



# On perturbative $PSU(N)$ invariants of rational homology 3-spheres<sup>☆</sup>

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## Abstract

We construct power series invariants of rational homology 3-spheres from quantum  $PSU(n)$ -invariants. The power series can be regarded as perturbative invariants corresponding to the contribution of the trivial connection in the hypothetical Witten's integral. This generalizes a result of Ohtsuki (the  $n = 2$  case) which led him to the definition of finite type invariants of 3-manifolds. The proof utilizes some symmetry properties of quantum invariants (of links) derived from the theory of affine Lie algebras and the theory of the Kontsevich integral. © 2000 Elsevier Science Ltd. All rights reserved.

*Keywords:* Homology 3-spheres; Quantum invariants; Perturbative invariants; The Kontsevich integral

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## 0. Introduction

In this paper we construct power series invariants of rational homology 3-spheres from the quantum  $PSU(n)$ -invariants. This generalizes a result of Ohtsuki (the  $n = 2$  case) which led him to the definition of finite type invariants of 3-manifolds [27].

For a fixed compact Lie group and an integer  $r$ , Witten defined an invariant of 3-manifolds using Feynman path integral which is not mathematically rigorous. Reshetikhin and Turaev [30], and later Turaev and Wenzl [36] defined (mathematically) quantum invariants of 3-manifolds associated with simple Lie algebras (see [35] and references therein). The quantum invariants can be thought of as a mathematical realization of Witten's path integral.

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One could use formal perturbation theory to approximate the Witten integral, i.e. to expand the integral around  $q = 1$ , where  $q$  is the “quantum” parameter. The coefficients of the approximations are known as perturbative invariants, which also need a mathematically rigorous definition. One might want to get perturbative invariants directly from (Reshetikhin–Turaev) quantum invariants. The difficulty here is that quantum invariants can be defined only when  $q$  is a root of unity, while in order to get perturbative invariants, one needs to make the formal expansion  $q = \exp(h)$ , with  $h$  an *indeterminate*. This substitution does not make sense if  $q$  takes only values which are roots of unity.

Ohtsuki [26] showed that one can extract power series invariants (of homology 3-spheres) from quantum  $SU(2)$ -invariants which can be considered as the perturbative invariants. Ohtsuki’s result led him to the important definition of finite type invariants of 3-manifolds, a counterpart of Vassiliev invariants for homology 3-spheres. The Ohtsuki series can be considered, on the physics level, as the contribution of the trivial connection in the perturbative expansion of the Witten integral (see [31]). Further investigation of Ohtsuki’s series was carried out by Ohtsuki [26,28], Lawrence and Rozansky [13,14,31,32], Lin and Wang [21], Kricker and Spence [12], and others. A result of Murakami [25] says that the first term of Ohtsuki’s series is 6 times the Casson–Walker invariant.

In [19] it was conjectured that similar power series invariants exist for Lie groups  $SU(n)$ ,  $n > 2$ . It is the main goal of this paper to prove this conjecture. We show that one can extract power series invariants of *rational homology 3-spheres* from quantum  $PSU(n)$ -invariants. Here the quantum  $PSU(n)$ -invariant is a refined version of the  $SU(n)$ -invariant which was introduced by Kirby and Melvin for the  $n = 2$  case [8], and by Kohno and Takata for the  $n > 2$  case [10]. It will be clear from the proof that for rational homology 3-spheres, one should use the  $PSU(n)$  version instead of the  $SU(n)$  one, because of some integrality and symmetry properties of quantum link invariants related to the  $PSU(n)$  case. In a sense, the  $SU(n)$  version is based on the weight lattice, and the  $PSU(n)$  version is based on the root lattice; and quantum invariants of links on the root lattice are nicer: they have integrality properties and more symmetry.

A few words about the proof. The proof of the main result differs from that of Ohtsuki for the case  $n = 2$ , since Ohtsuki [26] used some identities which are either specific to the  $n = 2$  case or hard to generalize to the  $n > 2$  case (see also [21,31]). Some of our arguments are generalizations of those in [21,26,31] which treat the  $n = 2$  case. We will make use of some symmetry properties of quantum invariants of links which are derived from affine Lie algebra theory and the Kontsevich integral theory. Along the way we also investigate the dependence of quantum invariants of links on the colors of the link components. The proof also suggests how to generalize the result to other Lie algebras, and we will try to formulate everything in terms of Lie algebra theory.

In Section 1 we introduce the notation and formulate the main result. In Section 2 we give the proof of the main theorem using some results proved in Sections 3 and 4. In Section 3 we investigate the dependence of quantum invariants of links on the colors (which are modules of Lie algebras). In Section 4 we prove some properties of the quadratic Gauss sums based on the root lattice. In Appendix we show that our definition of the quantum  $PSU(n)$ -invariant coincides with that of Kohno and Takata, and calculate the value of a multi-variable quadratic Gauss sum.

# 1. Preliminaries. Main result

## 1.1. Notations.

We will use the following notations for objects related to the Lie algebra  $sl_n$ . See, for example, [3,6], for the terminologies used here.

$\alpha_1, \dots, \alpha_{n-1}$ : standard root basis

$\Phi$ : root system

$\Phi_+$ : set of positive roots. There are  $n(n - 1)/2$  positive roots

$\lambda_1, \dots, \lambda_{n-1}$ : standard fundamental weights

$\Lambda$ : integral weight lattice,  $\Lambda = \{\sum_{i=1}^{n-1} k_i \lambda_i, k_i \in \mathbb{Z}\}$ . All the weights  $\lambda_i$  and roots  $\alpha_j$  are elements of  $\Lambda$ .

In  $\Lambda \otimes \mathbb{R}$  there is the standard scalar product, for which  $(\alpha_i | \alpha_j) = A_{ij}$ . Here  $A$  is the Cartan matrix. One has  $(\alpha_i | \lambda_j) = \delta_{ij}$

$\rho$ : half-sum of positive roots,  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$ . One has that  $\rho = \lambda_1 + \dots + \lambda_{n-1}$ , and  $(\rho | \rho) = n(n^2 - 1)/12$

$\mathbb{Z}_+$ : set of non-negative integers

$\Lambda_+$ : set of dominant weights,  $\Lambda_+ = \{k_1 \lambda_1 + \dots + k_{n-1} \lambda_{n-1} \mid k_1, \dots, k_{n-1} \in \mathbb{Z}_+\}$

$\Lambda_{++}$ : set of strongly dominant weights,  $\Lambda_{++} = \rho + \Lambda_+$

$\Lambda^{\text{root}}$ : root lattice,  $\Lambda^{\text{root}} = \{\sum_{i=1}^{n-1} k_i \alpha_i, k_i \in \mathbb{Z}\}$ . The root lattice is a subgroup of the weight lattice  $\Lambda$  of index  $n$

$\theta$ : longest root,  $\theta = \alpha_1 + \dots + \alpha_{n-1}$

$\mathcal{A}$ : ring of finite dimensional  $sl_n$ -modules

$V_\mu$ , with  $\mu \in \Lambda_{++}$ : the irreducible  $sl_n$ -module with highest weight  $\lambda - \rho$ . Note that this is slightly different from the usual notation. The shift by  $\rho$  is more convenient for us

The Weyl group  $\mathcal{W}$  acts on  $\Lambda \otimes \mathbb{R}$ , and a fundamental domain is the fundamental chamber

$$C = \{z \in \Lambda \otimes \mathbb{R} \mid (z, \alpha_i) \geq 0, i = 1, \dots, n - 1\} = \left\{ \sum_{i=1}^{n-1} a_i \lambda_i \mid a_i \geq 0, a_i \in \mathbb{R} \right\}.$$

So,  $\Lambda_+$  is the intersection of  $\Lambda$  and  $C$ . The Weyl group is generated by reflections along the boundary facets of  $C$ . For an element  $w \in \mathcal{W}$ , let  $\text{sn}(w)$  be the sign of  $w$ . Similarly, for a non-zero number  $b$ , let  $\text{sn}(b)$  be the sign of  $b$ .

Throughout the paper, the number  $n$  (of  $sl_n$ ) is fixed. We will use  $r$  to denote the “level” of the theory (it is just a positive integer). Let

$$\bar{r} = \frac{r - 1 - n(n - 1)}{2}.$$

Note that if  $r$  is odd, which we always assume, then  $\bar{r}$  is an integer.

We will use  $q, x$  as indeterminates, with  $q = 1 + x$ . We use  $\zeta$  to denote the  $r$ th root of unity,  $\zeta = \exp(2\pi i/r)$ . When  $r$  is an odd prime, we will identify  $\mathbb{Z}[\zeta]$  with  $\mathbb{Z}[q]/(q^{r-1} + \dots + q + 1)$ .

For any positive integer  $\ell$ , let

$$\mathbb{Z}_{(\ell)} = \mathbb{Z} \left[ \frac{1}{(\ell - 1)!n!} \right].$$

For a rational homology 3-sphere  $M$  (i.e. a closed oriented 3-manifold whose homology group  $H_1(M, \mathbb{Z})$  is finite), let  $\mathbb{Q}[[x]]_{(M)}$  be the set of all power series  $\sum d_\ell x^\ell$  such that  $d_\ell$  is a number in

$$\mathbb{Z} \left[ \frac{1}{(2\ell + n(n - 1))! |H_1(M, \mathbb{Z})|} \right].$$

## 1.2. Quantum invariants of oriented framed links and 3-manifolds

For a framed oriented link  $L$  with  $m$  ordered components in  $S^3$ , let  $J_L(\mu_1, \dots, \mu_m; q)$  be the  $sl_n$ -quantum invariant of the link [29,35], where the components of  $L$  are colored by  $V_{\mu_1}, \dots, V_{\mu_m}$ . Here  $q$  is an indeterminate and the  $\mu_j$ 's are in  $\Lambda_{++}$ , so that  $V_{\mu_j}$  are finite-dimensional irreducible  $sl_n$ -modules of highest weight  $\mu_j - \rho$ . Actually,  $J_L$  can be regarded as a linear homomorphism from  $\mathcal{R}^{\otimes m}$  to  $\mathbb{Z}[q^{\pm 1/2n}]$ .

We extend the definition of  $J_L(\mu_1, \dots, \mu_m; q)$  to the case when the  $\mu_j$ 's are in  $\Lambda$  as follows. If one of the  $\mu_j$  is on the boundary of a chamber  $w(C)$ , where  $w$  is in the Weyl group, put  $J_L(\mu_1, \dots, \mu_m; q) = 0$ . Otherwise, there exist unique  $w_1, \dots, w_m$  in the Weyl group  $\mathcal{W}$  such that the elements  $w_j(\mu_j)$  are in  $\Lambda_{++}$ . Then put

$$J_L(\mu_1, \dots, \mu_m; q) = \text{sn}(w_1) \dots \text{sn}(w_m) J_L(w_1(\mu_1), \dots, w_m(\mu_m); q).$$

It is more convenient to use the following normalization of  $J_L$ :

$$Q_L(\mu_1, \dots, \mu_m; q) = J_L(\mu_1, \dots, \mu_m; q) J_U(\mu_1; q) \dots J_U(\mu_m; q),$$

where  $U$  is the unknot with framing 0. The invariant  $Q_L$  enjoys the symmetry described in the following proposition, which follows immediately from the definition.

**Proposition 1.1.** (a)  $Q_L(\mu_1, \dots, \mu_{n-1}; q)$  is invariant under the action of the Weyl group, i.e. for every  $w_1, \dots, w_m \in \mathcal{W}$ , one has

$$Q_L(\mu_1, \dots, \mu_m; q) = Q_L(w_1(\mu_1), \dots, w_m(\mu_m); q).$$

(b) If one of the  $\mu_j, j = 1, \dots, m$ , is on the boundary of the fundamental chamber  $C$ , then  $Q_L(\mu_1, \dots, \mu_m; q) = 0$ .

For a positive integer  $r$ , let

$$F'_L(r) = \sum_{\mu \in \Lambda'_r, 1 \leq j \leq m} Q_L(\mu_1, \dots, \mu_m; \zeta^a),$$

where  $\zeta^a = \exp(2a\pi i/r)$  for every rational number  $a$ , and

$$\Lambda'_r = \{ \mu \in \Lambda_+ \mid (\mu|\theta) \leq r \} = \left\{ \sum_{i=1}^{n-1} k_i \lambda_i \in \Lambda_+ \mid \sum k_i \leq r \right\}.$$

Suppose the closed oriented 3-manifold  $M^3$  is obtained from  $S^3$  by surgery along a framed unoriented link  $L$ . Providing  $L$  with arbitrary orientation, the quantum  $SU(n)$ -invariant of  $M$  at the  $r$ th root of unity is defined by (see, for example, [9,36])

$$\tau_r^{SU(n)}(M) = \frac{F'_L(r)}{F'_{U_+}(r)^{\sigma_+} F'_{U_-}(r)^{\sigma_-}}.$$

Here  $\sigma_+, \sigma_-$  are the numbers of positive and negative eigenvalues of the linking matrix of  $L$  and  $U_{\pm}$  are the trivial knots with framing  $\pm 1$ .

1.3. Quantum invariants of links at roots of unity

Let  $C_r$  be the simplex which is the convex hull of the points  $0, r\lambda_1, \dots, r\lambda_{n-1}$ . In other words,

$$C_r = \{z \in C \mid (z, \theta) \leq r\} = \left\{ \sum_{i=1}^{n-1} a_i \lambda_i \in C \mid \sum_{i=1}^{n-1} a_i \leq r \right\},$$

(we call it the fundamental alcove of level  $r$ ). Then  $A'_r = C_r \cap A_+$ . The affine Weyl group at level  $r$ , by definition, is the group  $\mathcal{W}_{(r)}$  generated by all the reflections along boundary facets of the simplex  $C_r$ . It is known that  $\mathcal{W}_{(r)}$  is the semi-direct product of  $\mathcal{W}$  and the translation group  $rA^{\text{root}}$  [6, Chapter 6].

**Lemma 1.2.** *Each of the lattices  $A^{\text{root}}$  and  $\rho + A^{\text{root}}$  is invariant under the action of  $\mathcal{W}_{(r)}$ , for every positive integer  $r$ .*

**Proof.** The root lattice  $A^{\text{root}}$  is certainly invariant under the action of the Weyl group and the translation group  $rA^{\text{root}}$ . Hence  $A^{\text{root}}$  is invariant under the action of  $\mathcal{W}_{(r)}$ .

If  $\alpha_i$  is a basis root, then the reflection along the hyperplane perpendicular to  $\alpha_i$  maps  $\rho$  to  $\rho - \alpha_i$ , which belongs to  $\rho + A^{\text{root}}$ . These reflections generate the Weyl group, hence  $\rho + A^{\text{root}}$  is invariant under the action of the Weyl group. It is obviously invariant under the translation group  $rA^{\text{root}}$ .  $\square$

When specialized to the  $r$ th root of unity,  $Q_L$  has more symmetry. The proof of the following important proposition is given in Section 3.

**Proposition 1.3.** *Let  $\zeta = \exp(2\pi i/r)$ , where  $r$  is a positive integer.*

(a)  $Q_L(\mu_1, \dots, \mu_m; \zeta)$  is invariant under the action of the affine Weyl group  $\mathcal{W}_{(r)}$ , i.e. for every  $w_1, \dots, w_m$  in  $\mathcal{W}_{(r)}$ , one has

$$Q_L(\mu_1, \dots, \mu_m; \zeta) = Q_L(w_1(\mu_1), \dots, w_m(\mu_m); \zeta).$$

(b) If one of the  $\mu_j, j = 1, \dots, m$ , is on the boundary of the fundamental alcove  $C_r$ , then  $Q_L(\mu_1, \dots, \mu_m; \zeta) = 0$ .

#### 1.4. On the definition of $PSU(n)$ -quantum invariants

Quantum  $PSU(n)$ -invariants were introduced by Kirby–Melvin [8] in the  $n = 2$  case, and by Kohno–Takata [10] in the  $n > 2$  case. We give here another definition which is more natural from our point of view. We will prove that our definition coincides with that of Kohno and Takata in Appendix.

