Linear Algebra and its Applications 432 (2010) 566–570

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Signless Laplacian spectral radius and Hamiltonicity☆

Bo Zhou

Department of Mathematics, South China Normal University, Guangzhou 510631, PR China

ARTICLE INFO

Article history:
Received 23 May 2009
Accepted 1 September 2009
Available online 28 October 2009

Submitted by R.A. Brualdi

AMS classification:
05C50
05C35

Keywords:
Signless Laplacian spectral radius
Hamiltonian path
Hamiltonian cycle

ABSTRACT

We give tight conditions on the signless Laplacian spectral radius of a graph for the existence of Hamiltonian paths and cycles.

1. Introduction

Let G be a simple graph on n vertices. Let $\mathbf{A}(G)$ be the adjacency matrix of G. The spectral radius of G is the largest eigenvalue of $\mathbf{A}(G)$, denoted by $\mu(G)$, see [3].

Let K_n be the complete graph on n vertices. Write $K_{n-1}+v$ for K_{n-1} together with an isolated vertex, and $K_{n-1}+e$ for K_{n-1} with a pendent edge. Recently, Fiedler and Nikiforov [5] gave tight conditions on spectral radius of a graph for the existence of Hamiltonian paths and cycles:

Theorem 1. *Let G be a graph on n vertices.*

(i) *If $\mu(G) \geqslant n-2$ and $G \neq K_{n-1}+v$, then G contains a Hamiltonian path.*
(ii) *If $\mu(G) > n-2$ and $G \neq K_{n-1}+e$, then G contains a Hamiltonian cycle.*

☆ The work was supported by National Natural Science Foundation of China (10671076).
E-mail address: zhoubo@scnu.edu.cn

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doi:10.1016/j.laa.2009.09.004

Theorem 2. *Let G be a graph on n vertices with complement $\overline{G}$.*

(i) *If $\mu(\overline{G}) \leqslant \sqrt{n-1}$ and $G \neq K_{n-1} + v$, then G contains a Hamiltonian path.*
(ii) *If $\mu(\overline{G}) \leqslant \sqrt{n-2}$ and $G \neq K_{n-1} + e$, then G contains a Hamiltonian cycle.*

Let G be a graph on n vertices. Let $\mathbf{D}(G)$ be the diagonal matrix of order n whose (i, i)-entry is the degree (number of neighbors) of the ith vertex of G. The matrix $\mathbf{L}^+(G) = \mathbf{D}(G) + \mathbf{A}(G)$ is the signless Laplacian matrix, for details see [4]. The signless Laplacian spectral radius of G is the largest eigenvalue of $\mathbf{L}^+(G)$, denoted by $\gamma(G)$.

In this note we give tight conditions on the signless Laplacian spectral radius of a graph for the existence of Hamiltonian paths and cycles.

2. Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $e(G) = |E(G)|$. Let $d_G(u)$ be the degree of vertex u in G.

For an integer $k \geqslant 0$, the k-closure of the graph G is a graph obtained from G by successively joining pairs of nonadjacent vertices whose degree sum is at least k (in the resulting graph at each stage) until no such pair remains [1]. Write $C_k(G)$ for the k-closure of G. Note that $d_{C_k(G)}(u) + d_{C_k(G)}(v) \leqslant k - 1$ for any pair of nonadjacent vertices u and v in $C_k(G)$. The following lemma is due essentially to Ore [8].

Lemma 1. (i) *A graph G on n vertices has a Hamiltonian path if and only if $C_{n-1}(G)$ has one.*
(ii) *A graph G on n vertices has a Hamiltonian cycle if and only if $C_n(G)$ has one.*

Lemma 2 [8]. *Let G be a graph on n vertices.*

(i) *If $d_G(u) + d_G(v) \geqslant n - 1$ for any pair of nonadjacent vertices u and v in G, then G contains a Hamiltonian path.*
(ii) *If $d_G(u) + d_G(v) \geqslant n$ for any pair of nonadjacent vertices u and v in G, then G contains a Hamiltonian cycle.*

For a graph G, let $Z(G)$ be the sum of the squares of the degrees of G, i.e., $Z(G) = \sum_{u \in V(G)} d_G(u)^2$. Obviously, $Z(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$. This quantity has been studied in the literature, see [2, 7,9]. Let $\mathcal{L}(G)$ be the line graph of the graph G.

