorem.

Theorem 4.2. *Let T be an integral operator with L_ϱ-kernel $T(x, y)$ and let $R(T) \subset L_\varrho{}^a$. Then T is compact if and only if*

(i) *$\{Tf : \varrho(f) \leq 1\}$ is a set of uniformly absolutely continuous norm.*

(ii) *Every sequence $f_n (n=1, 2, \ldots)$ with $\varrho(f_n) \leq 1$ contains a subsequence f_{1n} such that Tf_{1n} converges pointwise μ-a.e. on X.*

Proof. We first prove that the conditions are sufficient. This readily follows from the preceding lemma, for if f_n is a sequence such that $\varrho(f_n)\leqslant 1$, then we have that Tf_{1n} converges pointwise μ-a.e. on X for some subsequence f_{1n} of f_n (by condition (ii)). By condition (i) we have that the set $\{Tf_{1n}: n=1, 2, \ldots\}$ is of uniformly absolutely continuous norm and so we conclude from the lemma that the sequence Tf_{1n} converges in norm to some element of L_ϱ. This shows that T is compact.

To prove the necessity of condition (i) suppose that T is compact but that (i) does not hold. Then there exist a sequence of μ-measurable sets $E_n \subset X$ with $E_n \downarrow \emptyset$, a number $\varepsilon>0$, and a sequence f_n with $\varrho(f_n)\leqslant 1$, such that $\varrho(\chi_{E_n}Tf_n)>\varepsilon$ for $n=1, 2, \ldots$. On account of the compactness of T, we may assume that $\varrho(Tf_n-g)\to 0$ for some $g\in L_\varrho{}^a$. Hence, $\varrho(Tf_n-g)<\varepsilon/2$ for all $n\geqslant N_1$, and $\varrho(\chi_{E_n}g)<\varepsilon/2$ for all $n\geqslant N_2$. But then, for $n\geqslant \max(N_1, N_2)$, we have

$$\varrho(\chi_{E_n}Tf_n)\leqslant \varrho\{\chi_{E_n}(Tf_n-g)\}+\varrho(\chi_{E_n}g)<\varepsilon,$$

which yields a contradiction. It follows that the set $\{Tf: \varrho(f)\leqslant 1\}$ is of uniformly absolutely continuous norm.

That the compactness of T implies (ii) is a direct consequence of the fact that every convergent sequence in L_ϱ contains a subsequence which is pointwise convergent μ-a.e. on X.

To apply the above theorem successfully, we should have available conditions under which (i) and (ii) hold.

It is easily seen that if there exists a function $\tau(x)\in L_\varrho{}^a$ such that, for all $f\in L_\varrho$ with $\varrho(f)\leqslant 1$, we have

$$|\textstyle\int T(x, y)f(y)d\mu(y)|\leqslant|\tau(x)|, \tag{1}$$

then the set $\{Tf: \varrho(f)\leqslant 1\}$ is of uniformly absolutely continuous norm, i.e., (i) holds. We shall discuss this matter further when considering an example.

The next lemma presents a sufficient condition in order that condition (ii) of our theorem holds.

Lemma 4.3. *Let T be an integral operator with L_ϱ-kernel $T(x, y)$ satisfying*

$$\int |T(x, y)g(x)|d\mu(x)\in L_{\varrho'}{}^a \text{ for every } g\in L_\varrho{}'.$$

Then every sequence f_n $(n=1, 2, \ldots)$ in L_ϱ with $\varrho(f_n)\leqslant 1$ contains a subsequence $f_n{}'$ such that $Tf_n{}'$ converges pointwise μ-a.e. on X.