Let  $A_r^{\text{root}}$  be the set of all elements of the root lattice in the half-open parallelepiped spanned by  $r\alpha_1, \dots, r\alpha_{n-1}$ , i.e.

$$A_r^{\text{root}} = \{k_1\alpha_1 + \dots + k_{n-1}\alpha_{n-1} \in A^{\text{root}} \mid 0 \leq k_i < r\}.$$

Define  $F_L(r)$  using the same formula as for  $F'_L(r)$ , only replacing  $A'_r$  by  $\rho + A_r^{\text{root}}$ :

$$F_L(r) = \sum_{\mu_j \in (\rho + A_r^{\text{root}}), 1 \leq j \leq m} Q_L(\mu_1, \dots, \mu_m; \zeta).$$

Suppose that  $r$  and  $n$  are coprime. We will show later that  $F_{U_{\pm}}(r) \neq 0$ . Suppose that  $M$  is obtained by surgery along the framed link  $L$ . Define the quantum  $PSU(n)$ -invariant  $\tau_r^{PSU(n)}(M)$  by

$$\tau_r^{PSU(n)}(M) = \frac{F_L(r)}{F_{U_+}(r)^{\sigma_+} F_{U_-}(r)^{\sigma_-}}. \quad (1.1)$$

**Remark.** (a) In the set  $\rho + A_r^{\text{root}}$  there are weights which are not highest weights of any finite-dimensional  $sl_n$ -modules.

(b) In the definition of  $F_L$ , the  $\mu_j$ 's run the set  $\rho + A_r^{\text{root}}$  since we use  $V_{\mu}$  for the  $sl_n$ -module with highest weight  $\mu - \rho$ . Also if  $n$  is odd, then  $\rho$  is in the root lattice, hence  $\rho + A_r^{\text{root}} = A_r^{\text{root}}$ ; and using the symmetry of  $Q_L$  we can replace  $\rho + A_r^{\text{root}}$  by  $A_r^{\text{root}}$ .

(c) If in the definition of  $F_L$ , we let the  $\mu_j$ 's run the set of all integral weights in the same parallelepiped, then from the above formula we get exactly the  $SU(n)$ -invariant. So both  $SU(n)$  and  $PSU(n)$ -invariants can be defined by one formula, only for the  $SU(n)$  invariant, we use the weight lattice, while for the  $PSU(n)$  invariant, we use the root lattice. It is known that  $A/A^{\text{root}}$  is the cyclic group of order  $n$ .

(d) Kohno and Takata showed that the  $PSU(n)$  version is finer than the  $SU(n)$  one.

The following important integrality property was established by Murakami [25] (see also [22]) for  $n = 2$ , and by Takata–Yokota [34] and Masbaum–Wenzl [23] for  $n > 2$ ; this result can also be obtained by the method of this paper.

**Theorem 1.4.** *Suppose  $r$  is a prime not dividing  $n! |H_1(M, \mathbb{Z})|$ . Then the  $PSU(n)$ -quantum invariant  $\tau_r^{PSU(n)}(M)$  is a number in  $\mathbb{Z}[\zeta]$ . Here  $\zeta = \exp(2\pi i/r)$ .*

#### 1.5. Existence of perturbative expansion

Unlike the link case, quantum invariants of 3-manifolds can be defined only at roots of unity. In perturbative theory, we want to expand  $q = e^h$ , with  $h$  an indeterminate. However, in the 3-manifold case,  $q$  must be specified at the  $r$ th root of unity, and it seems difficult to make meaning

the substitution  $q = e^h$ , with  $h$  an indeterminate. One way around is the following. Instead of  $h$ , we use  $x = q - 1$ . Suppose  $r$  is a prime not dividing  $n! |H_1(M, \mathbb{Z})|$ .

Let  $f(q) \in \mathbb{Z}[q]$  be a representative of  $\tau_r^{PSU(n)}(M) \in \mathbb{Z}[\zeta] = \mathbb{Z}[q]/(q^{r-1} + \dots + q + 1)$ . Using the substitution  $q = 1 + x$ , we get

$$f(q)|_{q=x+1} = c_{r,0} + c_{r,1}x + \dots + c_{r,\ell}x^\ell + \dots$$

The integer coefficients  $c_{r,\ell}$  are, in general, dependent on the representative  $f(q)$ . However, it is easy to prove the following [26].

**Lemma 1.5.** *For  $\ell \leq r - 2$ , the classes  $c_{r,\ell} \pmod{r}$  do not depend on the representative  $f(q)$  in  $\mathbb{Z}[q]$ .*

Hence for each prime  $r$ , the classes  $c_{r,\ell} \pmod{r}$ ,  $\ell = 0, 1, \dots, r - 2$ , are invariants of  $M$ . Let us fix  $\ell$  and let  $r$  vary (but  $r$  must be an odd prime). For each  $r$  we have a residue class  $c_{r,\ell}(M) \pmod{r}$ . We want to show that these classes (when  $r$  varies but  $\ell$  fixed) can be unified in the following sense: they are derived from the same rational number not depending on  $r$ . More precisely (see Theorem 1.7 below) we will show that there is a rational number

$$c_\ell(M) \in \mathbb{Z} \left[ \frac{1}{(2\ell + n(n - 1))! |H_1(M, \mathbb{Z})|} \right]$$

such that

$$c_\ell \equiv \left( \frac{|H_1(M, \mathbb{Z})|}{r} \right)^{n-1} c_{r,\ell} \pmod{r} \tag{1.2}$$

for sufficiently large prime  $r$  (in fact,  $r > 2\ell + n(n - 1)$  is enough). Here  $\left( \frac{|H_1(M, \mathbb{Z})|}{r} \right)$  is the Legendre symbol, which is either 1 or  $-1$ . The following is easy to prove [26].

**Lemma 1.6.** *For each  $\ell$ , there exists at most one rational number*

$$c_\ell \in \mathbb{Z} \left[ \frac{1}{(2\ell + n(n - 1))! |H_1(M, \mathbb{Z})|} \right]$$

such that (1.2) holds for all sufficiently large prime  $r$ .

It follows that if such  $c_\ell$  exist, then it is an invariant of  $M$ . The existence of such  $c_\ell$  is a quite non-trivial fact. If all the  $c_\ell$ 's do exist, then the series  $\sum c_\ell x^\ell$  should be considered the perturbative expansion of  $\tau_r^{PSU(n)}(M)$ .

Another way to look at the existence of  $c_\ell$  is the following. For  $f(q) \in \mathbb{Z}[q]$  (or in  $\mathbb{Z}_{(r)}[q]$ ) we will define  $p_r(f(q))$  as follows. First we substitute  $q = x + 1$  in  $f$ , then cancel any powers of  $x$  with degree greater than  $\bar{r} = (r - n(n - 1) - 1)/2$ , finally reduce all the coefficients modulo  $r$ . The result is  $p_r(f)$ . So  $p_r$  is an algebra homomorphism

$$p_r : \mathbb{Z}_{(r)}[q] \rightarrow \mathbb{Z}[x]/(r, x^{\bar{r}+1}).$$

It is easy to see that  $\mathfrak{p}_r$  descends to an algebra homomorphism, also denoted by  $\mathfrak{p}_r$ , on

$$\mathbb{Z}_{(r)}[q]/(q^{r-1} + \dots + q + 1) = \mathbb{Z}_{(r)}[\zeta].$$

Hence there is defined  $\mathfrak{p}_r(\tau_r^{PSU(n)}(M))$ . Certainly  $\mathfrak{p}_r(x^d) = 0$  if  $d > \bar{r}$ .

Recall that  $\mathbb{Q}[[x]]_{(M)}$  is the set of all formal power series  $\sum d_\ell x^\ell$  such that

$$d_\ell \in \mathbb{Z} \left[ \frac{1}{(2\ell + n(n-1))! |H_1(M, \mathbb{Z})|} \right].$$

So, when  $r$  is a prime greater than both  $|H_1(M, \mathbb{Z})|$  and  $2\ell + n(n-1)$ , we can reduce  $d_\ell$  modulo  $r$ .

For a series  $g = \sum d_\ell x^\ell \in \mathbb{Q}[[x]]_{(M)}$ , where  $|H_1(M, \mathbb{Z})| < r$ , we can also define  $\mathfrak{p}_r(g)$  in a similar way, as follows. First cancel powers of  $x$  with degree greater than  $\bar{r}$ , then reduce all the coefficients modulo  $r$ . The result is  $\mathfrak{p}_r(g)$ . A similar operator was also considered by Ohtsuki and Rozansky in connection with the  $n = 2$  case.

Then the equality

$$c_\ell \left( \frac{|H_1(M, \mathbb{Z})|}{r} \right)^{n-1} \equiv c_{r,\ell} \pmod{r},$$

for every  $\ell$  and  $r > 2\ell + n(n-1)$ , means that

$$\left( \frac{|H_1(M, \mathbb{Z})|}{r} \right)^{n-1} \mathfrak{p}_r(\tau_r^{PSU(n)}(M)) = \mathfrak{p}_r \left( \sum c_\ell x^\ell \right).$$

The following is the main result of this paper.

**Theorem 1.7.** *Let  $M$  be a rational homology 3-sphere. There exists a power series*

$$\tau^{PSU(n)}(M) = \sum_{\ell=0}^{\infty} c_\ell(M) x^\ell \in \mathbb{Q}[[x]]_{(M)}$$

such that for any prime number  $r$  greater than  $\max\{n(n-1), |H_1(M, \mathbb{Z})|\}$ , one has

$$\mathfrak{p}_r(\tau^{PSU(n)}(M)) = \left( \frac{|H_1(M, \mathbb{Z})|}{r} \right)^{n-1} \mathfrak{p}_r(\tau_r^{PSU(n)}(M)).$$

Here  $\tau_r^{PSU(n)}(M)$  is the  $PSU(n)$ -quantum invariant of  $M$  at the  $r$ th root of unity.

**Remark.** (a) In the case  $n = 2$ , the existence of  $c_\ell$  was proved by Ohtsuki [26].

(b) The above equality means that  $\tau^{PSU(n)}(M)$  is the Fermat limit of  $\tau_r^{PSU(n)}(M)$ , in the terminology of [21].

The theorem shows that we can recover part of the quantum invariant  $\tau_r^{PSU(n)}(M)$  from the perturbative invariant  $\tau^{PSU(n)}(M)$ . We have the following conjecture.

**Conjecture 1.8.** *The perturbative invariant  $\tau^{PSU(n)}(M)$  dominates the quantum invariants  $\tau_r^{PSU(n)}(M)$ , for every positive integer  $r$ , not necessarily prime. That is, if  $\tau^{PSU(n)}(M) = \tau^{PSU(n)}(M')$ , then  $\tau_r^{PSU(n)}(M) = \tau_r^{PSU(n)}(M')$ , for every  $r$  for which  $\tau_r^{PSU(n)}$  exists.*



Another conjecture is the integrality of  $c_\ell(M)$ , which is a generalization of conjectures made and investigated by Lawrence et al. (see [13,14,32,21]).

**Conjecture 1.9.** *The perturbative invariants  $c_\ell(M)$  are in  $\mathbb{Z}[1/n!|H_1(M, \mathbb{Z})|]$ .*

One can show that  $c_1(M) = |H_1(M, \mathbb{Z})|^{-n(n-1)/2} n(n^2 - 1) \lambda_C(M)$ , where  $\lambda_C$  is the Casson–Walker invariant in Lescop normalization [20].

## 2. Proof of the main theorem

We will give the proof of the main theorem based on some technical results which will be proved later.

### 2.1. Integrality of the exponents

It is known that for fixed  $\mu_1, \dots, \mu_m$ , the quantum invariant  $J_L(\mu_1, \dots, \mu_m; q)$  is a polynomial in  $q^{1/2n}, q^{-1/2n}$  with integer coefficients (see, for example, [23], for a proof), i.e.  $J_L(\mu_1, \dots, \mu_m; q) \in \mathbb{Z}[q^{1/2n}, q^{-1/2n}]$ . When the linking matrix of  $L$  is 0, we have the following stronger result, whose proof will be presented in Section 3.

**Proposition 2.1.** *Suppose that the linking matrix of  $L$  is 0. Then  $Q_L(\mu_1, \dots, \mu_m; q)$  is a polynomial in  $q, q^{-1}$  with integer coefficients. In other words,  $Q_L(\mu_1, \dots, \mu_m; q)$  does not contain any non-integer fractional power of  $q$ .*

**Remark.** The proposition does not hold true if the linking matrix of  $L$  is not 0.

**Corollary 2.2.** *Suppose that the linking matrix of  $L'$  is diagonal and that  $\mu_1, \dots, \mu_m$  are in  $(\rho + A^{\text{root}})$ . Then  $Q_{L'}(\mu_1, \dots, \mu_m; q)$  is in  $\mathbb{Z}[q, q^{-1}]$ .*

**Proof.** Let  $L$  be the same link as  $L'$ , only with 0 framing on each component. By the previous proposition,  $Q_L(\mu_1, \dots, \mu_m)$  is  $\mathbb{Z}[q, q^{-1}]$ . From the general theory of quantum invariants, it is known that increasing the framing by 1 on the  $j$ th component results in a factor  $q^{(|\mu_j|^2 - |\rho|^2)/2}$  in  $Q_L$  (see, for example, [35]). Here  $|\mu|^2 = (\mu|\mu)$ . We have:

$$Q_{L'}(\mu_1, \dots, \mu_m; q) = \left( \prod_{j=1}^m q^{b_j(|\mu_j|^2 - |\rho|^2)/2} \right) Q_L(\mu_1, \dots, \mu_m; q), \tag{2.1}$$

where  $b_i$ 's are the framing of components of  $L'$ . It remains to notice that when  $\mu$  is in  $\rho + A^{\text{root}}$ ,  $(|\mu|^2 - |\rho|^2)/2$  is an integer.  $\square$

The corollary shows an advantage of using the root lattice and links with diagonal linking matrix (which correspond to rational homology 3-spheres). If  $\mu_j$  are in the bigger lattice  $\mathcal{A}$ , then, in general,  $Q_{L'}(\mu_1, \dots, \mu_m; q)$  contains fractional powers of  $q$ .

## 2.2. Polynomial functions on the weight lattice

Let us introduce a new indeterminate  $x = q - 1$ . Then  $q^{-1} = 1 - x + x^2 - x^3 + \dots$ , and  $Q_L$  becomes a formal power series in  $x$ :

$$Q_L(\mu_1, \dots, \mu_m; q)|_{q=x+1} = \sum_{\ell=0}^{\infty} f_{\ell}(\mu_1, \dots, \mu_m) x^{\ell}. \quad (2.2)$$

This kind of expansion, for the  $n = 2$  case, was first considered by Melvin and Morton [24]. Each monomial  $\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_s}$ , where  $\alpha_i$ 's are the basis roots, defines a function on  $\Lambda$  by

$$\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_s}(\mu) = (\alpha_{i_1} | \mu)(\alpha_{i_2} | \mu) \dots (\alpha_{i_s} | \mu),$$

for  $\mu \in \Lambda$ . Using linearity, every polynomial in the  $\alpha_i$ 's can be regarded as a function on  $\Lambda$ ; we call them the *polynomial functions* on  $\Lambda$ . The *degree* of the polynomial function is the degree of the corresponding polynomial. A function of many variables,  $f(\mu_1, \dots, \mu_m)$ , with  $\mu_1, \dots, \mu_m \in \Lambda$ , is a *polynomial function* if it is a linear combination of functions of the form  $g_1(\mu_1) \dots g_m(\mu_m)$ , where each  $g_j$  is a polynomial function.

Let  $\Delta$  be the following polynomial function on  $\Lambda$  of degree  $n(n-1)/2$ :

$$\Delta = \frac{\prod_{\alpha \in \Phi_+} \alpha}{\prod_{\alpha \in \Phi_+} (\rho | \alpha)}.$$

The denominator is  $\prod_{\alpha \in \Phi_+} (\rho | \alpha) = 1^{n-1} 2^{n-2} \dots (n-1)^1$ .

**Lemma 2.3.** (a) If  $\mu \in \Lambda_{++}$ , then the dimension of  $V_{\mu}$  is given by  $\dim(V_{\mu}) = \Delta(\mu)$ .

(b) If  $\mu$  is on the boundary of the chamber  $C$ , then  $\Delta(\mu) = 0$ . For every  $w \in \mathcal{W}$ ,

$$\Delta(w(\mu)) = \text{sn}(w)\Delta(\mu).$$

Part (a) is the famous dimension formula of Weyl. Part (b) follows from elementary Lie algebra theory [3].

The proof of the following proposition will be given in Section 3.

**Proposition 2.4.** Suppose that the link  $L$  has  $m$  components and 0 linking matrix.

(a) All the functions  $f_i$  in (2.2) are polynomial functions divisible by  $\Delta^2(\mu_1) \dots \Delta^2(\mu_m)$ :

$$f_{\ell}(\mu_1, \dots, \mu_m) = \Delta^2(\mu_1) \dots \Delta^2(\mu_m) f'_{\ell}(\mu_1, \dots, \mu_m).$$

Moreover,  $f_{\ell}$  are polynomial functions taking integer values when  $\mu_1, \dots, \mu_m \in \Lambda$ .

