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THE MARKED LENGTH-SPECTRUM OF A SURFACE OF NONPOSITIVE CURVATURE†

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If M is a manifold and g_1, g_2 are two Riemannian metrics, we say that they have the same marked length spectrum if in each homotopy class of closed curves in M the infinimum of g_1-lengths of curves and the infinimum of g_2-lengths of curves are the same. The marked length spectrum problem in general is to show that two metrics with the same marked length spectrum are isometric. Of course, this cannot hold for arbitrary metrics (for example if M is simply connected). This problem was stated as a conjecture in [1] in the case where M is a closed surface and g_1 and g_2 are of negative curvature. This conjecture was solved by J. P. Otal [12] and independently by C. Croke [3]—see also [4]. Previous work on the problem was done by Guillemin and Kazhdan [7].

In this work, using Otal's approach, we improve some of these results by proving the following theorem:

Theorem A. *Let M be a closed surface and let g_1, g_2 be Riemannian metrics on M, with g_1 of nonpositive curvature and g_2 without conjugate points. If g_1 and g_2 have the same marked length-spectrum then they are isometric by an isometry homotopic to the identity.*

We will also prove the following fact, which reduces the length spectrum and curvature condition to the assumption that the Morse correspondence preserves angles—see §1 for the definition of the Morse correspondence.

Theorem B. *Let M be a closed surface of genus ≥ 2, and let g_1, g_2 be Riemannian metrics without conjugate points on M. If g_1 and g_2 have the same marked length-spectrum and the Morse correspondence preserves angles then they are isometric by an isometry homotopic to the identity.*

Finally, we obtain a third result of a more dynamical nature. This is a generalization of a question raised in [5, 6.3, p. 70], see also [3] and [2] where this question is solved.

Theorem C. *Let (M_1, g_1) and (M_2, g_2) be Riemannian closed surfaces of genus ≥ 2 without conjugate points. If one of the two metrics has nonpositive curvature, then any time preserving semi-conjugacy from the geodesic flow of (M_1, g_1) to the geodesic flow of (M_2, g_2) comes from a Riemannian submersion composed with a shift by some fixed time.*

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We introduce here some definitions and notation. If g is a Riemannian metric on the surface M we will denote by $\kappa_g(m)$ the curvature of g at a point $m \in M$. The lift of g to the universal cover $\tilde{M}$ of M will be denoted by $\tilde{g}$. A $\tilde{g}$-strip in $\tilde{M}$ is a closed subset of $\tilde{M}$ homeomorphic to $\mathbf{R} \times [0, 1]$ whose boundary consists of two $\tilde{g}$-geodesics which remain at bounded distance from each other. Any two disjoint $\tilde{g}$-geodesics which remain at bounded distance from one another and are closed as subsets of $\tilde{M}$ bound a $\tilde{g}$-strip. A $\tilde{g}$-strip is flat if the curvature of $\tilde{g}$ is zero on the strip. If two $\tilde{g}$-geodesics G and G' intersect at a unique point, we will denote by $\angle_{\tilde{g}}(G, G') \in]0, \pi[$ the angle at the point of intersection.

1. BACKGROUND

We fix a reference Riemannian metric g_0 of (strictly) negative curvature on M. The following theorem is due to Morse [10].

THEOREM 1.1. (Morse). *Let g be a Riemannian metric on M. Let $\tilde{g}$ and $\tilde{g}_0$ be the lifts of g and of the Riemannian metric g_0 of (strictly) negative curvature to the universal cover $\tilde{M}$ of M. Then there exists a constant $K > 0$, which depends only on g and g_0, such that any $\tilde{g}_0$-geodesic contains in its K-neighborhood a minimizing $\tilde{g}$-geodesic, and any minimizing $\tilde{g}$-geodesic contains a unique $\tilde{g}_0$-geodesic in its K-neighborhood. The map $\tilde{\mathcal{S}}: \tilde{\mathcal{M}} \to \tilde{\mathcal{G}}_0$ from the space $\tilde{\mathcal{M}}$ of $\tilde{g}$-minimizing geodesics to the space $\tilde{\mathcal{G}}_0$ of $\tilde{g}_0$-geodesics which sends a minimizing $\tilde{g}$-geodesic to the $\tilde{g}_0$-geodesic in its K-neighborhood is continuous and proper. (When g has no conjugate points then $\tilde{\mathcal{M}}$ is of course the space $\tilde{\mathcal{G}}$ of all $\tilde{g}$-geodesics.) Moreover, the map $\tilde{\mathcal{S}}$ is $\pi_1(M)$ equivariant.*

We will call the map $\tilde{\mathcal{S}}$ a Morse map.