Proof. It is known (see [3], lemma 7.2) that under the stated conditions we have, for any μ-measurable set $E\subset X$ such that $\varrho'(\chi_E)<\infty$, that the sequence $\chi_E Tf_n$ is conditionally sequentially compact with respect to the L_1-norm. Let $(\pi): X_k\uparrow X$ be any L_ϱ-admissable as well as $L_\varrho{}'$-admissable sequence. Then Tf_n has a subsequence converging on X_1 with respect to the L_1-norm, and hence Tf_n has a subsequence Tf_{1n} converging pointwise μ-a.e. on X_1. Similarly, Tf_{1n} contains a subsequence Tf_{2n} converging pointwise μ-a.e. on X_2, and so on. The diagonal sequence Tf_{nn} converges pointwise μ-a.e. on X. Hence, if we put $f_n' = f_{nn}$, then f_n' is the required subsequence of f_n.

W. A. J. LUXEMBURG and A. C. ZAANEN proved in [3] that if $T(x, y)$ is a $(\mu \times \mu)$-measurable function satisfying

$$(2) \quad \begin{cases} \text{(a)} & \int |T(x, y)f(y)| d\mu(y) \in L_\varrho^a \text{ for every } f \in L_\varrho, \\ \text{(b)} & \int |T(x, y)g(x)| d\mu(x) \in L_{\varrho'}^a \text{ for every } g \in L_\varrho', \end{cases}$$

then T is compact if and only if the set $\{Tf: \varrho(f) \leqslant 1\}$ is of uniformly absolutely continuous norm. It is clear that this statement is an immediate consequence of theorem 4.2 and lemma 4.3. (Note that condition (2) (a) implies that $R(T) \subset L_\varrho^a$.)

By implementing the results of section 2, we can supply other conditions which imply the pointwise convergence required in condition (ii) of theorem 4.2. The following two theorems show how this can be done.

Theorem 4.4. *Let T be an integral operator with L_ϱ-kernel $T(x, y)$ such that $T_x(y) = T(x, y) \in L_{\varrho'}^a$ for almost every $x \in X$, and let $R(T) \subset L_\varrho^a$. Then T is compact if and only if $\{Tf: \varrho(f) \leqslant 1\}$ is of uniformly absolutely continuous norm.*

Proof. Let f_n be a sequence such that $\varrho(f_n) \leqslant 1$. By theorem 2.2 the set $\{f_n : n = 1, 2, \dots\}$ is conditionally $\sigma(L_\varrho, L_{\varrho'}^a)$ sequentially compact. Hence, f_n contains a subsequence f_{1n} such that f_{1n} is $\sigma(L_\varrho, L_{\varrho'}^a)$ convergent. Since $T_x(y) \in L_{\varrho'}^a$ for almost every $x \in X$, we have for these x that

$$\int T_x(y) f_{1n}(y) d\mu(y) = \int T(x, y) f_{1n}(y) d\mu(y)$$

is a convergent sequence of complex numbers. In other words,

$$Tf_{1n} = \int T(x, y) f_{1n}(y) d\mu(y)$$

converges pointwise μ-a.e. on X. This shows that condition (ii) of theorem 4.2 holds. By hypothesis, (i) holds as well, so we may conclude that T is compact. As in theorem 4.2, the compactness of T implies that $\{Tf: \varrho(f) \leqslant 1\}$ is of uniformly absolutely continuous norm.

Theorem 4.5. *Let $\{Tf: \varrho(f) \leqslant 1\}$ be a set of uniformly absolutely continuous norm and let $T_x(y) \in L_\varrho'$ for almost every $x \in X$. Then the operator T^2 is compact.*

Proof. We prove that the operator T^2 satisfies conditions (i) and (ii) of theorem 4.2, and that $R(T^2) \subset L_\varrho^a$.

To prove that T^2 satisfies (i), we observe that

$$\{T^2 f: \varrho(f) < 1\} \subset \{Tg: \varrho(g) < \|T\|\},$$

and it is easily seen that this last set is of uniformly absolutely continuous norm whenever $\{Tf: \varrho(f) \leqslant 1\}$ is of uniformly absolutely continuous norm. This shows that the set $\{T^2 f: \varrho(f) \leqslant 1\}$ has the required property.