(b) Suppose the degree of  $\mu_j$  in  $f'_{\ell}$  is  $s_j$ . Then  $\sum_{j=1}^m s_j \leq 3\ell/2$  and  $\max_j s_j \leq 2\ell - \sum_{j=1}^m s_j$ . For  $\ell = 0$  we have  $f'_0 = 1$ .

## 2.3. An expression of $Q_L$ in terms of exponential functions

Let  $q^{\beta}$ , for  $\beta \in \Lambda^{\text{root}}$ , be the formal power series in  $x$  obtained by formally expanding  $q^{\beta} = (1+x)^{\beta}$ , the coefficients being polynomial in  $\beta$ :

$$q^{\beta} = \sum_{\ell=0}^{\infty} \binom{\beta}{\ell} x^{\ell}.$$

Here  $\binom{\beta}{\ell}$  is the usual binomial polynomial:

$$\binom{\beta}{\ell} = \frac{\beta(\beta - 1) \cdots (\beta - \ell + 1)}{\ell!}.$$

Then  $q^\beta(\mu)$ , for  $\beta \in A^{\text{root}}$  and  $\mu \in \Lambda$ , is a formal power series in  $x$  with integer coefficients:

$$q^\beta(\mu) = \sum_{\ell=0}^{\infty} \binom{\beta}{\ell}(\mu)x^\ell \in \mathbb{Z}[[x]],$$

since  $(\beta | \mu)$  is an integer for every  $\mu \in \Lambda$  and  $\beta \in A^{\text{root}}$ .

For each  $i = 1, \dots, n - 1$ , the element

$$\eta_i = 1 - q^{-\alpha_i}$$

can be regarded as a function from  $\Lambda$  to  $\mathbb{Z}[[x]]$ . Moreover, the degree 0 term of  $\eta_i(\mu)$  is 0, i.e.  $\eta_i(\mu)$  is divisible by  $x$ .

For  $\mathbf{a} = (a_1, \dots, a_{n-1}) \in (\mathbb{Z}_+)^{n-1}$ , let  $|\mathbf{a}| = a_1 + \dots + a_{n-1}$ , and  $\boldsymbol{\eta}^{\mathbf{a}} = \prod_{i=1}^{n-1} \eta_i^{a_i}$ . Then for  $\mu \in \Lambda$ ,  $\boldsymbol{\eta}^{\mathbf{a}}(\mu) \in x^{|\mathbf{a}|} \mathbb{Z}[[x]]$ .

**Proposition 2.5.** *Suppose that  $L$  is a framed link with  $m$  components and 0 linking matrix. As formal power series in  $x$  we have*

$$Q_L(\mu_1, \dots, \mu_m; x + 1) = \sum_{c_{a_1, \dots, a_m; \ell}} c_{a_1, \dots, a_m; \ell} \boldsymbol{\eta}^{a_1}(\mu_1) \dots \boldsymbol{\eta}^{a_m}(\mu_m) x^{\ell - \sum_j |a_j|}, \tag{2.3}$$

where the sum is over the set of  $\mathbf{a}_j \in (\mathbb{Z}_+)^{n-1}$  and  $\ell \in \mathbb{Z}_+$  such that

$$\sum_{j=1}^m (|\mathbf{a}_j| - n(n - 1)) \leq 3\ell/2, \tag{2.4}$$

$$\max_j (|\mathbf{a}_j| - n(n - 1)) \leq 2\ell - \sum_j (|\mathbf{a}_j| - n(n - 1)). \tag{2.5}$$

Moreover, the coefficient  $c_{a_1, \dots, a_m; \ell}$  is in  $\mathbb{Z}_{(2\ell + (m+1)n(n-1) - \sum |a_j| + 1)}$ .

**Remark.** Since  $\boldsymbol{\eta}^{\mathbf{a}}$  is a power series divisible by  $x^{|\mathbf{a}|}$ , the right-hand side of (2.3) contains only non-negative powers of  $x$ . The range of the sum is a bit complicated, the reader should just keep in mind that  $Q_L$  has the above form with appropriate restrictions on the indices and on denominators of the coefficients.

**Proof.** The proof consists of a few changes of variables. It is well known that if a polynomial  $p(t_1, \dots, t_s)$  takes integer values whenever  $t_i$ 's are integers, then  $p(t_1, \dots, t_s)$  is a  $\mathbb{Z}$ -linear combination of terms of the form

$$\binom{t_1}{l_1} \cdots \binom{t_s}{l_s},$$

where  $l_1, \dots, l_s$  are non-negative integers.

Recall that  $(\alpha_i | \lambda_j) = \delta_{ij}$ . For  $\mathbf{a} = (a_1, \dots, a_{n-1}) \in (\mathbb{Z}_+)^{n-1}$ , let

$$\binom{\boldsymbol{\alpha}}{\mathbf{a}} = \prod_{i=1}^{n-1} \binom{\alpha_i}{a_i}.$$

By Proposition 2.4,  $f_\ell(\mu_1, \dots, \mu_m)$  is a polynomial function on  $\mu_1, \dots, \mu_m$  which takes integer values whenever  $\mu_j \in \Lambda$ . Hence

$$f_\ell(\mu_1, \dots, \mu_m) = \sum^{(\ell)} c'_{\mathbf{a}_1, \dots, \mathbf{a}_m; \ell} \binom{\boldsymbol{\alpha}}{\mathbf{a}_1}(\mu_1) \dots \binom{\boldsymbol{\alpha}}{\mathbf{a}_m}(\mu_m), \tag{2.6}$$

with  $c'_{\mathbf{a}_1, \dots, \mathbf{a}_m; \ell}$  in  $\mathbb{Z}$ . Here  $\sum^{(\ell)}$  means the sum is over the set of all  $\mathbf{a}_j \in (\mathbb{Z}_+)^{n-1}$  and  $\ell \in \mathbb{Z}_+$  satisfying (2.4) and (2.5). These restrictions on  $|\mathbf{a}_j|$  and  $\ell$  follow from the restriction on the degree of  $f'_\ell$  (see Proposition 2.4, here  $|\mathbf{a}_j| = s_j + n(n-1)$ ).

Recall that  $\eta_i = 1 - (1+x)^{-\alpha_i}$ . We can easily express  $\alpha_i$  in terms of  $x$  and  $\eta_i$ :

$$\alpha_i = -\frac{\ln(1-\eta_i)}{\ln(1+x)}$$

and hence

$$x\alpha_i = -\frac{x \ln(1-\eta_i)}{\ln(1+x)}. \tag{2.7}$$

Note that the right-hand side can be expressed as a formal power series in  $x$  and  $\eta_i$ ; moreover, the coefficient of a term of total degree  $d$  (in  $x$  and  $\eta_i$ ) is in  $\mathbb{Z}_{(d+1)}$ .

Using (2.6) in formula (2.2), we get

$$\begin{aligned} Q_L(\mu_1, \dots, \mu_m; q)|_{q=x+1} &= \sum_{\ell=0}^{\infty} \sum^{(\ell)} c'_{\mathbf{a}_1, \dots, \mathbf{a}_m; \ell} x^\ell \prod_{j=1}^m \binom{\boldsymbol{\alpha}}{\mathbf{a}_j}(\mu_j) \\ &= \sum_{\ell=0}^{\infty} \sum^{(\ell)} c'_{\mathbf{a}_1, \dots, \mathbf{a}_m; \ell} x^{\ell - \sum_j |\mathbf{a}_j|} \prod_{j=1}^m \binom{\boldsymbol{\alpha}}{\mathbf{a}_j}(\mu_j) x^{|\mathbf{a}_j|}. \end{aligned}$$

Using the change of variable (2.7) in the above formula, with some simple calculation, we get (2.3). The denominators of the coefficients come from the denominators of the expressions  $\binom{\boldsymbol{\alpha}}{\mathbf{a}_j}$  and from the denominators of the right-hand side of (2.7). It is easy to check that  $c_{\mathbf{a}_1, \dots, \mathbf{a}_m; \ell}$  is in  $\mathbb{Z}_{(2\ell + (m+1)n(n-1) - \sum_j |\mathbf{a}_j| + 1)}$ .  $\square$

### 2.4. Linear operator $\Gamma_b$

We now introduce the linear operator  $\Gamma_b$  which is a kind of Laplace transform. For  $\mu \in \Lambda$ , let

$$\psi(\mu; q) = \prod_{\alpha \in \Phi_+} (1 - q^{-\langle \mu | \alpha \rangle}) = \prod_{\alpha \in \Phi_+} (1 - q^{-\alpha})(\mu). \tag{2.8}$$

This function plays important role in Lie theory, it appeared in Weyl’s character formula.

Let  $\mathcal{S}$  be the free  $\mathbb{Z}$ -module generated by  $q^\beta$ , with  $\beta \in A^{\text{root}}$ . For each integer  $b$  not divisible by the odd prime  $r$ , let  $\Gamma_b$  be the linear operator

$$\Gamma_b : \mathcal{S} \rightarrow \mathbb{Z} \left[ \frac{1}{n!b} \right] [[x]], \text{ defined by}$$

$$\Gamma_b(q^\beta) = (1 + x)^{(-|\beta|^2/2b)} y_b.$$

Here  $y_b$  is an element in  $\mathbb{Z} \left[ \frac{1}{n!} \right] [[x]]$ , not depending on  $\beta$ , and is given by

$$y_b = \frac{1}{n!} (1 + x)^{(\text{sn}(b) - b)/2} \psi(-\text{sn}(b)\rho; 1 + x),$$

( $\text{sn}(b)$  is the sign of  $b$ ).

Note that  $\eta^a$  is in  $\mathcal{S}$ , hence there is defined  $\Gamma_b(\eta^a)$ . A property of  $\Gamma_b$  is stated in the following proposition, and the proof will be presented in Section 6.

**Proposition 2.6.** *The power series  $\Gamma_b(\eta^a)$  is divisible by  $x^{n(n-1)/2 + \lfloor (|a| + 1)/2 \rfloor}$ . Here  $\lfloor z \rfloor$  is the integer part of  $z$ .*

2.5. Formula for perturbative invariants  $\tau^{\text{PSU}(n)}$

We continue to assume that  $L$  has 0 linking matrix with  $m$  components. Suppose that  $L'$  is the same as  $L$ , except that the framing on components are *non-zero* integers  $b_1, \dots, b_m$ . Suppose for  $L$  we have the expansion (2.3) of  $Q_L$ .

Let  $\tau(L')$  be obtained from  $Q_L$  (i.e. the right-hand side of (2.3)) by replacing  $\eta^a(\mu_j)$  by  $\Gamma_{b_j}(\eta^a)$ :

$$\tau(L') = \sum_{c_{a_1, \dots, a_m; \ell}} \Gamma_{b_1}(\eta^{a_1}) \cdots \Gamma_{b_m}(\eta^{a_m}) x^{\ell - \sum_j |a_j|}. \tag{2.9}$$

The range of the sum is the same as in formula (2.3), i.e. the sum is over the set of all  $\mathbf{a}_j \in \mathbb{Z}_+^{n-1}$  and  $\ell \in \mathbb{Z}_+$  satisfying (2.4) and (2.5).

By Proposition 2.6, the term  $\Gamma_{b_1}(\eta^{a_1}) \cdots \Gamma_{b_m}(\eta^{a_m}) x^{\ell - \sum_j |a_j|}$  is divisible by the power of  $x$  with exponent

$$\ell - \sum_j |a_j| + m \frac{n(n-1)}{2} + \sum_j \left\lfloor \frac{|a_j| + 1}{2} \right\rfloor$$

which is greater than or equal to  $\ell - \sum_j |a_j|/2 + mn(n-1)/2$ . The latter, by (2.5), is greater than or equal to  $\frac{1}{2} \max_j (|a_j| - n(n-1))$ . For each fixed number  $a$ , there are only a finite number of  $\mathbf{a}_j \in (\mathbb{Z}_+)^{n-1}$  such that  $\frac{1}{2} \max_j (|a_j| - n(n-1)) < a$ . Hence the right-hand side of (2.9) is really a formal power series in  $x$ .

The denominators of coefficients of  $\tau(L')$  come from the factor  $n!b_j$  and the denominators of  $c_{a_1, \dots, a_m; \ell}$ . Hence, using the property of the denominators of  $c_{a_1, \dots, a_m; \ell}$  in Proposition 2.5 one can easily show that in  $\tau(L')$  the coefficient of  $x^\ell$  is in  $\mathbb{Z}[1/(2\ell + n(n-1))!b_1 \cdots b_m]$ . In other words,  $\tau(L')$  is in  $\mathbb{Q}[[x]]_M$ , where  $M$  is the 3-manifold obtained by surgery along  $L'$ .

We will show that  $\tau(L')$  is an invariant of the rational homology 3-sphere  $M$  which is obtained from  $S^3$  by surgery along  $L'$ . Later in Section 4.5 we will show that the constant term of the power series  $\tau(L')$  is  $|H_1(M, \mathbb{Z})|^{-n(n-1)/2}$ . Hence there always exists the inverse  $\tau(L')^{-1} \in \mathbb{Q}[[x]]_{(M)}$ .

Let  $M$  be an arbitrary rational homology 3-sphere. Ohtsuki showed (see [26], see also [25]) that there are lens spaces  $M(d_1), \dots, M(d_s)$  such that  $M \# M(d_1) \# \dots \# M(d_s)$  can be obtained from  $S^3$  by surgery on a link  $L'$  with diagonal linking matrix. Here  $M(d)$  is the lens space obtained by surgery along  $U_d$ , the unknot with framing  $d$ . Moreover each  $d_i$  is less than or equal to  $|H_1(M; \mathbb{Z})|$ . Define

$$\tau^{PSU(n)}(M) = \tau(L')\tau(U_{d_1})^{-1} \dots \tau(U_{d_s})^{-1}.$$

We have not proved that  $\tau^{PSU(n)}(M)$  is an invariant of  $M$  yet. So, for the time being,  $\tau^{PSU(n)}(M)$  means any formal power series obtained by the above procedure, which a priori depends on the choice of  $d_1, \dots, d_s$  and  $L'$ .

The main theorem can be formulated in a more precise form as follows.

**Theorem 2.7.** *Suppose  $M$  is rational homology 3-sphere, and  $r$  a prime number greater than  $\max\{n(n-1), |H(M, \mathbb{Z})|\}$ . Then*

$$\left(\frac{|H_1(M, \mathbb{Z})|}{r}\right)^{n-1} \mathfrak{p}_r(\tau_r^{PSU(n)}(M)) = \mathfrak{p}_r(\tau^{PSU(n)}(M)).$$

Now the uniqueness of Lemma 1.6, with this theorem, shows that  $\tau^{PSU(n)}(M)$  is really an invariant of the 3-manifold  $M$ , i.e. it does not depend on the choice of  $d_1, \dots, d_s$  and  $L'$  in the construction of  $\tau^{PSU(n)}(M)$ .

Since the series  $\Gamma_b(\boldsymbol{\eta}^a)$  has coefficients in  $\mathbb{Z}[1/n!b]$ , we see that if all the coefficients  $c_{a_1, \dots, a_m; \ell}$  are in  $\mathbb{Z}[1/n!]$ , then the power series  $\tau^{PSU(n)}(M)$  has coefficients in  $\mathbb{Z}[1/n!|H_1(M, \mathbb{Z})|]$ . Hence Conjecture 1.9 follows from the following conjecture.

**Conjecture 2.8.** *All the coefficients  $c_{a_1, \dots, a_m; \ell}$  are in  $\mathbb{Z}[1/n!]$ .*

This conjecture, for the special case  $n = 2$  and in a slightly different form, had been formulated by Rozansky. He also proved that the conjecture holds true in the special case  $n = 2$  and the link  $L$  has one component (a knot), using explicit formula of the  $R$ -matrix (see [32]).

## 2.6. Proof of the main theorem

*Case 1:*  $M$  is obtained by surgery along a framed link  $L'$  with diagonal linking matrix and  $\tau^{PSU(n)}(M) = \tau(L')$ . Let  $L$  be the same link as  $L'$  except that the framing of each component is 0. Suppose that the framing on components of  $L'$  are  $b_1, \dots, b_m$ . We will fix an odd prime number  $r$  greater than  $\max\{n(n-1), |H_1(M, \mathbb{Z})|\}$ . Let us consider expression (2.3) of  $Q_L(\mu_1, \dots, \mu_m; x+1)$ , and let  $Q_L^{(1)}$  be the part of  $Q_L$  in (2.3), where the indices satisfy

$$2\ell - \sum_j (|\mathbf{a}_j| - n(n-1)) \leq r - n(n-1). \quad (2.10)$$

In other words,

$$Q_L^{(1)}(\mu_1, \dots, \mu_m; x+1) = \sum^{(1)} c_{a_1, \dots, a_m; \ell} \boldsymbol{\eta}^{a_1}(\mu_1) \dots \boldsymbol{\eta}^{a_m}(\mu_m) x^{\ell - \sum_j |\mathbf{a}_j|}, \quad (2.11)$$

where  $\sum^{(1)}$  means the sum over the set of all  $\mathbf{a}_j \in \mathbb{Z}_+^{n-1}$  and  $\ell \in \mathbb{Z}_+$  satisfying (2.4), (2.5) and (2.10). The reason we choose the new restriction (2.10) is that, by Proposition 2.5, all the coefficients  $c_{a_1, \dots, a_m; \ell}$  in (2.11) are in  $\mathbb{Z}_{(r)}$ . Note that  $\sum^{(1)}$  is a finite sum.