COROLLARY 1.2. *For every Riemannian metric g without conjugate points on M, there is a constant C such that any $\tilde{g}$ strip in $\tilde{M}$ has width $\leq C$.*

Proof. The Morse map sends all $\tilde{g}$-geodesics entirely contained in a given strip to the same $\tilde{g}_0$-geodesic. □

It will be convenient to introduce the following concept. If g_1 and g_2 are metrics on M, we say that the $\tilde{g}_1$-geodesic G_1 and the $\tilde{g}_2$-geodesic G_2, both in $\tilde{M}$, are Morse correspondent and we will write $G_1 \sim G_2$, if they remain at bounded distance, i.e. if we have:

$$\sup_{m \in G_1} d(m, G_2) < \infty \qquad \text{and} \qquad \sup_{m \in G_2} d(m, G_1) < \infty,$$

where d is a metric on $\tilde{M}$ coming from a Riemannian metric on M. Since M is compact, the condition does not depend on the choice of d. It is not assumed that the Riemannian metrics g_1 and g_2 are distinct. As before g_0 will be a fixed reference Riemannian metric of (strictly) negative curvature on M. Suppose g_1 and g_2 are Riemannian metrics without conjugate points on M. Call $\tilde{\mathcal{S}}_1: \tilde{\mathcal{G}}_1 \to \tilde{\mathcal{G}}_0$ (resp. $\tilde{\mathcal{S}}_2: \tilde{\mathcal{G}}_2 \to \tilde{\mathcal{G}}_0$) the Morse map from the set of $\tilde{g}_1$-geodesics (resp. $\tilde{g}_2$-geodesics) onto the set of $\tilde{g}_0$-geodesics. If G_1 is a $\tilde{g}_1$-geodesic and G_2 is a $\tilde{g}_2$-geodesic, then $G_1 \sim G_2$ if and only if $\tilde{\mathcal{S}}_1(G_1) = \tilde{\mathcal{S}}_2(G_2)$. The following two statements are easy consequences of the work of Leon Green [6, Corollary 3.2 p. 536 and Theorem 4.1 p. 559]:

PROPOSITION 1.3. *Let g be a Riemannian metric without conjugate points on M. Let G and G' be two $\tilde{g}$-geodesics (in $\tilde{M}$). If $G \sim G'$ then either $G = G'$ or G and G' do not intersect;*

moreover, in the latter case they bound a $\tilde{g}$-strip, and through each point y of this $\tilde{g}$-strip there passes a unique $\tilde{g}$-geodesic G_y such that $G \sim G_y$. If g is of nonpositive curvature then all $\tilde{g}$-strips are flat.

COROLLARY 1.4. *Let g_1 and g_2 be Riemannian metrics without conjugate points of M. Let G_1 and G'_1 (resp. G_2 and G'_2) be two $\tilde{g}_1$-geodesics (resp. $\tilde{g}_2$-geodesics) in $\tilde{M}$. If $G_1 \sim G_2$ and $G'_1 \sim G'_2$ then G_1 and G'_1 intersect transversally if and only if G_2 and G'_2 intersect transversally.*

We will now show how to adapt Otal's arguments [12] to prove the following lemma:

LEMMA 1.5. *Let g_1 and g_2 be two Riemannian metrics without conjugate points on M. Suppose that g_2 is of nonpositive curvature and that g_1 and g_2 have the same length spectrum. Then:*

For every pair (G_1, G'_1) of $\tilde{g}_1$-geodesics, and every pair (G_2, G'_2) of $\tilde{g}_2$-geodesics, if $G_1 \sim G_2$ and $G'_1 \sim G'_2$, then $\angle_{\tilde{g}_1}(G_1, G'_1) = \angle_{\tilde{g}_2}(G_2, G'_2)$.

Consequently the metric g_1 also has nonpositive curvature.

Sketch of Proof. We will use the setting of [4] to show how to adapt the arguments of Otal. Let g_0 be a metric of (strictly) negative curvature on M. Let $\tilde{\mathscr{S}}_1: \tilde{\mathscr{G}}_1 \to \tilde{\mathscr{G}}$ (resp. $\tilde{\mathscr{S}}_2: \tilde{\mathscr{G}}_2 \to \tilde{\mathscr{G}}$) be the Morse map from the space of $\tilde{g}_1$-geodesics (resp. $\tilde{g}_2$-geodesics) onto the space of $\tilde{g}_0$-geodesics, as described above. As in [4], using $\tilde{\mathscr{S}}_1$ and $\tilde{\mathscr{S}}_2$, we obtain, from the Liouville measures, geodesic currents $\bar{\lambda}_{g_1}$ and $\bar{\lambda}_{g_2}$. By [12, théorème 2], we obtain $\bar{\lambda}_{g_1} = \bar{\lambda}_{g_2}$. We define for each pair (G, G') of transversally intersecting $\tilde{g}_0$-geodesics the angle $\theta(G, G')$ as the $\tilde{g}_2$ angle of any pair (G_2, G'_2) of $\tilde{g}_2$-geodesics such that $\tilde{\mathscr{S}}_2(G_2) = G$ and $\tilde{\mathscr{S}}_2(G'_2) = G'$. The fact that this angle is independent of the choices follows from the flatness of the $\tilde{g}_2$-strips—see Proposition 1.3. It is not very difficult now to adapt Otal's arguments [12] as in [4], to prove the angle condition of the lemma.