Lemma 3. *Let G be a graph with at least one edge. Then*

$$\gamma(G) \geqslant \frac{Z(G)}{e(G)}.$$

Proof. Let $m = e(G)$. Let $\mathbf{B}(G)$ be the vertex–edge incidence matrix of G. Then $\mathbf{L}^+(G) = \mathbf{B}(G)\mathbf{B}(\mathbf{G})^t$ and $\mathbf{B}(G)^t\mathbf{B}(\mathbf{G}) = 2\mathbf{I}_m + \mathbf{A}(\mathcal{L}(G))$ (see [3]), where $\mathbf{I}_m$ stands for the unit matrix of order m. Since $\mathbf{B}(G)\mathbf{B}(G)^t$ and $\mathbf{B}(G)^t\mathbf{B}(G)$ have the same non-zero eigenvalues, we have

$$\gamma(G) = \mu(\mathcal{L}(G)) + 2.$$

Note that for a graph H, $\mu(H)$ is bounded from below by the average degree $a(H)$ of H [3]. Obviously, $a(\mathcal{L}(G)) = \frac{2}{e(G)} \sum_{u \in V(G)} \binom{d_G(u)}{2} = \frac{Z(G)}{e(G)} - 2$. Then the result follows. □

We mention that the bound in previous lemma is attained if and only if $\mathcal{L}(G)$ is regular, as for the graph H, $\mu(H) = a(H)$ if and only if H is regular. A semi-regular graph is a bipartite graph for which every vertex in the same partite set has the same degree. For a connected graph G, $\mathcal{L}(G)$ is regular if and only if $d_G(u) + d_G(v)$ is a constant for any edge $uv \in E(G)$ if and only if G is regular or semi-regular.

Lemma 4 [6]. *Every k-regular graph on $2k+1$ vertices contains a Hamiltonian cycle, where $k \geqslant 2$.*

3. Result

Let G and H be vertex-disjoint graphs. The join of G and H is the graph formed from the (vertex-disjoint) union of G and H by adding all possible edges between them.

Let $\mathbb{EP}_n$ be the set of graphs of the following three types of graphs on n vertices: (a) a regular graph of degree $\frac{n}{2}-1$, (b) a graph consisting of two complete components, and (c) the join of a regular graph of degree $\frac{n}{2}-1-r$ and a graph on r vertices, where $1 \leqslant r \leqslant \frac{n}{2}-1$.

Let $\mathbb{EC}_n$ be the set of graphs of the following two types of graphs on n vertices: (a) the join of a trivial graph and a graph consisting of two complete components, and (b) the join of a regular graph of degree $\frac{n-1}{2}-r$ and a graph on r vertices, where $1 \leqslant r \leqslant \frac{n-1}{2}$.

Our result is:

Theorem 3. *Let G be a graph on n vertices with complement $\overline{G}$.*

(i) *If $\gamma(\overline{G}) \leqslant n$ and $G \notin \mathbb{EP}_n$, then G contains a Hamiltonian path.*
(ii) *If $n \geqslant 3$, $\gamma(\overline{G}) \leqslant n-1$ and $G \notin \mathbb{EC}_n$, then G contains a Hamiltonian cycle.*

Proof. We use the techniques from Fiedler and Nikiforov [5]. Let $H = C_{n-1}(G)$. If $H = K_n$, then the result follows from Lemma 1(i). Suppose that $H \neq K_n$ and G has no Hamiltonian path. By Lemma 1(i), H has no Hamiltonian path either. By Lemma 2(i) and the property of $(n-1)$-closure of G, $d_H(u) + d_H(v) \leqslant n-2$ for any pair of nonadjacent vertices u and v (always existing) in H. Thus

$$d_{\overline{H}}(u) + d_{\overline{H}}(v) = n-1-d_H(u)+n-1-d_H(v) \geqslant n$$

for any edge $uv \in E(\overline{H})$. It follows that

$$Z(\overline{H}) = \sum_{uv\in E(\overline{H})} \left(d_{\overline{H}}(u) + d_{\overline{H}}(v)\right) \geqslant ne(\overline{H}).$$

By Lemma 3, we have

$$\gamma(\overline{H}) \geqslant \frac{Z(\overline{H})}{e(\overline{H})} \geqslant n.$$