**Lemma 2.9.** *There is the following splitting for  $Q_L$  (with  $q = x + 1$ ):*

$$Q_L(\mu_1, \dots, \mu_m; q) = Q_L^{(1)}(\mu_1, \dots, \mu_m; q) + Q_L^{(2)}(\mu_1, \dots, \mu_m; q) + Q_L^{(3)}(\mu_1, \dots, \mu_m; q),$$

in which

- (a)  $Q_L^{(1)}$  is defined by (2.11),
- (b)  $Q_L^{(2)}$  is a polynomial in  $(q - 1) = x$  of the form

$$Q_L^{(2)}(\mu_1, \dots, \mu_m; q) = \sum^{(2)} \tilde{c}_{a_1, \dots, a_m; \ell} \binom{\alpha}{a_1}(\mu_1) \cdots \binom{\alpha}{a_m}(\mu_m) x^\ell, \tag{2.12}$$

where  $\sum^{(2)}$  is the sum over all  $\mathbf{a}_j \in \mathbb{Z}_+^{n-1}$  and  $\ell \in \mathbb{Z}_+$  satisfying (2.4), (2.5), and, in addition,

$$2\ell - \sum_j (|a_j| - n(n - 1)) > r - n(n - 1), \tag{2.13}$$

$$\ell < N. \tag{2.14}$$

- (c)  $Q_L^{(3)}(\mu_1, \dots, \mu_m; q)$  is a polynomial in  $q, q^{-1}$  which is divisible by  $(q - 1)^N$  in  $\mathbb{Z}_{(r)}[q, q^{-1}]$ . Here  $N = m(r - n - 1)(n - 1)/2 + \bar{r} + 1$ .

Note that (2.10) and (2.13) are complementary.

**Proof of Lemma 2.9.** With the substitution  $q = x + 1$  and

$$\eta_i(\mu) = 1 - (1 + x)^{-\langle \alpha, \mu \rangle}, \tag{2.15}$$

$Q_L^{(1)}$  becomes a formal power series in  $x$ . The right-hand side of (2.15) is a power series with integer coefficients, hence the denominators of the power series  $Q_L^{(1)}$  comes from the coefficients  $c_{a_1, \dots, a_m; \ell}$  which are in  $\mathbb{Z}_{(r)}$ . Hence  $Q_L^{(1)}$  is a power series in  $x$  with coefficients in  $\mathbb{Z}_{(r)}$ . On the other hand,  $Q_L$  is a power series in  $x$  with integer coefficients. Hence, for fixed  $\mu_1, \dots, \mu_m$ ,  $Q_L - Q_L^{(1)}$  is a power series in  $x$  with coefficients in  $\mathbb{Z}_{(r)}$ . It follows that

$$Q_L - Q_L^{(1)} = \sum \tilde{c}_{a_1, \dots, a_m; \ell} \binom{\alpha}{a_1}(\mu_1) \cdots \binom{\alpha}{a_m}(\mu_m) x^\ell, \tag{2.16}$$

where the sum is over the set of all  $\mathbf{a}_j \in \mathbb{Z}_+^{n-1}$  and  $\ell \in \mathbb{Z}_+$  satisfying (2.4),(2.5) and (2.13), and the coefficients  $\tilde{c}_{a_1, \dots, a_m; \ell}$  are in  $\mathbb{Z}_{(r)}$ .

Let us define  $Q_L^{(2)}$  by using the right-hand side of the (2.16), imposing additional restriction (2.14) on the indices  $\mathbf{a}_j; \ell$ . It is easy to see that there are only a finite number of terms in  $Q_L^{(2)}$ . Finally, let  $Q_L^{(3)} = Q_L - Q_L^{(1)} - Q_L^{(2)}$ . A term of the right-hand side of (2.16), for which (2.14) is not satisfied, must be divisible by the power of  $x$  with exponent  $\ell + 1$  which is greater than or equal to  $N$ . Hence  $Q_L^{(3)}$  is divisible by  $x^N$ .

Let us return to the indeterminate  $q = x + 1$ . The advantage of using this indeterminate is that all  $Q_L, Q_L^{(1)}, Q_L^{(2)}, Q_L^{(3)}$  are polynomials in  $q, q^{-1}$  (i.e. they do not have infinite order), which can be seen as follows.

First of all,  $Q_L$  is a polynomial in  $q$  and  $q^{-1}$  with coefficients in  $\mathbb{Z}$ , by Proposition 2.1.

For fixed  $\mu_1, \dots, \mu_m$ , replacing  $x = q - 1$  in the expression of  $Q_L^{(2)}$ , we see that  $Q_L^{(2)}$  is a polynomial in  $q$  with coefficients in  $\mathbb{Z}_{(r)}$ .

Note that, for  $i = 1, \dots, n - 1$ , the number  $(\alpha_i | \mu)$  is always an integer for every  $\mu \in \mathcal{A}$ . Hence  $\eta_i(\mu) = 1 - q^{-(\alpha_i | \mu)}$  is a polynomial in  $q, q^{-1}$  with integer coefficients, and  $\eta_i$  is divisible by  $(q - 1)$  in  $\mathbb{Z}[q, q^{-1}]$ . Now formula (2.11) shows that  $Q_L^{(1)}$  is a polynomial in  $q, q^{-1}$  with coefficients in  $\mathbb{Z}_{(r)}$ .

It follows that the remaining  $Q_L^{(3)}$  is also a polynomial in  $q, q^{-1}$  with coefficients in  $\mathbb{Z}_{(r)}$ . Moreover,  $Q_L^{(3)}$  is divisible by  $(q - 1)^N$  in  $\mathbb{Z}_{(r)}[q, q^{-1}]$ :

$$Q_L^{(3)}(\mu_1, \dots, \mu_m; q) = (q - 1)^N \sum_{s=-d}^d g(\mu_1, \dots, \mu_m) q^s,$$

for some positive integer  $d$ . Here  $g(\mu_1, \dots, \mu_m)$ , possibly not a polynomial function, takes values in  $\mathbb{Z}_{(r)}$  when  $\mu_1, \dots, \mu_m \in \mathcal{A}$ . This completes the proof of the proposition.  $\square$

Recall that  $L'$  differs from  $L$  only by the framing on components. By formula (2.1):

$$L'(\mu_1, \dots, \mu_m; q) = \prod_{j=1}^m q^{b_j(|\mu_j|^2 - |\rho|^2)/2} Q_L(\mu_1, \dots, \mu_m; q),$$

and hence

$$F_{L'}(r) = \sum_{\mu_j \in (\rho + \mathcal{A}_r^{\text{root}})} \prod_{j=1}^m q^{b_j(|\mu_j|^2 - |\rho|^2)/2} Q_L(\mu_1, \dots, \mu_m; \zeta). \tag{2.17}$$

Let  $F_{L'}^{(1)}(r), F_{L'}^{(2)}(r), F_{L'}^{(3)}(r)$  be the value of the right-hand side of (2.17), if we replace  $Q_L$  by, respectively,  $Q_L^{(1)}, Q_L^{(2)}, Q_L^{(3)}$ . Then we have

$$\begin{aligned} F_{L'} &= F_{L'}^{(1)} + F_{L'}^{(2)} + F_{L'}^{(3)}, \\ \tau_r^{PSU(m)}(M) &= \frac{F_{L'}(r)}{F_{U_+}(r)^{\sigma_+} F_{U_-}(r)^{\sigma_-}} \\ &= \tau^{(1)} + \tau^{(2)} + \tau^{(3)}, \end{aligned}$$

where

$$\tau^{(j)} = \frac{F_{L'}^{(j)}(r)}{F_{U_+}(r)^{\sigma_+} F_{U_-}(r)^{\sigma_-}}.$$

The following three lemmas prove the main theorem (Theorem 2.7) in case 1.

**Lemma 2.10.** *The number  $\tau^{(3)}$  is divisible by  $(\zeta - 1)^{r+1}$  in  $\mathbb{Z}_{(r)}[\zeta]$ . Hence  $\mathfrak{p}_r(\tau^{(3)}) = 0$ .*

**Lemma 2.11.** *The number  $\tau^{(2)}$  is divisible by  $(\zeta - 1)^{r+1}$  in  $\mathbb{Z}_{(r)}[\zeta]$ . Hence  $\mathfrak{p}_r(\tau^{(2)}) = 0$ .*

**Lemma 2.12.** *One has that*

$$\left( \frac{|H_1(M, \mathbb{Z})|}{r} \right)^{n-1} \mathfrak{p}_r(\tau^{(1)}) = \mathfrak{p}_r(\tau(L')).$$

The expression  $F_{U_{\pm}}(r)$  appears in the denominators of the formulas of  $\tau^{(j)}$ . We will need the following proposition whose proof will be given in Section 4.



**Proposition 2.13.** Each of  $F_{U_+}(r)$  and  $F_{U_-}(r)$  is proportional to  $(\zeta - 1)^{(n-1)(r-n-1)/2}$  by a proportional factor which is a unit in  $\mathbb{Z}_{(1)}[\zeta] = \mathbb{Z}[1/n!][\zeta]$ .

**Proof of Lemma 2.10.** Since each  $b_j$  is non-zero,  $\sigma_+ + \sigma_- = m$ . By Proposition 2.13,  $F_{U_+}(r)^{\sigma_+} F_{U_-}(r)^{\sigma_-}$  is proportional to  $(\zeta - 1)^{m(n-1)(r-n-1)/2}$  by a unit in  $\mathbb{Z}_{(r)}[\zeta]$ . But  $F_L^{(3)}(r)$  is divisible by  $(\zeta - 1)^N$  in  $\mathbb{Z}_{(r)}[\zeta]$  by Lemma 2.9. Hence

$$\tau^{(3)} = \frac{F_L^{(3)}(r)}{F_{U_+}(r)^{\sigma_+} F_{U_-}(r)^{\sigma_-}}$$

is divisible by the power of  $\zeta - 1$  with exponent  $N - m(n - 1)(r - n - 1)/2 = \bar{r} + 1$ .  $\square$

To prove Lemma 2.11 we need the following lemma whose proof will be given in Section 4.

**Lemma 2.14.** Let  $\mathbf{a} \in (\mathbb{Z}_+)^{n-1}$  and  $b \in \mathbb{Z}_+$ . Then the number

$$\sum_{\mu \in \rho + \mathcal{A}_r^{\text{root}}} \zeta^{b(|\mu|^2 - |\rho|^2)/2} \binom{\mathbf{a}}{\mu}$$

is proportional to  $(\zeta - 1)^{(n-1)(r-1)/2 - \lfloor |a|/2 \rfloor}$  by a factor in  $\mathbb{Z}_{(r)}[\zeta]$ .

**Proof of Lemma 2.11.** Using expression (2.12) we have that

$$\begin{aligned} F_L^{(2)}(r) &= \sum_{\mu_j \in \rho + \mathcal{A}_r^{\text{root}}} \prod_{j=1}^m \zeta^{b_j(|\mu_j|^2 - |\rho|^2)/2} \sum \binom{(2)}{a_1, \dots, a_m; \ell} \binom{\mathbf{a}}{a_1}(\mu_1) \cdots \binom{\mathbf{a}}{a_m}(\mu_m) X^\ell \\ &= \sum \binom{(2)}{a_1, \dots, a_m; \ell} X^\ell \prod_{j=1}^m \sum_{\mu_j \in \rho + \mathcal{A}_r^{\text{root}}} \zeta^{b_j(|\mu_j|^2 - |\rho|^2)/2} \binom{\mathbf{a}}{a_j}(\mu_j). \end{aligned}$$

Applying Lemma 2.14, we see that  $F_L^{(2)}(r)$  is divisible, in  $\mathbb{Z}_{(r)}[\zeta]$ , by the power of  $\zeta - 1$  with the exponent

$$\ell + m \frac{(n-1)(r-1)}{2} - \sum_j \left\lfloor \frac{|a_j|}{2} \right\rfloor,$$

which, by (2.13), is greater than or equal to  $N$ . As in the proof of Lemma 2.10, it follows that  $\tau^{(2)}$  is divisible by  $(\zeta - 1)^{\bar{r}+1}$ .  $\square$

In order to prove Lemma 2.12, we will need the following proposition, whose proof will be presented in Section 4.

**Proposition 2.15.** Let

$$g_b(\mathbf{a}) = \frac{\sum_{\mu \in (\rho + \mathcal{A}_r^{\text{root}})} \zeta^{b(|\mu|^2 - |\rho|^2)/2} \boldsymbol{\eta}^{\mathbf{a}}(\mu)}{F_{U_{\text{sn}(b)}}(r)}.$$

Then

$$\left(\frac{|b|}{r}\right)^{n-1} \mathfrak{p}_r(g_b(\mathbf{a})) = \mathfrak{p}_r(\Gamma_b(\boldsymbol{\eta}^{\mathbf{a}})).$$

**Proof of Lemma 2.12.** Recall that

$$Q_L^{(1)} = \sum^{(1)} c_{a_1, \dots, a_m; \ell} \boldsymbol{\eta}^{a_1}(\mu_1) \cdots \boldsymbol{\eta}^{a_m}(\mu_m) (q-1)^{\ell - \sum_j |a_j|}.$$

Here  $c_{a_1, \dots, a_m; \ell} \in \mathbb{Z}(r)$ . Hence

$$F_{L'}^{(1)}(r) = \sum^{(1)} c_{a_1, \dots, a_m; \ell} (\zeta - 1)^{\ell - \sum_j |a_j|} \prod_{j=1}^m \sum_{\mu_j \in (\rho + A_r^{\text{root}})} \zeta^{b_j(|\mu_j|^2 - |\rho|^2)/2} \boldsymbol{\eta}^{a_j}(\mu_j),$$

and

$$\tau^{(1)} = \frac{F_{L'}^{(1)}(r)}{F_{U_+}(r)^{\sigma_+} F_{U_-}(r)^{\sigma_-}} = \sum^{(1)} c_{a_1, \dots, a_m; \ell} (\zeta - 1)^{\ell - \sum_j |a_j|} \prod_{j=1}^m g_{b_j}(\mathbf{a}_j).$$

By Proposition 2.15,  $(|b|/r)^{n-1} \mathfrak{p}_r(g_b(\mathbf{a})) = \mathfrak{p}_r(\Gamma_b(\boldsymbol{\eta}^{\mathbf{a}}))$ . It follows that

$$\left( \frac{|b_1 \cdots b_m|}{r} \right)^{n-1} \mathfrak{p}_r(\tau^{(1)}) = \mathfrak{p}_r \left( \sum^{(1)} c_{a_1, \dots, a_m; \ell} X^{\ell - \sum_j |a_j|} \prod_{j=1}^m \Gamma_{b_j}(\boldsymbol{\eta}^{\mathbf{a}_j}) \right).$$

The argument of  $\mathfrak{p}_r$  on the right-hand side of this formula, by definition, is a part of  $\tau(L')$  (see formula (2.9) for the definition of  $\tau(L')$ ). This part differs from  $\tau(L')$  only by elements of degree  $> \bar{r}$ , which are annihilated by  $\mathfrak{p}_r$ , hence

$$\left( \frac{|b_1 \cdots b_m|}{r} \right)^{n-1} \mathfrak{p}_r(\tau^{(1)}) = \mathfrak{p}_r(\tau(L')).$$

This completes the proof of Lemma 2.12, and hence that of the main theorem, case 1.