This angle condition, taken with the fact that g_2 has nonpositive curvature, implies that the sum of the angles of any triangle whose sides are $\tilde{g}_1$-geodesics is $\leq \pi$. It follows from the Gauss–Bonnet theorem that g_1 also has nonpositive curvature. □

2. PROOF OF THEOREM A

Because, the sphere is simply connected the genus of M has to be ≥ 1. A theorem of Hopf says that a metric without conjugate points on a torus or a Klein bottle is flat. Theorem A follows for the torus and the Klein bottle—see [3, pp. 167–168]. So we assume for the rest of the section that the genus of M is ≥ 2. Since we want to use the work done in this section to prove other theorems, we will use general arguments as often as possible.

Definition 2.1. Suppose g_1 and g_2 are Riemannian metrics without conjugate points on M. We define a partial relation $\mathscr{R}$ on $\tilde{M}$ in the following way $m\mathscr{R}m'$, if every $\tilde{g}_2$-geodesic through m' is at bounded distance from some $\tilde{g}_1$-geodesic through m.

LEMMA 2.2. *Suppose g_1 and g_2 are Riemannian metrics without conjugate points on M. If $m\mathscr{R}m'$, then every $\tilde{g}_1$-geodesic through m is at bounded distance from some $\tilde{g}_2$-geodesic through m'.*

Proof. Let g_0 be a metric of (strictly) negative curvature on M. As before, let $\tilde{\mathscr{S}}_1 : \tilde{\mathscr{G}}_1 \to \tilde{\mathscr{G}}$ (resp. $\tilde{\mathscr{S}}_2 : \tilde{\mathscr{G}}_2 \to \tilde{\mathscr{G}}$) be the Morse map from the space of $\tilde{g}_1$-geodesics (resp. $\tilde{g}_2$-geodesics) onto the space of $\tilde{g}_0$-geodesics. We have $m\mathscr{R}m'$ if and only if $\tilde{\mathscr{S}}_2\{G \in \tilde{\mathscr{G}}_2 \mid m' \in G\} \subset \tilde{\mathscr{S}}_1\{G \in \tilde{\mathscr{G}}_1 \mid m \in G\}$. But by Proposition 1.3, the map $\tilde{\mathscr{S}}_1$ (resp. $\tilde{\mathscr{S}}_2$) is injective on $\{G \in \tilde{\mathscr{G}}_1 \mid m \in G\}$ (resp. $\{G \in \tilde{\mathscr{G}}_2 \mid m' \in G\}$). Hence we have a natural 1–1 continuous map from the circle of $\tilde{g}_2$-geodesics through m' to the circle of $\tilde{g}_1$-geodesics through m. Such a map must be a homeomorphism, so we are done. □

LEMMA 2.3. *Suppose g_1 and g_2 are Riemannian metrics without conjugate points on M, and every $\tilde{g}_1$-strip is flat. If $m_1\mathscr{R}m'$ and $m_2\mathscr{R}m'$, then $m_1 = m_2$. If every $\tilde{g}_2$-strip is also flat, then $\mathscr{R}$ is the graph of a bijection φ between the domain $\tilde{D}_1$ of $\mathscr{R}$ and its range $\tilde{D}_2$. Both $\tilde{D}_1$ and $\tilde{D}_2$ are invariant under the action of $\pi_1(M)$, and moreover, the map φ is $\pi_1(M)$ equivariant.*

Proof. From Lemma 2.2 and the definition of $\mathscr{R}$, it follows that if G is a $\tilde{g}_1$-geodesic through m_1 it is at bounded distance from some $\tilde{g}_1$-geodesic G' through m_2. If $m_1 \neq m_2$ and G is not the geodesic through m_1 and m_2, then G and G' bound a $\tilde{g}_1$-strip which by hypothesis must be flat. This implies that the curvature of $\tilde{g}_1$ along every $\tilde{g}_1$-geodesic through m_1 is 0 (by continuity this is also true for the geodesic between m_1 and m_2). The completeness of g_1 implies that g_1 is flat which is impossible since the genus of M is ≥ 2. The equivariance is obvious. □

Let us now suppose that the Riemannian metrics g_1 and g_2 without conjugate points on M have the same length spectrum and one of them has non positive curvature. By 1.5, both of them have nonpositive curvature, and by 1.3 all $\tilde{g}_1$ and $\tilde{g}_2$ strips are flat and by 2.3 the relation $\mathscr{R}$ is the graph of a bijection.