The set $S = \{Tf: \varrho(f) \leqslant 1\}$ is norm bounded and of uniformly absolutely continuous norm so that, by theorem 2.3, S is conditionally $\sigma(L_\varrho, L_\varrho')$ sequentially compact. Hence, if f_n is a sequence such that $\varrho(f_n) \leqslant 1$, then Tf_n contains a subsequence Tf_{1n} which is $\sigma(L_\varrho, L_\varrho')$ convergent. By hypothesis $T_x(y) \in L_\varrho'$, and so

$$\begin{aligned} T^2 f_{1n}(x) &= \int \{ \int T(x, y) T(y, z) d\mu(y) \} f_{1n}(z) d\mu(z) \\ &= \int T(x, y) \{ \int T(y, z) f_{1n}(z) d\mu(z) \} d\mu(y) \\ &= \int (Tf_{1n})(y) T_x(y) d\mu(y), \end{aligned}$$

converges pointwise μ-a.e. on X. We conclude that condition (ii) holds for T^2.

Finally, let $0 \neq f \in R(T)$, i.e., let $f = Tg$ for some $g \in L_\varrho$, $g \neq 0$. Consider the element $g' = g/\varrho(g)$. If $E_n \downarrow \emptyset$ is any sequence of measurable sets then, since $\varrho(g') = 1$, we have $\varrho(\chi_{E_n} Tg') \downarrow 0$, i.e., $[\varrho(g)]^{-1} \varrho(\chi_{E_n} f) \downarrow 0$ so that $\varrho(\chi_{E_n} f) \downarrow 0$. Since this holds for any sequence of measurable sets $E_n \downarrow \emptyset$, and for any $f \in R(T)$, we conclude that $R(T)$, and hence also $R(T^2)$, is contained in L_ϱ^a.

Our final conclusion is therefore that the operator T^2 is compact.

The last theorem is obviously a generalization of the fact that the square of an integral operator of finite double-norm in the Lebesgue space L_1 is compact.

By way of example we finally consider integral operators of finite double-norm in L_p $(1 \leqslant p \leqslant \infty)$. An integral operator T with kernel $T(x, y)$ is said to be of finite double-norm in L_p $(1 \leqslant p \leqslant \infty)$ whenever

$$\|\tau(x)\|_p = \| \, \|T(x, y)\|_q \|_p < \infty \qquad (p^{-1} + q^{-1} = 1).$$

Note that if T is of finite double-norm then $\tau(x) = \|T(x, y)\|_q$ is an element of L_p. This implies that, for $1 \leqslant p < \infty$, the set $\{Tf: \|f\|_p \leqslant 1\}$ is of uniformly absolutely continuous norm, for

$$|(Tf)(x)| \leqslant \int |T(x, y) f(y)| d\mu(y) \leqslant \|T(x, y)\|_q \cdot \|f\|_p \leqslant \tau(x) \in L_p = L_p^a.$$

(See formula (1)). Hence, any integral operator of finite double-norm in L_p $(1 \leqslant p < \infty)$ satisfies condition (i) of theorem 4.2. Furthermore, for $1 < p < \infty$, we have that $T_x(y) \in L_q = L_q^a$ if T is of finite double-norm. Theorem 4.4 therefore shows that T is compact. If $p = 1$, then $T_x(y) \in L_\infty$, so $T_x(y)$ is usually not of absolutely continuous norm (i.e., if μ does not have atoms), unless of course $T_x(y) = 0$ for almost every $x \in X$. However, T satisfies all the conditions of theorem 4.5, so we conclude that T^2 is compact. We note that for $1 < p < \infty$, $T(x, y)$ also satisfies the condition

(2), so that we may also conclude from the theorem of Luxemburg and Zaanen that T is compact. In general, however, an operator T may have a kernel $T(x, y)$ satisfying (2) but not the conditions that $T_x(y) \in L_{\varrho'}{}^a$ or $T_x(y) \in L_\varrho{}'$, and vice versa.

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