*Case 2:*  $M$  is an arbitrary rational homology 3-sphere,  $M' = M \# M(d_1) \# \cdots \# M(d_s)$  is obtained by surgery along a link  $L'$  with diagonal linking matrix, and

$$\tau^{PSU(n)}(M) = \tau(L') \tau(U_{d_1})^{-1} \cdots \tau(U_{d_s})^{-1}.$$

The multiplicative property of quantum invariants says that

$$\tau_r^{PSU(n)}(M') = \tau_r^{PSU(n)}(M) \tau_r^{PSU(n)}(M(d_1)) \cdots \tau_r^{PSU(n)}(M(d_s)). \quad (2.18)$$

Applying the result of case 1 to  $L'$  and  $U_{d_1}, \dots, U_{d_s}$ , and using (2.18) we see that  $(|H_1(M, \mathbb{Z})|/r)^{n-1} \mathfrak{p}_r(\tau_r^{PSU(n)}(M)) = \mathfrak{p}_r(\tau^{PSU(n)}(M))$ . This completes the proof of the main theorem.  $\square$

### 3. Quantum invariants of links and the Kontsevich integral

In this section we always assume that  $L$  is a framed oriented link of  $m$  components, and that  $r$  is a positive integer. We will investigate the dependence of the quantum invariant  $J_L(\mu_1, \dots, \mu_m; q)$  on the  $\mu_j$ 's and prove Propositions 1.3, 2.1, and 2.4.

#### 3.1. Some general facts about quantum invariants

Recall that  $J_L(\mu_1, \dots, \mu_m; q)$ , or  $J_L(V_{\mu_1}, \dots, V_{\mu_m}; q)$  is the quantum invariant of  $L$ , when the components of  $L$  are colored by the  $sl_n$ -modules  $V_{\mu_1}, \dots, V_{\mu_m}$ . Here  $V_{\mu}$  is the irreducible module of highest weight  $\mu - \rho$ , for  $\mu \in \Lambda_{++}$ .

Recall that for  $\mu \in \Lambda$ ,

$$\psi(\mu; q) = \prod_{\alpha \in \Phi_+} (1 - q^{-\langle \mu, \alpha \rangle}) = \prod_{\alpha \in \Phi_+} (1 - q^{-\alpha})(\mu).$$

From the Weyl denominator formula (see, for example, [6]), we have the following.

**Lemma 3.1.** *One has that  $\psi(\mu; q) = \sum_{w \in W} \text{sn}(w) q^{\langle \mu | w(\rho) - \rho \rangle}$ .*

The value of the quantum invariant  $J_L$  of the unknot  $U$  is (see, for example, [5]):

$$J_U(\mu; q) = q^{\langle \mu - \rho | \rho \rangle} \frac{\psi(\mu; q)}{\psi(\rho; q)}. \tag{3.1}$$

Note that  $\langle \mu | \alpha \rangle \in \mathbb{Z}$  for  $\mu \in \Lambda$  and  $\alpha \in \Lambda^{\text{root}}$ . Moreover, it is known that  $2\rho \in \Lambda^{\text{root}}$ , and hence  $\langle \mu | \rho \rangle \in \frac{1}{2}\mathbb{Z}$ . It follows that  $J_U(\mu; q)$  is either in  $q^{1/2}\mathbb{Z}[q, q^{-1}]$  or in  $\mathbb{Z}[q, q^{-1}]$ .

Let  $L^{(2)}$  be the framed link obtained from  $L$  by replacing the first component by two of its parallel pushoffs (using the frame). Suppose that the color of these 2 push-offs are modules  $V$  and  $V'$ . Then we have the following formula [35]:

$$J_{L^{(2)}}(V, V', \dots; q) = J_L(V \otimes V', \dots; q).$$

Hence for  $Q_L$  we have a similar result:

$$Q_{L^{(2)}}(V, V', \dots; q) = Q_L(V \otimes V', \dots; q). \tag{3.2}$$

In particular, when  $L = U$ , we have that

$$J_U(V \otimes V'; q) = J_{U^{(2)}}(V, V'; q) = J_U(V; q)J_U(V'; q).$$

This means

$$J_U(\cdot; q): \mathcal{R} \rightarrow \mathbb{Z}[q^{\pm 1/2n}]$$

is a ring homomorphism. Recall that  $\mathcal{R}$  is the ring of finite-dimensional  $sl_n$ -modules. Let  $I_r$  be the ideal in  $\mathcal{R}$  generated by all the  $V_\mu$ , with  $\langle \mu | \theta \rangle = r$  (here  $\mu \in \Lambda_{++}$ ). Recall that  $\theta = \alpha_1 + \dots + \alpha_{n-1}$  is the longest root. The quotient ring  $\mathcal{R}/I_r$  is known as the fusion algebra (see [2,8] and reference therein).

**Lemma 3.2.** *If  $\mu$  is in  $I_r$ , then  $J_U(\mu; \zeta) = 0$ .*

**Proof.** Since  $J_U(\cdot; \zeta): \mathcal{R} \rightarrow \mathbb{C}$  is a ring homomorphism, it suffices to show that  $J_U(\mu; \zeta) = 0$  if  $\langle \mu | \theta \rangle = r$ . We have that

$$1 - \zeta^{-\langle \mu | \theta \rangle} = 1 - \zeta^{-r} = 0.$$

Since  $\theta$  is one of the positive roots, it follows from the definition of  $\psi$  that  $\psi(\mu; \zeta) = 0$ , and hence  $J_U(\mu; \zeta) = 0$ .  $\square$

**Proposition 3.3.** *Suppose that one of  $\mu_1, \dots, \mu_m$  is in  $I_r$ . Then  $J_L(\mu_1, \dots, \mu_m; \zeta) = 0$ .*

**Proof.** A proof for the case  $n = 2$  is given in [8, Lemma 3.29]. Suppose  $\mu_1 \in I_r$ . The argument of Lemma 3.29 of [8] shows that  $J_L(\mu_1, \dots, \mu_m; \zeta)$  is proportional to  $J_U(\mu_1; \zeta)$  which is 0. Hence  $J_L(\mu_1, \dots, \mu_m; \zeta) = 0$ .  $\square$

**Remark.** This proposition also follows from the theory of the Kontsevich integral, more precisely, from Proposition 3.6 and formula (3.8) below.

### 3.2. Proof of Proposition 1.3

We will fix  $\mu_2, \dots, \mu_m$  and will write  $J_L(\mu_1; q)$  instead of  $J_L(\mu_1, \mu_2, \dots, \mu_m; q)$ .

Proposition 3.3 shows that  $J_L(\mu_1; \zeta) = J_L(\mu'_1; \zeta)$  if  $\mu_1 \equiv \mu'_1 \pmod{I_r}$ .

From the theory of fusion algebra [2] it is known that for every  $w \in \mathcal{W}_{(r)}$ ,

$$V_\mu \equiv \text{sn}(w)V_{w(\mu)} \pmod{I_r}.$$

Hence

$$J_L(\mu; \zeta) = \text{sn}(w)J_L(w(\mu); \zeta).$$

Put  $L = U$  we get

$$J_U(\mu; \zeta) = \text{sn}(w)J_U(w(\mu); \zeta).$$

Take the product of the last two identities, we get

$$Q_L(\mu; \zeta) = Q_L(w(\mu); \zeta),$$

which proves part (a) of Proposition 1.3. Part (b) follows from (a) and Proposition 3.3.

### 3.3. Proof of Proposition 2.1

We have to show that  $Q_L(\mu_1, \dots, \mu_m; q) \in \mathbb{Z}[q^{\pm 1}]$  if the linking matrix of  $L$  is 0. It is enough to consider the case when  $\mu_j \in A_{++}$ , since  $Q_L$  is invariant under the action of the Weyl group and equal to 0 if one of the  $\mu_j$  is on the boundary of the fundamental chamber  $C$ .

The invariant  $Q_L$  satisfies the doubling formula (3.2), and the representation ring  $\mathcal{R}$  is generated (as algebra over  $\mathbb{Z}$ ) by  $V_{\lambda_1 + \rho}, \dots, V_{\lambda_{n-1} + \rho}$ . Hence it suffices to prove that  $Q_L(\mu_1, \dots, \mu_m; q) \in \mathbb{Z}[q^{\pm 1}]$  for the case when each  $\mu_j$  is in  $\{\lambda_1 + \rho, \dots, \lambda_{n-1} + \rho\}$ . This fact follows immediately from the following lemma.

**Lemma 3.4.** *Suppose that the linking matrix of  $L$  is 0 and  $\mu_j = \lambda_{k_j} + \rho$ , with  $j = 1, \dots, m$ . Then  $J_L(\mu_1, \dots, \mu_m; q)$  is in  $q^{(k_{j_1} + \dots + k_{j_m})(n-1)/2} \mathbb{Z}[q, q^{-1}]$ .*

**Remark.** Without the assumption that the linking matrix is 0, the lemma does not hold true.

**Proof of Lemma 3.4.** For any framed oriented link  $K$  let us define  $J_K^{\text{Hom}}(q)$  as follows. First let  $K'$  be any link obtained from  $K$  by changing the framings so that  $K' \cdot K' = 0$ , i.e. the sum of all the entries of the linking matrix of  $K'$  is 0. Then let  $J_K^{\text{Hom}}(q) = J_{K'}^{\text{Hom}}(\mu_1, \dots, \mu_m; q)$  with all the  $\mu_j$  equal to  $\lambda_1 + \rho$  (i.e. when each  $V_{\mu_j}$  is the fundamental representation of  $sl_n$ ). This is a non-framed version of

quantum invariants. For the link  $K$  with 0 linking matrix we choose  $K' = K$ . It is known that  $J^{Hom}$  is a version of the Homfly polynomial and can be calculated by the skein theory as follows.

$$J_{L_1 \sqcup L_2}^{Hom}(q) = J_{L_1}^{Hom}(q)J_{L_2}^{Hom}(q),$$

$$q^{n/2}J_{L_+}^{Hom} - q^{-n/2}J_{L_-}^{Hom} = (q^{1/2} - q^{-1/2})J_{L_0}^{Hom},$$

where  $(L_0, L_-, L_+)$  is the standard skein triple. By using the skein relation and induction on the number of crossing points of the link diagram, one easily proves

**Lemma 3.5.** *If  $L$  has  $m$  components, then  $J_L^{Hom}(q) \in q^{m(n-1)/2} \mathbb{Z}[q^{\pm 1}]$ .*

This shows that Lemma 3.4 holds true when all the  $\mu_j$ 's are equal to  $\lambda_1 + \rho$ .

The case when all the  $\mu_j$ 's are in the set  $\{\lambda_1 + \rho, \dots, \lambda_{n-1} + \rho\}$  can be reduced to the case when all  $\mu_j$  are equal to  $\lambda_1 + \rho$  by using cabling as follows. (See similar arguments in Appendix A of [23]).

Let  $B_k$  be the braid group on  $k$  strands. Let  $\mathbb{Z}(q^{1/2})[B_k]$  be the set of linear combinations of elements in  $B_k$  with coefficients in  $\mathbb{Z}(q^{1/2})$ , the ring of rational functions in  $q^{1/2}$  with integer coefficients. The anti-symmetrizer  $g^{(k)} \in \mathbb{Z}(q^{1/2})[B_k]$  is defined, for  $k = 1, \dots, n - 1$ , by induction as follows [38]:  $g^{(1)} = 1 \in B_1$ ,

$$g^{(k)} = \frac{1 - q^{-1}}{1 - q^{-k}}(g^{(k-1)} \otimes 1) - q^{(n-1)/2} \frac{1 - q^{1-k}}{1 - q^{-k}}(g^{(k-1)} \otimes 1)\sigma_{k-1}(g^{(k-1)} \otimes 1), \tag{3.3}$$

where  $\sigma_1, \dots, \sigma_{k-1}$  are the standard generators of the braid group  $B_k$ , and  $z \otimes 1$  is obtained from  $z$  by adding a vertical strand to the right of the braid  $z$ .

Now suppose  $\mu_j = \lambda_{k_j} + \rho$ , where  $1 \leq k_j \leq n - 1$ . Then [38,23]

$$J_L(\mu_1, \dots, \mu_m; q) = J_{\mathcal{L}}(\rho + \lambda_1, \dots, \rho + \lambda_1; q), \tag{3.4}$$

where  $\mathcal{L}$  is a linear combination of links with coefficients in  $\mathbb{Z}(q^{1/2})$  obtained as follows. First we replace the  $j$ th component of  $L$  by  $k_j$  of its parallel push-offs, then we cut these push-offs at one place and glue in the anti-symmetrizer element  $g^{(k_j)}$ . One gets a linear combination of links. Modify the links in this linear combination by changing all the framings to 0. The result is  $\mathcal{L}$ .

Note that since  $L$  has 0 linking matrix, the right-hand side of (3.4) is equal to  $J_{\mathcal{L}}^{Hom}(q)$ .

**Claim.**  $g^{(k)} = \sum_z f_z(q)z$ , where the sum is over a finite subset of the braid group  $B_k$ , and  $f_z(q)$  is a rational function in  $q^{(1 - \text{sn}(z))(n-1)/4} \mathbb{Z}(q)$ .

Here  $\text{sn}(z)$  is the sign of the permutation corresponding to the braid  $z$ . Note that  $1 - \text{sn}(z)$  is always even.

The claim can be proved easily by induction on  $k$ , using (3.3) and noting that  $\text{sn}(zz') = \text{sn}(z)\text{sn}(z')$ .

For a braid  $z \in B_k$  let  $c(z)$  be the number of cycles in the permutation corresponding to  $z$ . The closure of  $z$  is a link of  $c(z)$  components.

It is easy to see that, for  $z \in B_k$ ,

$$q^{(1 - \text{sn}(z))(n-1)/4} \mathbb{Z}(q) = q^{(k - c(z))(n-1)/2} \mathbb{Z}(q). \tag{3.5}$$

Hence the right-hand side of (3.4), by Lemma 3.5 and equality (3.5), must belong to  $q^{(k_{j_1} + \dots + k_{j_m})(n-1)/2} \mathbb{Z}(q)$ . On the other hand,  $J_L(\mu_1, \dots, \mu_m; q)$  is a polynomial in  $q^{1/2n}, q^{-1/2n}$ . It follows that  $J_L(\mu_1, \dots, \mu_m; q)$  is in  $q^{(k_{j_1} + \dots + k_{j_m})(n-1)/2} \mathbb{Z}[q, q^{-1}]$ .

### 3.4. The Kontsevich integral and weight systems

The Kontsevich integral is a very powerful invariant of links which was found by Kontsevich [11]. We will use here the version for *framed* oriented links introduced in [17,18] (with exactly the same normalization). In [18] the invariant was denoted by  $\hat{Z}_f(L)$ , but here for simplicity we will use the notation  $Z(L)$ . This invariant  $Z(L)$  takes values in the (completed) graded vector space  $\mathcal{A}(\bigsqcup^m S^1)$  of *web diagrams* (also known as Chinese character diagrams) on  $m$  circles, i.e. the support of the web diagrams is  $m$  circles. Here  $m$  is the number of components of  $L$ . In brief, a *web diagram* consists of a 1-dimensional compact oriented manifold  $X$  and a graph, every vertex of which is either univalent or trivalent. The components of  $X$  are supposed to be ordered; and a cyclic order at every trivalent vertex of the graph is fixed. All the univalent vertices of the graph must be on the manifold  $X$ . Usually the components of  $X$  are drawn using solid line, while the components of the graph is drawn by dashed lines. So we sometimes refer to components of  $X$  as solid components, and components of the graph as dashed components. A univalent (resp. trivalent) vertex of the graph is also called an external (resp. internal) vertex of the web diagram. The degree of a web diagram is half the number of (internal and external) vertices. The space of web diagrams on  $X$  is the vector space generated by web whose solid part is  $X$ , subject to the antisymmetry and Jacobi (also known as IHX) relations.

On the Lie algebra  $sl_n$  there is the standard invariant bilinear form defined by  $(y|z) = tr(yz)$ , where  $tr$  is the trace in the fundamental representation.

Suppose  $D$  is a web diagram on  $m$  solid circles. Using the above-defined bilinear form, we can define the weight  $W_D(V_1, \dots, V_m)$ , which is a number, of the web diagram  $D$  when the  $m$  components of  $D$  are colored by  $sl_n$ -modules  $V_1, \dots, V_m$  (see, for example, [11,18]). In fact,  $W_D$  can be regarded as a multi-linear map:

$$W_D: \mathcal{R}^{\otimes m} \rightarrow \mathbb{Q}.$$

For  $\mu_1, \dots, \mu_m \in \Lambda_{++}$ , let  $W_D(\mu_1, \dots, \mu_m)$  stand for  $W_D(V_{\mu_1}, \dots, V_{\mu_m})$ .

If  $Y$  is a linear combination of chord diagrams on  $m$  solid circles, then we can define  $W_Y$  using linearity. The relation between quantum invariants of framed oriented links and the Kontsevich integral is expressed in the following proposition.

**Proposition 3.6.** *For  $\mu_1, \dots, \mu_m \in \Lambda_{++}$ , one has that*

$$J_L(\mu_1, \dots, \mu_m; q)|_{q=e^h} = \sum_{\ell=0}^{\infty} W_{Z_\ell(L)}(\mu_1, \dots, \mu_m) h^\ell. \quad (3.6)$$

Here  $Z_\ell$  is the degree  $\ell$  part of the Kontsevich integral  $Z$ .