Let $\tilde{U}_i$, $i = 1, 2$, be $\{m \in \tilde{M} \mid \kappa_{\tilde{g}_i}(m) \neq 0\}$.

LEMMA 2.4. *Under the hypothesis of Theorem A, let G_1 and G'_1 be $\tilde{g}_1$-geodesics which intersect transversally at a point m which is in $\tilde{U}_1$. Suppose that G_2 and G'_2 are $\tilde{g}_2$-geodesics with $G_1 \sim G_2$ and $G'_1 \sim G'_2$. If m' is the point of intersection of G_2 and G'_2 then $m\mathscr{R}m'$. In particular, the set $\tilde{U}_1$ is contained in the domain $\tilde{D}_1$ of the map φ given by* 2.3.

Moreover, every $\tilde{g}_1$-geodesic which is at bounded distance from a $\tilde{g}_1$-geodesic through m must also pass through m, and every $\tilde{g}_2$-geodesic which is at bounded distance from a $\tilde{g}_2$-geodesic through m' must also pass through m'.

Proof. If H is a $\tilde{g}_1$-geodesic which is at bounded distance from some $\tilde{g}_2$-geodesic which passes through m', the angle condition of Lemma 1.5 shows that G_1, G'_1 and H bound a triangle T whose sum of angles is π. But g_1 is of nonpositive curvature and one of the vertices of the triangle, namely m, satisfies $\kappa_{\tilde{g}_1}(m) \neq 0$, so from the Gauss–Bonnet theorem it follows that T is degenerate and that H goes through m. This proves that $m\mathscr{R}m'$.

Since $\tilde{g}_1$-strips are flat, the point m cannot be contained in a $\tilde{g}_1$-strip. So every $\tilde{g}_1$-geodesic which is at bounded distance from some $\tilde{g}_1$-geodesic through m must also pass through m.

It remains to prove that every $\tilde{g}_2$-geodesic which is at bounded distance from a $\tilde{g}_2$-geodesic through m' must also pass through m'. Let H'_2 and H''_2 be two $\tilde{g}_2$-geodesics that remain at bounded distance and suppose that $m' \in H'_2$, $m' \notin H''_2$ and $m'' \in H''_2$. We know from the first part that both H'_2 and K, the $\tilde{g}_2$-geodesic through m' and m'', are at bounded distance from $\tilde{g}_1$-geodesics that pass through m. It follows that the pair H''_2, K of transverse $\tilde{g}_2$-geodesics are at bounded distance from $\tilde{g}_1$-geodesics through m. By the first part of the lemma $m\mathscr{R}m''$. From Lemma 2.3, we obtain $m' = m''$. This is a contradiction. □

LEMMA 2.5. *Under the hypothesis of Theorem A, if* $m, m' \in \tilde{U}_1$, *then* $d_{\tilde{g}_1}(m, m') = d_{\tilde{g}_2}(\varphi(m), \varphi(m'))$, *where* φ *is given by Lemma 2.3. In particular, the map* φ *induces an isometry between* $\tilde{U}_1$ *and* $\tilde{U}_2$.

Proof. Fix a Riemannian metric g_0 on M of (strictly) negative curvature and let $\tilde{\mathscr{S}}_1: \tilde{\mathscr{G}}_1 \to \tilde{\mathscr{G}}$ and $\tilde{\mathscr{S}}_2: \tilde{\mathscr{G}}_2 \to \tilde{\mathscr{G}}$ the Morse maps described above. It is not difficult to see, using Lemma 2.4, that there exists a set $\mathscr{A} \subset \tilde{\mathscr{G}}$ such that $\tilde{\mathscr{S}}_1^{-1}(\mathscr{A})$ (resp. $\tilde{\mathscr{S}}_2^{-1}(\mathscr{A})$) is the subset of $\tilde{\mathscr{G}}_1$ (resp. $\tilde{\mathscr{G}}_2$) consisting of $\tilde{g}_1$-geodesics (resp. $\tilde{g}_2$-geodesics) that intersect the $\tilde{g}_1$-geodesic (resp. $\tilde{g}_2$-geodesic) segment between m and m' (resp. $\varphi(m)$ and $\varphi(m')$). Using the fact that the Liouville currents obtained from g_1 and g_2 are the same, an application of the Crofton formula finishes the proof of the first part.