Since $\overline{H}$ is a subgraph of $\overline{G}$, by the Perron–Frobenius theorem,

$$\gamma(\overline{G}) \geqslant \gamma(\overline{H}) \geqslant n.$$

Since $\gamma(\overline{G}) \leqslant n$, we have $\gamma(\overline{G}) = \gamma(\overline{H}) = \frac{Z(\overline{H})}{e(\overline{H})} = n$, and then $d_{\overline{H}}(u) + d_{\overline{H}}(v) = n$ for any $uv \in E(\overline{H})$, implying that $\overline{H}$ contains exactly one nontrivial component F, which is either regular or semi-regular, where $\frac{n}{2}+1 \leqslant |V(F)| \leqslant n$. Suppose that F is semi-regular. Then $F = \overline{H}$ is a complete bipartite graph. Since $\gamma(\overline{G}) = \gamma(\overline{H})$, $\overline{H}$ is a subgraph of $\overline{G}$, we have by applying the Perron–Frobenius theorem (to the signless Laplacian matrices of $\overline{G}$ and $\overline{H}$) that $\overline{G} = \overline{H}$, and then G consists of two complete components, which contradicts the condition that G is not such a graph. Thus F is a regular graph of degree $\frac{n}{2}$ that is not semi-regular.

If $F = \overline{H}$, then since $\gamma(\overline{G}) = \gamma(\overline{H})$ and $\overline{H}$ is a subgraph of $\overline{G}$, we have by the Perron–Frobenius theorem that $\overline{G} = \overline{H}$, and thus $G(= H)$ is a regular graph of degree $\frac{n}{2}-1$, which contradicts the condition that G is not such a graph. Thus $\overline{H}$ consists of F and additional $r = n - |V(F)|$ isolated vertices, where $1 \leqslant r \leqslant \frac{n}{2}-1$. Note that $\gamma(\overline{G}) = \gamma(\overline{H})$ and $\overline{H}$ is a subgraph of $\overline{G}$. By the Perron–Frobenius theorem, $\overline{G}$ consists of vertex-disjoint graph F and a graph F_1 on r vertices. Thus G is the join of $\overline{F}$ (a regular graph of degree $\frac{n}{2}-1-r$) and $\overline{F_1}$ (a graph on r vertices), which contradicts the condition that G is not such a graph. This proves (i).

Now we prove (ii). Let $H' = C_n(G)$. If $H' = K_n$, then the result follows from Lemma 1(ii). Suppose that $H' \neq K_n$ and G has no Hamiltonian cycle. By Lemma 1(ii), H has no Hamiltonian cycle either. By Lemma 2(ii) and the property of n-closure of G, $d_{H'}(u) + d_{H'}(v) \leqslant n-1$ for any pair of nonadjacent vertices u and v (always existing) in H'. Thus

$$d_{\overline{H'}}(u) + d_{\overline{H'}}(v) = n-1-d_{H'}(u) + n-1-d_{H'}(v) \geqslant n-1$$

for any edge $uv \in E(\overline{H'})$. It follows that

$$Z(\overline{H'}) = \sum_{uv\in E(\overline{H'})} \left(d_{\overline{H'}}(u) + d_{\overline{H'}}(v)\right) \geqslant (n-1)e(\overline{H'}).$$

By Lemma 3, we have

$$\gamma(\overline{H'}) \geqslant \frac{Z(\overline{H'})}{e(\overline{H'})} \geqslant n-1.$$

Since $\overline{H'}$ is a subgraph of $\overline{G}$, by the Perron–Frobenius theorem, we have