This is a simple corollary of Drinfeld's theory of quasi-Hopf algebra and was observed by Le and Murakami [18] and Kassel [7]. This shows that all quantum invariants are special values of the Kontsevich integral.

The following is an easy exercise in the theory of the Kontsevich integral and we omit the proof (see similar results in [15], Lemma 5.6; [16], Proposition 5.3]).

**Lemma 3.7.** *Suppose that  $L$  has 0 linking matrix. Then the degree  $\ell$  part  $Z_\ell$  of the Kontsevich integral can be represented in the form*

$$Z_\ell(L) = \sum_D a_D D, \quad \text{with } a_D \in \mathbb{Q}$$

where the sum is over the set of web diagrams  $D$  which has at most  $3\ell/2$  external vertices. Moreover, if  $s_j$  is the number of external vertices of  $D$  on the  $j$ th component, then  $\max_j s_j \leq \ell - \sum_j s_j$ .

### 3.5. Dependence of $W_D$ on $\mu_1, \dots, \mu_m$

Suppose that  $D$  is a web diagram on  $m$  circles. At first we will fix  $\mu_2, \dots, \mu_m$  and investigate the dependence of  $W_D$  on  $\mu_1$ , we will write  $W_D(\mu_1)$  instead of  $W_D(\mu_1, \dots, \mu_m)$ .

Break the first component of  $D$  at an arbitrary point which is not an external vertex. The result is a web diagram  $D'$  whose support (i.e. the solid part) is the union of an interval and  $m - 1$  circles. It is known that  $D'$ , modulo the anti-symmetry and Jacobi relations, depends only on  $D$ , but not on the point where we break the first component (see [18, Section 1]).

Assign  $V_{\mu_2}, \dots, V_{\mu_m}$  to the circle components of  $D'$ , and take the weight of  $D'$  (see, for example, [7,18]). The result is not a number, but an element  $z(D')$  which lies in the center of the universal enveloping algebra  $U(\mathfrak{sl}_n)$  of  $\mathfrak{sl}_n$  (see, for example, [7, Proposition XX.8.2]). Since  $D'$  is determined by  $D$ , we will write  $z(D)$  for  $z(D')$ . By the definition of the weight, for every  $\mu_1 \in \Lambda_{++}$ , we have that

$$W_D(\mu_1) = \text{tr}_{\mu_1} z(D), \tag{3.7}$$

where  $\text{tr}_{\mu_1}$  is the trace taken in the representation  $V_{\mu_1}$ .

Since  $z(D)$  is a central element, and  $V_{\mu_1}$  is an irreducible  $\mathfrak{sl}_n$ -module,  $z(D)$  acts on  $V_{\mu_1}$  as a scalar times the identity. The scalar, denoted by  $\xi_D(\mu_1)$ , is a function on  $\mu_1$ , which is known as a *character* in Lie theory. Harish-Chandra theory (see, for example, [3]) says that  $\xi_D(\mu_1)$  is a polynomial function on  $\mu_1$  of degree not exceeding the degree of  $z(D)$  in  $U(\mathfrak{sl}_n)$ . Moreover the polynomial function  $\xi_D(\mu_1)$  is invariant under the action of the Weyl group.

Formula (3.7) now shows that

$$W_D(\mu_1) = \dim(V_{\mu_1}) \xi_D(\mu_1), \tag{3.8}$$

where the dimension of  $V_{\mu_1}$ , by the Weyl formula, is equal to  $\Delta(\mu_1)$  (see Lemma 2.3).

From the definition of the weight, it follows that if  $D$  has  $k$  external vertices on the first component, then  $z(D)$  has degree  $\leq k$ , and hence  $\xi_D(\mu_1)$  is a polynomial function of degree at most  $k$ .

Now we return to the general case, when the other variables  $\mu_2, \dots, \mu_m$  are not fixed. Since all the variables  $\mu_1, \dots, \mu_m$  are independent, we get the following proposition.

**Proposition 3.8.** *For every  $\mu_1, \dots, \mu_m \in \Lambda_{++}$ , one has that*

$$W_D(\mu_1, \dots, \mu_m) = \Delta(\mu_1) \dots \Delta(\mu_m) p_D(\mu_1, \dots, \mu_m), \tag{3.9}$$

where  $p_D(\mu_1, \dots, \mu_m)$  is a polynomial function invariant under the action of the Weyl group on each variable. Moreover, the total degree of  $p_D$  is less than or equal to the number of external vertices of  $D$ , and the degree of  $\mu_j$  in  $p_D$  is less than or equal to the number of external vertices on the  $j$ th component.

### 3.6. Proof of Proposition 2.4

From Proposition 3.8 and Lemma 3.7 we see that, for every  $\mu_1, \dots, \mu_m \in \Lambda_{++}$ ,

$$W_{Z_\ell(L)}(\mu_1, \dots, \mu_m) = \Delta(\mu_1) \cdots \Delta(\mu_m) p_\ell(\mu_1, \dots, \mu_m),$$

where  $p_\ell(\mu_1, \dots, \mu_m)$  is a polynomial function of degree  $\leq 3\ell/2$ , and if  $s_j$  is the degree of  $\mu_j$  in  $p_\ell$ , then  $\max_j s_j \leq 2\ell - \sum_j s_j$ .

Applying Proposition 3.6, we get

$$J_L(\mu_1, \dots, \mu_m; q)|_{q=e^h} = \Delta(\mu_1) \cdots \Delta(\mu_m) \sum_{\ell=0}^{\infty} p_\ell(\mu_1, \dots, \mu_m) h^\ell. \tag{3.10}$$

This is true for  $\mu_1, \dots, \mu_m \in \Lambda_{++}$ . We will show that (3.10) holds true for  $\mu_1, \dots, \mu_m \in \Lambda$ .

Suppose one of the  $\mu_j$ , say  $\mu_1$ , is on the boundary of the fundamental chamber  $C$ . Then the left-hand side is 0 by definition, while  $\Delta(\mu_1) = 0$  by Lemma 2.3. Hence (3.10) also holds true in this case.

If we replace  $\mu_1$  by  $w(\mu_1)$ , where  $w$  is an element of the Weyl group, then the left-hand side of (3.10) must be multiplied by  $\text{sn}(w)$ . On the right-hand side, all the functions  $p_\ell$ 's are invariant under the action of the Weyl group, while  $\Delta(w(\mu_1)) = \text{sn}(w)\Delta(\mu_1)$  (see Lemma 2.3). Hence if (3.10) is true for  $\mu_1$ , then it holds true for  $w(\mu_1)$ . Similarly for other  $\mu_j$ . It follows that (3.10) holds true for every  $\mu_1, \dots, \mu_m \in \Lambda$ .

Now substituting  $h = \ln(1 + x)$ , we get

$$\begin{aligned} Q_L(\mu_1, \dots, \mu_m; q)|_{q=x+1} &= J_L(\mu_1, \dots, \mu_m; q)|_{q=x+1} J_{U^{\text{sm}}}(\mu_1, \dots, \mu_m; q)|_{q=x+1} \\ &= \Delta^2(\mu_1) \cdots \Delta^2(\mu_m) \sum_{\ell=0}^{\infty} f'_\ell(\mu_1, \dots, \mu_m) x^\ell, \end{aligned} \tag{3.11}$$

for some polynomial functions  $f'_\ell(\mu_1, \dots, \mu_m)$ .

Note that  $h$ , as a formal power series in  $x$ , has the first non-trivial term  $x$ . Hence each  $f'_\ell$  is a polynomial function of total degree  $\leq 3\ell/2$ , and if  $s_j$  is the degree of  $\mu_j$  in  $f'_\ell$ , then  $s_j \leq 2\ell - \sum_j s_j$ .

That  $f(\mu_1, \dots, \mu_m) \in \mathbb{Z}$  when  $\mu_1, \dots, \mu_m \in \Lambda$  follows from the fact that  $Q_L(\mu_1, \dots, \mu_m)$  is a Laurent polynomial in  $q$  with integer coefficients. This completes the proof of Proposition 2.4.

## 4. Sums related to the root lattice

We will always assume that  $r$  is an odd prime, and  $\zeta = e^{2\pi i/r}$ . Recall that  $\rho$  is the half-sum of positive roots, and  $|\rho|^2 = n(n^2 - 1)/12$  is always in  $\frac{1}{2}\mathbb{Z}$ . If  $\beta$  is in the root lattice, then  $|\beta|^2$  is an even integer.



4.1. Gauss sum on the root lattice

For an integer  $b$  not divisible by  $r$  let

$$\mathcal{G}(b) = \sum_{\mu \in (\rho + A_r^{\text{root}})} \zeta^{b(|\mu|^2 - |\rho|^2)/2}.$$

Note that for every  $\mu \in (\rho + A_r^{\text{root}})$ , one has that  $(|\mu|^2 - |\rho|^2)/2 \in \mathbb{Z}$ , hence  $\mathcal{G}(b) \in \mathbb{Z}[\zeta]$ .

The fact that  $\zeta$  is an  $r$ th root of unity has the following consequence.

**Proposition 4.1.** *For every  $\beta$  in the root lattice  $A_r^{\text{root}}$ , one has that*

$$\sum_{\mu \in (\rho + A_r^{\text{root}})} \zeta^{b(|\mu + \beta|^2 - |\rho|^2)/2} = \mathcal{G}(b). \tag{4.1}$$

**Proof.** For  $i = 1, \dots, n - 1$ , we have

$$|\mu + r\alpha_i|^2 - |\rho|^2 = 2r(\mu | \alpha_i) + r^2|\alpha_i|^2.$$

Since  $|\alpha_i|^2 = 2$  is an even number and  $(\mu | \alpha_i) \in \mathbb{Z}$  for every  $\mu \in A$ , the right-hand side is divisible by  $2r$ . Hence

$$\zeta^{b(|\mu + r\alpha_i|^2 - |\rho|^2)/2} = \zeta^{b(|\mu|^2 - |\rho|^2)/2}. \tag{4.2}$$

Let us denote the left-hand side of (4.1) by  $\mathcal{G}(b; \beta)$ . It is enough to show that  $\mathcal{G}(b; \beta + \alpha_i) = \mathcal{G}(b; \beta)$  for every  $\beta \in A_r^{\text{root}}$  and  $i = 1, \dots, n - 1$ . We have

$$\sum_{\mu \in (\rho + A_r^{\text{root}})} \zeta^{b(|\mu + \beta|^2 - |\rho|^2)/2} = \sum_{\mu \in (\beta + \rho + A_r^{\text{root}})} \zeta^{b(|\mu|^2 - |\rho|^2)/2}.$$

By definition,  $\beta + \rho + A_r^{\text{root}}$  and  $\beta + \alpha_i + \rho + A_r^{\text{root}}$  have the same number of elements, and  $\mu$  belongs to the first set if and only if either  $\mu$  or  $\mu + r\alpha_i$  belongs to the second set (see the definition of  $A_r^{\text{root}}$  in Section 1.4). Hence, Eq. (4.2) shows that

$$\sum_{\mu \in (\beta + \rho + A_r^{\text{root}})} \zeta^{b(|\mu|^2 - |\rho|^2)/2} = \sum_{\mu \in (\beta + \alpha_i + \rho + A_r^{\text{root}})} \zeta^{b(|\mu|^2 - |\rho|^2)/2},$$

or  $\mathcal{G}(b; \beta) = \mathcal{G}(b; \beta + \alpha_i)$ .  $\square$

The exact value of  $\mathcal{G}(b)$  is given in the next proposition. For a rational number  $a/b$ , with  $a, b \in \mathbb{Z}$  and  $b$  not divisible by  $r$ , let  $(a/b)^\vee$  be the natural image of  $a/b$  in  $\mathbb{Z}/r\mathbb{Z}$ .

**Proposition 4.2.** *Suppose that  $b$  is not divisible by the prime  $r$  and  $n < r$ . Then  $\mathcal{G}(b)$  has absolute value  $r^{(n-1)/2}$ , and its phase is given by*

$$\frac{\mathcal{G}(b)}{r^{(n-1)/2}} = \left(\frac{n}{r}\right) \left[ \left(\frac{b}{r}\right) e^{\pi i(1-r)/4} \right]^{n-1} \zeta^{(-b|\rho|^2/2)^\vee}. \tag{4.3}$$

Here  $\binom{n}{r}$  and  $\binom{b}{r}$  are the Legendre symbols. In particular, (here  $\text{sn}(b)$  is the sign of  $b$ )

$$\frac{\mathcal{G}(b)}{\mathcal{G}(\text{sn}(b))} = \left(\frac{|b|}{r}\right)^{n-1} \zeta^{\binom{\text{sn}(b)-b}{2}|\rho|^2} \quad (4.4)$$

**Proof.** Recall that  $|\rho|^2 = n(n^2 - 1)/12$ . Note that  $r + 1$  is an even number, and hence  $|\rho|^2(r + 1)^2/2$  is an integer. Then we have

$$\left(-\frac{b|\rho|^2}{2}\right)^\vee \equiv -b|\rho|^2(r + 1)^2/2 \pmod{r}. \quad (4.5)$$

Direct calculation shows that

$$\frac{|\mu + \rho|^2 - |\rho|^2}{2} - \frac{|\mu + (r + 1)\rho|^2 - (r + 1)^2|\rho|^2}{2} = -r(\mu|\rho).$$

For  $\mu \in \Lambda^{\text{root}}$ ,  $(\mu|\rho)$  is in  $\mathbb{Z}$ , and hence the right-hand side is divisible by  $r$ . It follows that

$$\zeta^{b(|\mu + \rho|^2 - |\rho|^2)/2} = \zeta^{b(|\mu + (r + 1)\rho|^2 - (r + 1)^2|\rho|^2)/2} = \zeta^{b(|\mu + (r + 1)\rho|^2)/2} \zeta^{-(b|\rho|^2)/2} \quad (4.6)$$

where the second identity follows from (4.5). Now we have

$$\begin{aligned} \mathcal{G}(b) &= \sum_{\mu \in (\rho + \Lambda_r^{\text{root}})} \zeta^{b(|\mu|^2 - |\rho|^2)/2} \\ &= \sum_{\mu \in \Lambda^{\text{root}}} \zeta^{b(|\mu + \rho|^2 - |\rho|^2)/2} \\ &= \zeta^{(-b|\rho|^2/2)^\vee} \sum_{\mu \in \Lambda^{\text{root}}} \zeta^{b|\mu + (r + 1)\rho|^2/2} \\ &= \zeta^{(-b|\rho|^2/2)^\vee} \sum_{\mu \in \Lambda^{\text{root}}} \zeta^{b|\mu|^2/2} \end{aligned} \quad (4.7)$$

where the third identity follows from (4.6) and the fourth from Proposition 4.1, noting that  $(r + 1)\rho$  is in the root lattice (since  $r + 1$  is even).

Recall that  $(\alpha_i|\alpha_j) = A_{ij}$ , where  $A$  is the Cartan matrix. If  $\mu = k_1\alpha_1 + \dots + k_{n-1}\alpha_{n-1}$ , then  $|\mu|^2 = \mathbf{k}^t A \mathbf{k}$ , where  $\mathbf{k}$  is the column vector transpose to  $\mathbf{k}^t = (k_1, \dots, k_{n-1})$ . Hence

$$\sum_{\mu \in \Lambda_r^{\text{root}}} \zeta^{b|\mu|^2/2} = \sum_{\mathbf{k} \in (\mathbb{Z}/r\mathbb{Z})^{n-1}} \zeta^{b \mathbf{k}^t A \mathbf{k}/2}.$$

The right-hand side is a multi-variable Gauss sum, and its value is

$$\sum_{\mathbf{k} \in (\mathbb{Z}/r\mathbb{Z})^{n-1}} \zeta^{b \mathbf{k}^t A \mathbf{k}/2} = \left(\frac{n}{r}\right) \left[\left(\frac{b}{r}\right) e^{\pi i(1-r)/4} \sqrt{r}\right]^{n-1}. \quad (4.8)$$

(A proof of this formula is given in Appendix). This formula, together with (4.7), proves the proposition.  $\square$

**Corollary 4.3.** *The number  $\mathcal{G}(b)$  is proportional to  $(\zeta - 1)^{(n-1)(r-1)/2}$  by a unit in  $\mathbb{Z}[\zeta]$ .*

**Proof.** Note that  $\zeta$  is invertible in  $\mathbb{Z}[\zeta]$ , since  $\zeta^{-1} = \zeta^{r-1}$ . It is well-known that  $e^{(\pi i(1-r)/4)}\sqrt{r}$  is proportional to  $(\zeta - 1)^{(r-1)/2}$  by a unit in  $\mathbb{Z}[\zeta]$  (see, for example, [4,25]). The corollary now follows immediately from formula (4.3).  $\square$

4.2. Completing the square

**Proposition 4.4.** *For every  $\beta$  in the root lattice  $A_r^{\text{root}}$ , we have*

$$\sum_{\mu \in (\rho + A_r^{\text{root}})} \zeta^{b(|\mu|^2 - |\rho|^2)/2} \zeta^{(\mu|\beta)} = \zeta^{(-|\beta|^2/2b)^\vee} \mathcal{G}(b). \tag{4.9}$$

**Proof.** The proof uses the trick of completing the square. Suppose  $b^*$  is a number such that  $bb^* \equiv 1 \pmod{r}$ . Noting that  $(\mu|\beta) \equiv b(\mu|b^*\beta) \pmod{r}$ , we have

$$\frac{b}{2}(|\mu|^2 - |\rho|^2) + (\mu|\beta) \equiv \frac{b}{2}(|\mu + b^*\beta|^2 - |\rho|^2) - \left(\frac{|\beta|^2}{2b}\right)^\vee \pmod{r}. \tag{4.10}$$

It follows that the left-hand side of (4.9) is equal to

$$\zeta^{(-|\beta|^2/2b)^\vee} \sum_{\mu \in (\rho + A_r^{\text{root}})} \zeta^{b(|\mu + b^*\beta|^2 - |\rho|^2)/2}$$

which is equal to the right-hand side of (4.9) by Proposition 4.1.