It follows from [11] that φ is differentiable on $\tilde{U}_1$ and hence it is also a Riemannian isometry on U_1. This implies $\varphi(\tilde{U}_1) \subset \tilde{U}_2$. Exchanging the role of g_1 and g_2 gives $\varphi(\tilde{U}_1) = \tilde{U}_2$. □

LEMMA 2.6. *Let* g_1 *and* g_2 *be Riemannian metrics without conjugate points on* M, *for which* $\tilde{g}_1$ *and* $\tilde{g}_2$ *strips are flat. Suppose that the map* φ *induces a bijection between* $\tilde{U}_1 = \{m \in \tilde{M} \mid \kappa_{g_1}(m) \neq 0\}$ *and* $\tilde{U}_2 = \{m \in \tilde{M} \mid \kappa_{g_2}(m) \neq 0\}$, *and that for every* $m, m' \in \tilde{U}_1$, *we have* $d_{\tilde{g}_1}(m, m') = d_{\tilde{g}_2}(\varphi(m), \varphi(m'))$. *Then* φ *extends to a Riemannian isometry of* $(\tilde{M}, \tilde{g}_1)$ *onto* $(\tilde{M}, \tilde{g}_2)$ *which is equivariant under the action of* π_1. *Hence the Riemannian metrics* g_1 *and* g_2 *are isometric by an isometry homotopic to the identity of* M.

Proof. As above φ induces a Riemannian isometry on $\tilde{U}_1$. If $p \in \tilde{U}_1$, call $T_p\varphi$ the derivative of φ at p. One can check that the map $\bar{\varphi} = \exp^{g_2}_{\varphi p} T_p \varphi (\exp^{g_1}_p)^{-1}$ extends φ to $\tilde{M}$. Moreover, the extension $\bar{\varphi}$ preserves curvature, since along any geodesic through p the map will be an isometry at points of $\tilde{U}_1$, namely φ, and will take points of zero curvature to points of zero curvature. From the well-known relation between Jacobi fields and the derivative of the exponential map—see [8, Lemma 5.4.3 p. 102]—it follows that $\bar{\varphi}: (\tilde{M}, \tilde{g}_1) \to (\tilde{M}, \tilde{g}_2)$ is an isometry. The fact that $\bar{\varphi}$ is equivariant under $\pi_1(M)$ follows from the fact that φ is equivariant under the same action. □

The proof of Theorem A follows from the above lemmas.

3. PROOF OF THEOREM B

In this section we assume that g_1 and g_2 are Riemannian metrics without conjugate points on M, and that the angle hypothesis of Theorem B is satisfied, i.e.:

For every pair (G_1, G_1') of $\tilde{g}_1$-geodesics, and every pair (G_2, G_2') of $\tilde{g}_2$-geodesics,

$$G_1 \sim G_2 \text{ and } G_1' \sim G_2' \Rightarrow \angle_{\tilde{g}_1}(G_1, G_1') = \angle_{\tilde{g}_2}(G_2, G_2').$$

As before let $\tilde{U}_i = \{m \in \tilde{M} \mid \kappa_{\tilde{g}_i}(m) \neq 0\}$, $i = 1, 2$.

We first prove three more lemmas.

LEMMA 3.1. *All strips for* $\tilde{g}_1$ *and* $\tilde{g}_2$ *are flat. Consequently, no point of* $\tilde{U}_1$ *(resp.* $\tilde{U}_2$*) is contained in a* $\tilde{g}_1$ *(resp.* $\tilde{g}_2$*) strip.*

Proof. We will show the result for $\tilde{g}_1$. A consequence of the hypothesis of Theorem 3.1, is that if the $\tilde{g}_1$-geodesics G, G' remain at bounded distance then any other $\tilde{g}_1$-geodesic cuts them at the same angle. By the result of Leon Green, Proposition 1.3, any strip bounded by two $\tilde{g}_1$-geodesics can be foliated by infinite $\tilde{g}_1$-geodesics. It is easy to deduce that any point inside the strip is contained in arbitrarily small geodesic triangles whose sum of interior angles is π. It follows from the Gauss–Bonnet theorem that the strip is flat. □

LEMMA 3.2. *Let* $\tilde{\mathscr{U}}_1 = \{G_1 \in \tilde{\mathscr{G}}_1 \mid G_1 \cap \tilde{U}_1 \neq 0\}$ *and* $\tilde{\mathscr{U}}'_2 = \{G_2 \in \tilde{\mathscr{G}}_2 \mid \exists G_1 \in \tilde{\mathscr{U}}_1, G_1 \sim G_2\}$. *The formula* $G_2 \sim \tilde{\mathscr{S}} G_2$ *defines a continuous surjective map* $\tilde{\mathscr{S}}: \tilde{\mathscr{U}}'_2 \to \tilde{\mathscr{U}}_1$. *Of course* $\tilde{\mathscr{U}}_1$ *is open; moreover, the set* $\tilde{\mathscr{U}}'_2$ *is also open.*