$$\gamma(\overline{G}) \geqslant \gamma(\overline{H'}) \geqslant n-1.$$

Since $\gamma(\overline{G}) \leqslant n-1$, we have $\gamma(\overline{G}) = \gamma(\overline{H'}) = \frac{Z(\overline{H'})}{e(\overline{H'})} = n-1$, and then $d_{\overline{H'}}(u) + d_{\overline{H'}}(v) = n-1$ for any $uv \in E(\overline{H'})$, implying that $\overline{H'}$ contains exactly one nontrivial component F', which is either regular or semi-regular, where $\frac{n+1}{2} \leqslant |V(F')| \leqslant n$. Suppose that F' is semi-regular. Then F' contains at least $n-1$ vertices. Suppose that $\overline{H'}$ is connected. For $u \in V(\overline{H'})$, let $N(u)$ be the set of neighbors of u in $\overline{H'}$. For $uv \in E(\overline{H'})$, there is a vertex w such that $V(\overline{H'}) = N(u) \cup N(v) \cup \{w\}$. Then $(N(u), N(v) \cup \{w\})$ or $(N(u) \cup \{w\}, N(v))$, say the former is a bipartition of $\overline{H'}$, and thus $d_{\overline{H'}}(u)d_{\overline{H'}}(v) = (d_{\overline{H'}}(v)+1)d_{\overline{H'}}(u)$, which is a contradiction. Then $\overline{H'}$ is disconnected, and thus it consists of a complete bipartite graph F' and an additional isolated vertex. Since $\gamma(\overline{G}) = \gamma(\overline{H'})$ and $\overline{H'}$ is a subgraph of $\overline{G}$, we have by the Perron–Frobenius theorem that $\overline{G} = \overline{H'}$, and then G is the join of a trivial graph and a graph consisting of two complete components, which contradicts the condition that G is not such a graph. Thus F' is a regular graph of degree $\frac{n-1}{2}$ that is not semi-regular.

If $F' = \overline{H'}$, then by the Perron–Frobenius theorem, $\overline{G} = \overline{H'}$, and thus G $(= H')$ is a regular graph of degree $\frac{n-1}{2}$, which contradicts the conclusion of Lemma 4. Thus $\overline{H'}$ consists of F' and additional $r = n - |V(F')|$ isolated vertices, where $1 \leqslant r \leqslant \frac{n-1}{2}$. Note that $\gamma(\overline{G}) = \gamma(\overline{H'})$ and $\overline{H'}$ is a subgraph of $\overline{G}$. By the Perron–Frobenius theorem, $\overline{G}$ consists of vertex-disjoint graph F' and a graph F'_1 on r vertices. Thus G is the join of $\overline{F'}$ (a regular graph of degree $\frac{n-1}{2} - r$) and $\overline{F'_1}$ (a graph on r vertices), which contradicts the condition that G is not such a graph. This proves (ii). □

Let $K_{r,s}$ be the complete bipartite graph with r and s vertices in the two partite sets. Then $\mu(K_{r,s}) = \sqrt{rs}$.

For a bipartite graph G with n vertices, if $\overline{G}$ is not connected, then $\gamma(G) = n$.

Example. There are graphs to which Theorem 3 may apply but Theorems 1 and 2 may not.

(a) For odd $n \geqslant 5$, consider $G = K_{(n-1)/2,(n+1)/2}$. Then $\mu(G) = \frac{\sqrt{n^2-1}}{2}$. Since $\overline{G}$ consists of two components $K_{(n-1)/2}$ and $K_{(n+1)/2}$, and noting that $\mu(K_r) = r-1$, we have $\mu(\overline{G}) = \mu(K_{(n+1)/2}) = \frac{n-1}{2}$. Thus G does not satisfy the condition $\mu(G) \geqslant n-2$ of Theorem 1 (i) and (for $n \geqslant 7$) the condition $\mu(\overline{G}) \leqslant \sqrt{n-1}$ of Theorem 2(i). However, $\gamma(\overline{G}) = \gamma(K_{(n+1)/2}) = \frac{n-1}{2} + \mu(K_{(n+1)/2}) = n-1$, and thus G satisfies the conditions of Theorem 3(i), implying that G contains a Hamiltonian path.

(b) For even $n \geqslant 4$, consider $G' = K_{n/2,n/2}$. As above, we have $\mu(G') = \frac{n}{2}$, $\mu(\overline{G'}) = \mu(K_{n/2}) = \frac{n}{2} - 1$ and $\gamma(\overline{G'}) = \gamma(K_{n/2}) = n-2$. Thus G' does not satisfy the condition $\mu(G') > n-2$ of Theorem 1(ii) and (for $n \geqslant 8$) the condition $\mu(\overline{G'}) \leqslant \sqrt{n-2}$ of Theorem 2(ii), but satisfies the conditions of Theorem 3(ii), implying that G' contains a Hamiltonian cycle.

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