4.3. The value of  $F_{U_b}$

Recall that  $U_b$  is the unknot with framing  $b$ . We have

$$J_{U_b}(\mu; q) = q^{b(|\mu|^2 - |\rho|^2)/2} J_U(\mu; q).$$

Using the value of the unknot (3.1) and the definition of  $Q_L$  we get

$$Q_{U_b}(\mu; q) = q^{b(|\mu|^2 - |\rho|^2)/2} q^{2(\mu - \rho|\rho)} \frac{\psi(\mu; q)^2}{\psi(\rho; q)^2}.$$

Hence

$$F_{U_b}(r) = \sum_{\mu \in (\rho + A_r^{\text{root}})} Q_{U_b}(\mu; \zeta) = \frac{\zeta^{-2|\rho|^2}}{\psi(\rho; \zeta)^2} \sum_{\mu \in (\rho + A_r^{\text{root}})} \zeta^{b(|\mu|^2 - |\rho|^2)/2} [\zeta^{(\mu|\rho)} \psi(\mu; \zeta a)]^2. \tag{4.11}$$

**Proposition 4.5.** *The value of  $F_{U_b}(r)$  is given by*

$$F_{U_b}(r) = n! \mathcal{G}(b) \frac{\psi(b^*\rho; \zeta)}{\psi(\rho; \zeta)\psi(-\rho; \zeta)}. \tag{4.12}$$

*In particular, when  $b = \pm 1$ , we have*

$$F_{U_\pm} = \frac{n! \mathcal{G}(\pm 1)}{\psi(\mp \rho; \zeta)} \tag{4.13}$$

**Proof.** The Weyl denominator formula (see Lemma 3.1) shows that

$$\zeta^{(\mu|\rho)}\psi(\mu; \zeta) = \sum_{w \in \mathcal{W}^\vee} \text{sn}(w)\zeta^{(\mu|w(\rho))}. \tag{4.14}$$

Hence

$$[\zeta^{(\mu|\rho)}\psi(\mu; \zeta)]^2 = \sum_{w, w' \in \mathcal{W}^\vee} \text{sn}(ww')\zeta^{(\mu|w(\rho) + w'(\rho))}. \tag{4.15}$$

Although  $\rho$  may not be in the root lattice, the sum  $w(\rho) + w'(\rho)$  is always in  $\Lambda^{\text{root}}$ , since each of  $w(\rho), w'(\rho)$  is in  $\rho + \Lambda^{\text{root}}$  (see Lemma 1.2) and  $2\rho \in \Lambda^{\text{root}}$ .

Using (4.15) in (4.11) and applying Proposition 4.4, we get

$$F_{U_b}(r) = \frac{\zeta^{-2|\rho|^2}\mathcal{G}(b)}{\psi(\rho; \zeta)^2} \sum_{w, w' \in \mathcal{W}^\vee} \text{sn}(ww')\zeta^{(-|(w(\rho) + w'(\rho)|^2/2b)^\vee)}. \tag{4.16}$$

Note that

$$\begin{aligned} |w(\rho) + w'(\rho)|^2 &= |w(\rho)|^2 + |w'(\rho)|^2 + 2(w(\rho)|w'(\rho)) \\ &= 2|\rho|^2 + 2(\rho|w^{-1}w'(\rho)). \end{aligned}$$

It is easy to check that, modulo  $r$ ,

$$\left(\frac{2|\rho|^2 + 2(\rho|w^{-1}w'(\rho))}{2b}\right)^\vee \equiv 2b^*|\rho|^2 + (b^*\rho|w^{-1}w'(\rho) - \rho),$$

where  $b^*$  is any integer such that  $bb^* \equiv 1 \pmod{r}$ . Using this in the right-hand side of (4.16), we get

$$F_{U_b}(r) = \frac{\zeta^{(-2b^*|\rho|^2 - 2|\rho|^2)}\mathcal{G}(b)}{\psi(\rho; \zeta)^2} \sum_{w, w' \in \mathcal{W}^\vee} \text{sn}(ww')\zeta^{(-b^*\rho|w^{-1}w'(\rho) - \rho)}.$$

Note that  $\text{sn}(ww') = \text{sn}(w^{-1}w')$ . Hence

$$\sum_{w, w' \in \mathcal{W}^\vee} \text{sn}(ww')\zeta^{(-b^*\rho|w^{-1}w'(\rho) - \rho)} = |\mathcal{W}^\vee| \sum_{w \in \mathcal{W}^\vee} \text{sn}(w)\zeta^{(-b^*\rho|w(\rho) - \rho)}.$$

The right-hand side, again by Lemma 3.1 and  $|\mathcal{W}^\vee| = n!$ , is equal to  $n! \psi(-b^*\rho; \zeta)$ . This shows that

$$F_{U_b}(r) = n! \mathcal{G}(b) \zeta^{(-2|\rho|^2 - 2b^*|\rho|^2)} \frac{\psi(-b^*\rho; \zeta)}{\psi(\rho; \zeta)^2}. \tag{4.17}$$

For every number  $k$ ,

$$\begin{aligned} \psi(k\rho; \zeta) &= \prod_{\alpha \in \Phi_+} (1 - \zeta^{-(k\rho|\alpha)}) \\ &= \prod_{\alpha \in \Phi_+} \zeta^{-(k\rho|\alpha)} \prod_{\alpha \in \Phi_+} (\zeta^{(k\rho|\alpha)} - 1) \\ &= \zeta^{-2k|\rho|^2} (-1)^{n(n-1)/2} \psi(-k\rho; \zeta). \end{aligned}$$

Using this formula, with  $k = b^*$  and  $k = -1$ , in formula (4.17), we get (4.12).  $\square$

4.4. Proof of Proposition 2.13

If  $k$  is an integer between 1 and  $r - 1$ , then  $\zeta^k - 1$  is proportional to  $\zeta - 1$  by a unit in  $\mathbb{Z}[\zeta]$ . In fact, both

$$\frac{\zeta^k - 1}{\zeta - 1} \quad \text{and} \quad \frac{\zeta - 1}{\zeta^k - 1} = \frac{\zeta^{kk^*} - 1}{\zeta^k - 1}$$

are in  $\mathbb{Z}[\zeta]$ .

For every positive root  $\alpha$ , the number  $(\alpha | \rho)$  is an integer between 1 and  $n - 1$ . This fact follows from the explicit formulas of  $\alpha$  and  $\rho$  [3]. There are  $n(n - 1)/2$  positive roots, and hence if  $r > n$ , the number

$$\psi(\mp \rho; \zeta) = \prod_{\alpha \in \Phi_+} (1 - \zeta^{\pm(\rho|\alpha)})$$

is proportional to  $(\zeta - 1)^{n(n-1)/2}$  in  $\mathbb{Z}[\zeta]$ .

Formula (4.13) and Corollary 4.3 shows that  $F_{U_\pm}(r)$  is proportional to

$$\zeta^{(r-1)(n-1)/2 - n(n-1)/2}$$

by a proportional factor which is a unit in  $\mathbb{Z}[\frac{1}{nr}][\zeta]$ . This proves Proposition 2.13.

4.5. Simple lens spaces and the constant term

With the knowledge of  $F_{U_b}(r)$ , we can calculate the value  $\tau_r^{PSU(n)}$  and  $\tau^{PSU(n)}$  of lens spaces  $M(b)$ .

**Proposition 4.6.** *Let  $M(b)$  be the lens space obtained by surgery on the unknot with framing  $b$ , where  $b$  is a non-zero integer with absolute value less than the prime  $r$ . Then*

$$\begin{aligned} \tau_r^{PSU(n)}(M(b)) &= \left(\frac{|b|}{r}\right)^{n-1} \zeta^{\binom{\text{sn}(b)-b}{2}|\rho|^2} \frac{\psi(b^*\rho; \zeta)}{\psi(\text{sn}(b)\rho; \zeta)} \\ \tau^{PSU(n)}(M(b)) &= (1+x)^{\binom{\text{sn}(b)-b}{2}|\rho|^2} \frac{\psi(\rho/b; 1+x)}{\psi(\text{sn}(b)\rho; 1+x)} \end{aligned} \tag{4.18}$$

**Proof.** The first identity follows from  $\tau_r^{PSU(n)}(M(b)) = F_{U_b}(r)/F_{U_{\text{sn}(b)}}(r)$ , and the explicit formula (4.12) of  $F_{U_b}$ . Actually the value of  $\tau_r^{PSU(n)}(M(b))$  had been calculated by Takata [33] for the values of  $\tau_r^{PSU(n)}$  of Seifert fibered space.

The value of  $\tau^{PSU(n)}(M(b))$  can also be calculated directly from the explicit formula of  $Q_{U_b}(\mu; q)$ . However, it is simpler to notice that the right-hand side of (4.18), which we denote by  $g$  for a moment, satisfies

$$\mathfrak{p}_r(g) = \left(\frac{|b|}{r}\right)^{n-1} \mathfrak{p}_r(\tau_r^{PSU(n)}(M(b))).$$

Hence by the uniqueness (see Lemma 1.6), we can conclude that  $g = \tau^{PSU(n)}(M(b))$ .  $\square$

From formula (4.18) one can calculate the constant term of the power series  $\tau(U_b)$ ; the result is  $|b|^{-n(n-1)/2}$ . Suppose  $L'$  is a link with diagonal linking matrix, with  $b_1, \dots, b_m$  on the diagonal. From the definition it follows that the constant term of the power series  $\tau(L')$  is the same as that of  $\tau(U_{b_1}) \times \dots \times \tau(U_{b_m})$ . Hence the constant term of  $\tau(L')$  is  $|b_1 \dots b_m|^{-n(n-1)/2}$ . It follows that for every rational homology 3-sphere  $M$ , the constant term of the power series  $\tau^{PSU(n)}(M)$  is  $|H_1(M, \mathbb{Z})|^{-n(n-1)/2}$ .

4.6. Proofs of Proposition 2.6 and 2.15

The following follows directly from the definitions of  $\mathfrak{p}_r$ .

**Lemma 4.7.** *For every  $a, b \in \mathbb{Z}$  and every prime  $r$  not dividing  $b$ , one has that*

$$\mathfrak{p}_r((1 + x)^{a/b}) = \mathfrak{p}_r((1 + x)^{(a/b)^\vee}) = \mathfrak{p}_r(\zeta^{(a/b)^\vee}).$$

Let  $\beta \in \Lambda^{\text{root}}$  be an element of the root lattice. Then Proposition 4.4 and formula (4.13) show that

$$\begin{aligned} \frac{\sum_{\mu \in (\rho + \Lambda_r^{\text{root}}) \zeta^{b(|\mu|^2 - |\rho|^2)/2} \zeta^{(\mu|\beta)}}}{F_{U_{\text{sn}(b)}}} &= \frac{1}{n!} \frac{\mathcal{G}(b)}{\mathcal{G}(\text{sn}(b))} \psi(-\text{sn}(b)\rho; \zeta) \zeta^{(-|\beta|^2/2b)^\vee} \\ &= \left(\frac{|b|}{r}\right)^{n-1} \frac{\zeta^{((\text{sn}(b)-b)/2)|\rho|^2}^\vee}{n!} \psi(-\text{sn}(b)\rho; \zeta) \zeta^{(-|\beta|^2/2b)^\vee}. \end{aligned} \tag{4.19}$$

The second equality follows from (4.4).

The right-hand side, if the checks are dropped and  $\zeta$  replaced by  $x + 1$ , is, by definition, the same as  $(|b|/r)^{n-1} \Gamma_b(q^\beta)$ . Hence Lemma 4.7 and (4.19) show that

$$\left(\frac{|b|}{r}\right)^{n-1} \mathfrak{p}_r\left(\frac{\sum_{\mu \in (\rho + \Lambda_r^{\text{root}}) \zeta^{b(|\mu|^2 - |\rho|^2)/2} \zeta^{(\mu|\beta)}}}{F_{U_{\text{sn}(b)}}}\right) = \mathfrak{p}_r(\Gamma_b(q^\beta)). \tag{4.20}$$

This proves Proposition 2.15, since the set of all  $q^\beta$  spans  $\mathcal{S}$ .

Now we prove Proposition 2.6. Recall that

$$\Gamma_b(q^\beta) = q^{-|\beta|^2/2b} y_b,$$

where

$$y_b = \frac{(1 + x)^{(\text{sn}(b)-b)|\rho|^2/2}}{n!} \psi(-\text{sn}(b)\rho; 1 + x)$$

does not depend on  $\beta$ . Since  $1 - (1 + x)^{\text{sn}(b)(\alpha|\rho)}$  is divisible by  $x$ , and since there are  $n(n-1)/2$  positive roots,  $\psi(-\text{sn}(b)\rho; 1 + x)$  is divisible by  $x^{n(n-1)/2}$  in  $\mathbb{Z}[1/n!b][x]$ . Since  $y_b$  is divisible by  $\psi(-\text{sn}(b)\rho; 1 + x)$ , it is divisible by  $x^{n(n-1)/2}$  in  $\mathbb{Z}[1/n!b][x]$ .

Let  $\mathbf{a} = (a_1, \dots, a_{n-1})$ . Then

$$\begin{aligned} \eta^{\mathbf{a}} &= \prod_{i=1}^{n-1} (1 - q^{-\alpha_i})^{a_i} \\ &= \prod_{i=1}^{n-1} \sum_{s_i=0}^{a_i} \binom{a_i}{s_i} (-1)^{a_i - s_i} q^{\sum (s_i - a_i)\alpha_i} \\ &= \sum_{0 \leq \mathbf{s} \leq \mathbf{a}} (-1)^{|\mathbf{a} - \mathbf{s}|} \binom{\mathbf{a}}{\mathbf{s}} q^{\sum (s_i - a_i)\alpha_i}. \end{aligned}$$

Here  $\mathbf{s} = (s_1, \dots, s_{n-1})$ ,  $\mathbf{s} \leq \mathbf{a}$  means  $s_i \leq a_i$  for every  $i$ , and

$$\binom{\mathbf{a}}{\mathbf{s}} \text{ means } \prod_i \binom{a_i}{s_i}.$$

Then

$$\frac{\Gamma_b(\eta^{\mathbf{a}})}{y^b} = (-1)^{|\mathbf{a}|} \sum_{0 \leq \mathbf{s} \leq \mathbf{a}} (-1)^{|\mathbf{s}|} \binom{\mathbf{a}}{\mathbf{s}} (1+x)^{-|\sum (s_i - a_i)\alpha_i|^2/2b}. \tag{4.21}$$

So we need only to show that the right-hand side of (4.21) is divisible by  $x^{\lfloor (|\mathbf{a}|+1)/2 \rfloor}$ , or that the coefficient of  $x^d$  in the right-hand side of (4.21) is equal to 0 whenever  $2d < |\mathbf{a}|$ . The coefficient of  $x^d$  is

$$v_d = (-1)^{|\mathbf{a}|} \sum_{0 \leq \mathbf{s} \leq \mathbf{a}} (-1)^{|\mathbf{s}|} \binom{\mathbf{a}}{\mathbf{s}} \binom{-|\sum (s_i - a_i)\alpha_i|^2/2b}{d}$$

The expression

$$\binom{-|\sum (s_i - a_i)\alpha_i|^2/2b}{d}$$

is a polynomial in  $s_1, \dots, s_{n-1}$  of total degree  $2d$ . Hence when  $2d < |\mathbf{a}|$ ,  $v_d$  is equal to 0 by Lemma 4.9 below. This proves Proposition 2.6.