Proof. By Lemma 3.1, if G, G' are $\tilde{g}_1$-geodesics with $G \in \tilde{\mathscr{U}}_1$ and $G \sim G'$ then $G = G'$. As above, let $\tilde{\mathscr{S}}_1: \tilde{\mathscr{G}}_1 \to \tilde{\mathscr{G}}$ and $\tilde{\mathscr{S}}_2: \tilde{\mathscr{G}}_2 \to \tilde{\mathscr{G}}$ the Morse maps obtained in 1.1. From the observation just made, $\tilde{\mathscr{S}}_1$ induces a bijection from $\tilde{\mathscr{U}}_1 \to \tilde{\mathscr{S}}_1(\tilde{\mathscr{U}}_1)$ and $\tilde{\mathscr{S}}_1^{-1}\tilde{\mathscr{S}}_1(\tilde{\mathscr{U}}_1) = \tilde{\mathscr{U}}_1$. It is not difficult, using the fact that $\tilde{\mathscr{S}}_1$ is continuous and proper, to conclude that $\tilde{\mathscr{S}}_1(\tilde{\mathscr{U}}_1)$ is open and that $\tilde{\mathscr{S}}_1$ restricts to a homeomorphism from $\tilde{\mathscr{U}}_1$ onto $\tilde{\mathscr{S}}_1(\tilde{\mathscr{U}}_1)$. The lemma follows since $\tilde{\mathscr{U}}'_2 = \tilde{\mathscr{S}}_2^{-1}(\tilde{\mathscr{S}}_1(\tilde{\mathscr{U}}_1))$ and $\tilde{\mathscr{S}} = \tilde{\mathscr{S}}_1^{-1}\tilde{\mathscr{S}}_2$. □

LEMMA 3.3. *Suppose that the* $\tilde{g}_2$*-geodesics* G_2, G'_2 *are in* $\tilde{\mathscr{U}}'_2$ *and that* $\tilde{\mathscr{S}} G_2$ *and* $\tilde{\mathscr{S}} G'_2$ *intersect transversally at a point m which is in* $\tilde{U}_1$. *If m′ is the point of intersection of* G_2 *and* G'_2 *then* $m\mathscr{R}m'$. *Moreover, every* $\tilde{g}_1$*-geodesic which is at bounded distance from some* $\tilde{g}_1$*-geodesic through m must also pass through m, and every* $\tilde{g}_2$*-geodesic which is at bounded distance from some* $\tilde{g}_2$*-geodesic through m′ must also pass through m′.*

Proof. Let G_2 and G'_2 be $\tilde{g}_2$-geodesics passing through m' such that $G_2 \sim G_1$ and $G'_2 \sim G'_1$ where G_1 and G'_1 are $\tilde{g}_1$-geodesics passing through m. For $\theta \in [0, \pi]$, let G_2^θ (resp. $G_2'^\theta$) be the $\tilde{g}_2$-geodesic through m' making an angle θ with G_2 (resp. G'_2). Let T be the set of $\theta \in [0, \pi]$ such that there exists $\tilde{g}_1$-geodesics G_1^θ and $G_1'^\theta$ through m with $G_2^\theta \sim G_1^\theta$ and $G_2'^\theta \sim G_1'^\theta$.

Since two geodesics that stay at a bounded distance must stay within a constant distance depending only on g_0, g_1 and g_2 we see that T must be closed.

Since T is non-empty we need only show that T is open. Let $\psi \in T$. Since G_2^ψ and $G_2'^\psi$ are in $\tilde{\mathscr{U}}'_2$ and $\tilde{\mathscr{U}}'_2$ is open, if θ is close enough to ψ, the $\tilde{g}_2$-geodesics G_2^θ and $G_2'^\theta$ are also in $\tilde{\mathscr{U}}'_2$ and hence there exists unique geodesics G_1^θ and $G_1'^\theta$ with $G_2^\theta \sim G_1^\theta$ and $G_2'^\theta \sim G_1'^\theta$.

Let $\varepsilon > 0$ be so small that $B(m, \varepsilon)$, the ε-ball for $\tilde{g}_1$ about m, is convex and has non-zero curvature for $\tilde{g}_1$ at every point. By transversality and continuity of the map $\tilde{\mathscr{S}}: \tilde{\mathscr{U}}'_2 \to \tilde{\mathscr{U}}_1$ we see that for all θ sufficiently close to ψ the intersection points of G_1^ψ and $G_1'^\theta$ and $G_2'^\theta$, $G_1'^\psi$ and $G_1'^\psi$, G_1^θ and $G_1'^\theta$, and G_1^θ and $G_1'^\psi$ all lie in $B(m, \varepsilon)$.

We consider two cases. First assume G_1^θ intersects G_1^ψ inside $B(m, \varepsilon)$ (or similarly that $G_1'^\theta$ intersects $G_1'^\psi$ inside $B(m, \varepsilon)$). Then the geodesic triangles G_1^θ, G_1^ψ, $G_1'^\psi$ and G_1^θ, G_1^ψ, $G_1'^\theta$ both lie inside $B(m, \varepsilon)$. By preservation of angles both have interior angles that sum to π, but since the curvature is never zero in these triangles, the Gauss–Bonnet theorem forces them to be degenerate triangles which forces all these geodesics to pass through the common point m, so $m\mathscr{R}m'$.

If on the other hand, both the intersections of G_1^θ with G_1^ψ and $G_1'^\theta$ with $G_1'^\psi$ occur outside $B(m, \varepsilon)$ we see that G_1^θ, G_1^ψ, $G_1'^\theta$, $G_1'^\psi$ form a quadrilateral inside $B(m, \varepsilon)$. Again the Gauss–Bonnet theorem forces this quadrilateral to be degenerate and all geodesics pass through m, and again $m\mathscr{R}m'$.

Again by flatness of strips, every $\tilde{g}_1$-geodesic at a bounded distance from some $\tilde{g}_1$-geodesic through m passes through m. Let G'_2 and G''_2 be $\tilde{g}_2$-geodesics with $G'_2 \sim G''_2$ and $m' \in G'_2$, $m' \notin G''_2$. Let G_1 be the $\tilde{g}_1$-geodesic through m with $G_1 \sim G'_2$ hence $G_1 \sim G''_2$. Pick $m'' \in G''_2$ and let H_2 be the $\tilde{g}_2$-geodesic through m' and m''. We know that there is a $\tilde{g}_1$-

geodesic through m such that $H_1 \sim H_2$ since $m\mathscr{R}m'$. On the other hand, since $m'' = H \cap G_2''$ the first part of the lemma yields $m\mathscr{R}m''$. Now Lemma 2.3 yields $m' = m''$. □

COROLLARY 3.4. *Under the assumptions of this section $\tilde{U}_1$ is contained in the domain $\tilde{D}_1$ of th relation $\mathscr{R}$ and $\tilde{U}_2$ is contained in is range $\tilde{D}_2$.*

Proof. This is immediate from Lemma 3.3. □

Proof of Theorem B: We now proceed as in the proof of Theorem A, using Lemma 2.6. □

4. PROOF OF THEOREM C

Part of the arguments are already in [2] and [3]. It is easy to see, by taking an orientable cover, that one can reduce the proof to the case where M_1 is orientable. We will assume that this is the case in the sequel.

LEMMA 4.1. *Suppose $S(M)$ is the unit tangent bundle of the closed surface M of genus ≥ 2. If a subgroup of $\pi_1(M)$ has non trivial center then it is isomorphic to $\mathbf{Z}$. Call $p_M: S(M) \to M$ the canonical projection. The center of $\pi_1(S(M))$ is contained in the kernel of the induced map $p_{M_*}: \pi_1(S(M)) \to \pi_1(M)$. Moreover, if M is orientable the kernel of p_{M_*} is precisely the center of $\pi_1(S(M))$.*

Proof. This is well-known and can be proven using elementary hyperbolic geometry. □

LEMMA 4.2. *Suppose M_1 and M_2 are closed surfaces of genus ≥ 2 endowed respectively with Riemannian metrics g_1 and g_2. Suppose that h: $S(M_1) \to S(M_2)$ is a time preserving semi-conjugacy between the geodesic flow g_1^t of g_1 and the geodesic g_2^t of g_2. If g_2 has no conjugate points, then h maps the center of $\pi_1(S(M_1))$ in the center of $\pi_1(S(M_2))$, hence it induces a map $h_\#: \pi_1(M) \to \pi_1(M_2)$. The map $h_\#$ is injective.*