#### 4.7. A technical result

The following result is well-known, see [37, Lemma 5.19].

**Lemma 4.8.** *Suppose that  $a$  is a positive integer. Then for  $d = 1, 2, \dots, a - 1$ ,*

$$\sum_{s=0}^a (-1)^s \binom{\mathbf{a}}{\mathbf{s}} s^d = s.$$

From this lemma one can prove the following.

**Lemma 4.9.** *Suppose  $p(s_1, \dots, s_{n-1})$  is a polynomial of total degree less than  $|\mathbf{a}|$ , where  $\mathbf{a} \in (\mathbb{Z}_+)^{n-1}$ . Then*

$$\sum_{0 \leq s_i \leq a_i} (-1)^{|\mathbf{s}|} \binom{\mathbf{a}}{\mathbf{s}} p(s_1, \dots, s_{n-1}) = 0.$$

**Proof.** It suffices to consider the case when  $p(s_1, \dots, s_{n-1}) = s_1^{d_1} \cdots s_{n-1}^{d_{n-1}}$ . We have

$$\sum_{0 \leq s_i \leq a_i} (-1)^{|\mathbf{s}|} \binom{\mathbf{a}}{\mathbf{s}} p(s_1, \dots, s_{n-1}) = \prod_{i=1}^{n-1} \sum_{s_i=0}^{a_i} (-1)^{s_i} \binom{a_i}{s_i} s_i^{d_i}.$$

Since  $d_1 + \dots + d_{n-1} < a_1 + \dots + a_{n-1}$ , there must be an index  $i$  for which  $d_i < a_i$ . Apply the previous lemma we get the result.  $\square$

#### 4.8. Proof of Proposition 2.14

Proposition 2.14, for the case  $n = 2$ , is similar to Lemma 2.2 of [31]. We begin with the following lemma.

**Lemma 4.10.** *Let  $r$  be an odd prime and  $d \in \mathbb{Z}_+$ . Then*

$$\sum_{k=0}^{r-1} \binom{k}{d}$$

is divisible by  $(\zeta - 1)^{(r-1)/2 - \lfloor d/2 \rfloor}$  in  $\mathbb{Z}_{(r)}[\zeta]$ , in the sense that the quotient

$$\frac{\sum_{k=0}^{r-1} \binom{k}{d}}{(\zeta - 1)^{(r-1)/2 - \lfloor d/2 \rfloor}}$$

is in  $\mathbb{Z}_{(r)}[\zeta]$ .

**Proof.** If  $d \geq r - 1$ , then  $(r - 1)/2 - \lfloor d/2 \rfloor \leq 0$ . The above quotient is in  $\mathbb{Z}[\zeta]$ , and we are done.

Suppose that  $d \leq r - 2$ . Note that  $\binom{k}{d}$  is a polynomial in  $k$  of degree  $d$ . It is well-known that

$$\sum_{k=0}^{r-1} k^d = \frac{B_{d+1}(r) - B_{d+1}(0)}{d + 1}, \tag{4.22}$$

where  $B_{d+1}(z)$  is the Bernoulli polynomial (see, for example, [4]). It is also known that for  $d \leq r - 2$ , the polynomial  $B_{d+1}$  has coefficients in  $\mathbb{Z}_{(r)}$ . Hence the right-hand side of (4.22) is divisible by  $r$ , which is divisible by  $(\zeta - 1)^{(r-1)}$ , and hence by  $(\zeta - 1)^{(r-1)/2}$ , in  $\mathbb{Z}_{(r)}[\zeta]$ . This proves the lemma.  $\square$

**Corollary 4.11.** *Suppose  $p(k_1, \dots, k_{n-1})$  is a polynomial of degree  $d$  which takes values in  $\mathbb{Z}_{(r)}$  whenever  $k_i \in \mathbb{Z}$ . Then*

$$\sum_{i=1}^{n-1} \sum_{k_i=0}^{r-1} p(k_1, \dots, k_{n-1})$$

is divisible by  $(\zeta - 1)^{(n-1)(r-1)/2 - \lfloor d/2 \rfloor}$  in  $\mathbb{Z}_{(r)}[\zeta]$ .



**Proof.** The polynomial  $p(k_1, \dots, k_{n-1})$  is a  $\mathbb{Z}_{(r)}$ -linear combination of terms of the form

$$\binom{k_1}{d_1} \cdots \binom{k_{n-1}}{d_{n-1}}.$$

Applying Lemma 4.10, we get the result.  $\square$

**Proof of Proposition 2.14.** We have to show that the number

$$u = \sum_{\mu \in (\rho + A_r^{\text{root}})} \zeta^{b(|\mu|^2 - |\rho|^2)/2} \binom{\alpha}{a}(\mu)$$

is divisible by  $(\zeta - 1)^{(n-1)(r-1)/2 - \lfloor |a|/2 \rfloor}$  in  $\mathbb{Z}_{(r)}[\zeta]$ . Note that

$$u = \sum_{\mu \in A_r^{\text{root}}} \zeta^{b(|\mu + \rho|^2 - |\rho|^2)/2} \binom{\alpha}{a}(\mu + \rho).$$

Writing  $\zeta = 1 + (\zeta - 1)$  and using the expansion  $\zeta^c = \sum_{i=0}^c \binom{c}{i} (\zeta - 1)^i$  in the above formula, we can express  $u$  as a polynomial in  $\zeta - 1$ :

$$u = \sum_{\ell} u_{\ell} (\zeta - 1)^{\ell}.$$

The coefficient of  $(\zeta - 1)^{\ell}$  is

$$u_{\ell} = \sum_{\mu \in A_r^{\text{root}}} \binom{b(|\mu + \rho|^2 - |\rho|^2)/2}{\ell} \binom{\alpha}{a}(\mu + \rho).$$

Suppose  $\mu = k_1 \alpha_1 + \dots + k_{n-1} \alpha_{n-1}$ , then  $\sum_{\mu \in A_r^{\text{root}}}$  becomes  $\sum_{i=1}^{n-1} \sum_{k_i=0}^{r-1}$ , and  $u_{\ell}$  can be written as

$$u_{\ell} = \sum_{i=1}^{n-1} \sum_{k_i=0}^{r-1} p(k_1, \dots, k_{n-1}),$$

where  $p(k_1, \dots, k_{n-1})$  is a polynomial taking values in  $\mathbb{Z}_{(r)}$  whenever  $k_i \in \mathbb{Z}$ . The degree of  $p$  is  $2\ell + |a|$ . Hence by Corollary 4.11,  $u_{\ell}$  is divisible by  $(\zeta - 1)^{(n-1)(r-1)/2 - \lfloor |a|/2 \rfloor - \ell}$  in  $\mathbb{Z}_{(r)}[\zeta]$ .

Now remember that  $u_{\ell}$  is the coefficient of  $(\zeta - 1)^{\ell}$  in  $u$ . Hence taking  $(\zeta - 1)^{\ell}$  into account, we see that each term  $u_{\ell} (\zeta - 1)^{\ell}$ , and hence  $u$ , is divisible by  $(\zeta - 1)^{(n-1)(r-1)/2 - \lfloor |a|/2 \rfloor}$  in  $\mathbb{Z}_{(r)}[\zeta]$ . This completes the proof of Proposition 2.14.

## Appendix A

### A.1. On the definition of PSU(n)-quantum invariant

Here we prove that  $\tau_r^{PSU(n)}(M)$  is coincident with the one introduced by Kohno and Takata [10] (see also [38]). We first recall Kohno and Takata’s definition of quantum PSU(n) invariant, which we will denote by  $\tilde{\tau}_r^{PSU(n)}(M)$ .

Recall that  $A'_r$  is the intersection of the simplex  $C_r$  and the weight lattice  $\Lambda$ . Let us consider the subset  $A''_r$  of  $A'_r$  which consists of all weights  $\mu = \rho + k_1\lambda_1 + \dots + k_{n-1}\lambda_{n-1}$  such that

$$k_1 + 2k_2 + \dots + (n - 1)k_{n-1} \text{ is divisible by } n \tag{*}$$

Now we define  $F''_L(r)$  by the same formula as for  $F'_L(r)$ , only replacing  $A'_r$  by  $A''_r$ :

$$F''_L(r) = \sum_{\mu_j \in A''_r, 1 \leq j \leq m} Q_L(\mu_1, \dots, \mu_m; \zeta).$$

Then, by definition [10]

$$\tilde{z}_r^{PSU(n)}(M) = \frac{F''_L(r)}{F''_{U_+}(r)^{\sigma_+} F''_{U_-}(r)^{\sigma_-}}. \tag{A.1}$$

We have the following simple observation.

**Lemma A.1.** *The tuple  $(k_1, \dots, k_{n-1})$  satisfies (\*) if and only if  $k_1\lambda_1 + \dots + k_{n-1}\lambda_{n-1}$  is in the root lattice. In other words,*

$$A''_r = C_r \cap (\rho + \Lambda^{\text{root}}).$$

**Proof.** If  $\beta = l_1\alpha_1 + \dots + l_{n-1}\alpha_{n-1}$ , then  $\beta = k_1\lambda_1 + \dots + k_{n-1}\lambda_{n-1}$ , where  $\mathbf{k} = A\mathbf{l}$ . Here  $\mathbf{k} = (k_1, \dots, k_{n-1})$  and  $\mathbf{l} = (l_1, \dots, l_{n-1})$ , and  $A$  is the Cartan matrix. The inverse  $A^{-1}$  of  $A$  is the symmetric matrix whose entries are given by  $(A^{-1})_{ij} = (n - i)j/n$  for  $n - 1 \geq i \geq j \geq 1$ .

Now it is easy to show that  $\mathbf{k}$  satisfies (\*) if and only if  $\mathbf{l} = A^{-1}\mathbf{k}$  has integer entries.  $\square$

The translation group  $rA^{\text{root}}$  acts on  $\Lambda \otimes \mathbb{R}$ ; and a fundamental domain is the parallelepiped

$$P_r = \{t_1\alpha_1 + \dots + t_{n-1}\alpha_{n-1} \mid 1 \leq t_i < r + 1, t_i \in \mathbb{R}\}.$$

The affine group  $\mathcal{W}_{(r)}$  is the semi-direct product of the Weyl group  $\mathcal{W}$  and the translation group  $rA^{\text{root}}$ . Hence  $rA^{\text{root}}$  has index  $n!$  in  $\mathcal{W}_{(r)}$ . The group  $\mathcal{W}_{(r)}$  acts on  $\Lambda \otimes \mathbb{R}$ , and a fundamental domain in  $C_r$ .

In a sense,  $P_r$  is  $n!$  times  $C_r$ . More precisely, each point in the interior of  $C_r$  has exactly  $n!$  points in  $P_r$  in its  $\mathcal{W}_{(r)}$  orbit (see [6, Lemma 6.6]). For a point on the boundary of  $C_r$ , the cardinality of its  $\mathcal{W}_{(r)}$  orbits in  $P_r$  may be different from  $n!$ . However, if one of the  $\mu_j$  is on the boundary of  $C_r$ , then  $Q_L(\mu_1, \dots, \mu_m; \zeta) = 0$ , by Proposition 1.3. By that same proposition,  $Q_L(\mu_1, \dots, \mu_m; \zeta)$  is invariant under the action of  $\mathcal{W}_{(r)}$ .

Notice that  $A''_r = C_r \cap (\rho + \Lambda^{\text{root}})$ , while  $\rho + \Lambda^{\text{root}} = P_r \cap (\rho + \Lambda^{\text{root}})$ . Hence

$$\sum_{\mu_j \in (\rho + \Lambda^{\text{root}})} Q_L(\mu_1, \dots, \mu_m; \zeta) = (n!)^m \sum_{\mu_j \in A''_r} Q_L(\mu_1, \dots, \mu_m; \zeta).$$

Or  $F_L(r) = (n!)^m F''_L(r)$ . When the linking matrix of  $L$  has non-zero determinant, one has that  $\sigma_+ + \sigma_- = m$ . Hence

$$\frac{F_L(r)}{F_{U_+}(r)^{\sigma_+} F_{U_-}(r)^{\sigma_-}} = \frac{F''_L(r)}{F''_{U_+}(r)^{\sigma_+} F''_{U_-}(r)^{\sigma_-}}.$$

This shows that our definition agrees with that of Kohno and Takata.

A.2. The value of Gauss sums

Formula (4.8) can be deduced from the results of [1]. Here we give a simple proof. For the Legendre symbols we have  $(b/r)(b'/r) = (bb'/r)$ . Hence  $(b/r) = (b^*/r)$ , where  $b^*$  is any number such that  $bb^* \equiv 1 \pmod{r}$ .

The value of the quadratic Gauss sum

$$\gamma = \sum_{k \in \mathbb{Z}/r\mathbb{Z}} \zeta^{k^2},$$

where  $\zeta = e^{2\pi i/r}$ , is well-known (see, for example, [4]):

$$\gamma = \begin{cases} \sqrt{r} & \text{if } r \equiv 1 \pmod{4}, \\ i\sqrt{r} & \text{if } r \equiv 3 \pmod{4}. \end{cases}$$

Noting that  $(2/r) = (-1)^{(r^2-1)/2}$  (see [4]) we can reformulate the value of  $\gamma$  as follows.

**Lemma A.2.** For an odd prime  $r$ , the value of the quadratic Gauss sum is given by

$$\gamma = \left(\frac{2}{r}\right) e^{\pi i(1-r)/4} \sqrt{r}.$$

The following is well-known (see [4, Proposition 6.3.1]).

**Lemma A.3.** Suppose that  $b$  is an integer not divisible by  $r$ , then

$$\sum_{k \in \mathbb{Z}/r\mathbb{Z}} \zeta^{bk^2} = \left(\frac{b}{r}\right) \gamma.$$

Now let us consider the following multi-variable Gauss sum

$$\gamma_{A,b} = \sum_{k \in (\mathbb{Z}/r\mathbb{Z})^{n-1}} \zeta^{bk^t A k/2},$$

where  $A$  is the Cartan matrix of  $sl_n$ . Recall that  $A_{ii} = 2$ ,  $A_{i,i+1} = A_{i+1,i} = -1$ , and other entries are 0. Since the entries on the diagonal of  $A$  are even,  $k^t A k/2$  is always an integer. Hence  $\gamma_{A,b} \in \mathbb{Z}[\zeta]$ .

Let  $D$  be the diagonal  $(n-1) \times (n-1)$ -matrix with entries  $D_{jj} = (j+1)/2j$ . Let  $P$  be the upper-triangle  $(n-1) \times (n-1)$ -matrix with entries  $P_{ii} = 1$  for  $i = 1, \dots, n-1$ ,  $P_{i,i+1} = -i/(i+1)$  for  $i = 1, \dots, n-2$ , and other entries equal to 0. Then one has

$$\frac{1}{2}A = P^t D P \tag{A.2}$$

i.e., the bilinear form corresponding to  $A$  can be diagonalized (over  $\mathbb{Q}$ ) using  $P$ .

Recall that “check” is the natural reduction modulo  $r$  map from  $\mathbb{Z}_{(r)}$  to  $\mathbb{Z}/r\mathbb{Z}$ . Since all the entries of  $P, D, P^{-1}$  are in  $\mathbb{Z}_{(r)}$  (recall that  $r > n$ ), there are defined  $D^\vee$  and  $P^\vee, (P^{-1})^\vee$ . It follows that the symmetric bilinear form  $(A/2)^\vee$  can be diagonalized over  $\mathbb{Z}/r\mathbb{Z}$  by  $P^\vee$ . The resulting diagonal matrix is  $D^\vee$ . So we have

$$\gamma_{A,b} = \sum_{k \in (\mathbb{Z}/r\mathbb{Z})^{n-1}} \zeta^{bk^t D^\vee k}.$$

The matrix  $D^\vee$  is diagonal, with  $D_{jj}^\vee = (j+1)2^*j^*$ . Hence

$$\gamma_{A,b} = \prod_{j=1}^{n-1} \sum_{k \in \mathbb{Z}/r\mathbb{Z}} \zeta^{b(j+1)2^*j^*k^2}.$$

Using Lemmas A.2 and A.3 and the fact that  $(j/r)(j^*/r) = 1$  we get

$$\gamma_{A,b} = \left(\frac{n}{r}\right) \left[ \left(\frac{b}{r}\right) e^{\pi i(1-r)/4} \sqrt{r} \right]^{n-1}.$$

This proves formula (4.8).

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