Proof. Let us look at the composition $\theta: S(M_1) \to S(M_2) \to M_2$, where the first arrow is h and the second one is p_{M_2}. We want to show that $\theta_*: \pi_1 S((M_1)) \to \pi_1(M_2)$ sends the center of $\pi_1 S((M_1))$ to 0. Suppose this is not the case, then by 4.1 the image G of θ_* is a cyclic subgroup of $\pi_1(M_2)$ which is isomorphic to $\mathbf{Z}$. Let us call P: $C \to M_2$ the covering of M_2 such that $P_*(\pi_1(C)) = G$. It is easy to see that h can be written as a composition $S(P)\tilde{h}$ where $\tilde{h}$: $S(M_1) \to S(C)$ and $S(P)$: $S(C) \to S(M_2)$ is the tangent map obtained from P. If we lift the metric g_2 to a metric $\tilde{g}_2$ on C via P, we obtain that $\tilde{h}$ is a time preserving semi-conjugacy between flows. Using a little bit of the theory described in §1 and the fact that C is a cylinder or an open Möbius band without conjugate points, it is not difficult to realize that $\tilde{h}$ sends each g_1-geodesic to a $\tilde{g}_2$-geodesic that remain in the strip associated to a non-trivial closed $\tilde{g}_2$-geodesic G of minimum length in C. If H is a $\tilde{g}_2$-geodesic transversal to G, using the fact that g_2 has no conjugate points, all $\tilde{g}_2$-geodesics that remain in the strip of G are also transversal to H. By the connectedness of $S(M)$, we conclude that $\tilde{h}$ sends each oriented g_1-geodesic to a $\tilde{g}_2$-geodesic that always raps around C in the same sense. This is impossible, because a closed oriented g_1-geodesic and its opposite are in opposite homotopy classes of closed curves in $S(M)$.

To show that $h_\#$ is injective, let us start with γ in $\pi_1(M_1)$, we can find a closed g_1-geodesic G_1 in the free homotopy class of γ. Since h is a semi-conjugacy the image $h(G_1)$ is a closed g_2-geodesic. Since g_2 has no conjugate points it cannot be homotopic to 0. □

Suppose M_1 and M_2 are closed surfaces of genus ≥ 2 endowed respectively with Riemannian metrics g_1 and g_2. We assume that g_2 has no conjugate points. Since by 4.2, the map $h_\#$ is injective and M_1 and M_2 are closed surfaces, the subgroup $h_\#(\pi_1(M_1))$ has finite index in $\pi_1(M_2)$ (if not then by covering theory $\pi_1(M_1)$ would be the fundamental group of a connected non-compact surface, but such a group is free and the fundamental group of a closed surface is never free). Hence it is easy to reduce to the case where $h_\#$ is an isomorphism. Since all automorphisms of the fundamental group of a surface can be realized by diffeomorphisms, we can find a diffeomorphism f: $M_2 \to M_1$ such that the induced map f_* on π_1 is $h_\#^{-1}$. If we use the diffeomorphism f to transport the metric g_2 to a metric $\hat{g}_2$ on M_1, it is not difficult to see that $\hat{g}_2$ has no conjugate points and that g_1 and $\hat{g}_2$ have the same marked length-spectrum. If both g_1 and g_2 are without conjugate points and one of them is of nonpositive curvature, then we can apply Theorem A to g_1 and $\hat{g}_2$, so if we compose f with an isometry homotopic to the identity, we see that the proof of Theorem C is reduced to:

LEMMA 4.3. *Let g be a Riemannian metric with nonpositive curvature on a closed orientable surface M. If h: $S(M) \to S(M)$ is a self semiconjugacy of the geodesic flow of g such that $h_\#: \pi_1(M) \to \pi_1(M)$ is the identity then $h = g^{t_0}$ for some fixed time t_0.*

Proof. It is not difficult to see from the hypothesis on $h_\#$ that we can lift h to a map $\tilde{h}$: $S(\tilde{M}) \to S(\tilde{M})$ homotopic to the identity by a bounded homotopy, where $\tilde{M}$ is the universal cover of M. It follows that for any geodesic G of the lift $\tilde{g}$ of g to the universal cover $\tilde{M}$ the geodesic $\tilde{h}(G)$ is bounded distance from G. By Proposition 1.3, the geodesics G and $\tilde{h}(G)$ either coincide or bound a flat strip. By [6] or [9, p. 379, Theorem 3.9.17], there exists a geodesic dense in $S(M)$. Suppose that G is the lift to $\tilde{M}$ of such a geodesic dense in $S(M)$; then the second case cannot happen because G has to go through points of negative curvature. In fact, the geodesic G and $\tilde{h}(G)$ have to coincide as *oriented* geodesics since $\tilde{h}$ preserves time and is homotopic to the identity by a bounded homotopy. The lemma follows easily using the denseness of the image of G in M and the fact that h preserves time. □

Remark 4.4. Suppose M_1 and M_2 are closed surfaces of genus ≥ 2 endowed respectively with Riemannian metrics g_1 and g_2. We assume that g_2 has nonpositive curvature. If there exists a time preserving *conjugacy* (not necessarily C^1) between the geodesic flows of g_1 and g_2, then the arguments in [3, Lemma 3.2] show that g_1 has no conjugate points and we can apply theorem C to obtain [3, Theorem B] without the assumption that the conjugacy is C^1.

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