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A representation theorem for recovering contraction relations satisfying *wci* ☆

Zhaohui Zhu[a,b,*], Bin Li[b], Xi'an Xiao[c], Shifu Chen[b], Wujia Zhu[a]

[a] *Department of Computer Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, People's Republic of China*

[b] *State Key Lab of Novel Software Technology, Nanjing University, Nanjing 210093, People's Republic of China*

[c] *College of Science, PLA University of Science and Technology, Nanjing 211101, People's Republic of China*

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Abstract

A notion of an *image structure* associated with the canonical epistemic state is introduced. Based on it, we get a representation result for recovering contraction inference relations satisfying the condition *weak conjunctive inclusion* (*wci*) in terms of *F-standard epistemic AGM states*. In effect, this result establishes a representation theorem for belief contraction functions satisfying AGM postulates (k-1) – (k-7), and Rott's (*wci*) and (k-8c), and hence generalizes Rott's corresponding result in the finite framework. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Belief change is one of important researching topics in knowledge representation and reasoning. In certain sense, approaches suggested in this field can be broadly classified into two categories: coherentist and foundationalist approaches to belief change. In the

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* Corresponding author. Department of Computer Science, Nanjing University of Aeronautics and Astronautics, Nanjing 20016, People's Republic of China.

E-mail address: bnj4892856@jlonline.com (Z. Zhu).

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former category, the best known is AGM theory which is a milestone in this field (see [1]). In the latter category, Nebel, Fuhrmann and Hansson may be representative persons who suggest studying changes of base-generated belief sets (see [8,10,14]).

Coherentist approach takes deductively closed sets of propositions as belief states, and does not account for the fact that some of agent's beliefs may be reasons for other beliefs [2]. Contrastively, foundationalist approach regards the corpus of beliefs as generated by some set of basic propositions, and changes of these base-generated belief sets are determined by changes in their underlying base [2]. A more comprehensive comparison between coherentist and foundationalist approaches may be found in [11,17].

A common mechanism of the AGM and base-generation representation is a selection mechanism on certain sets of propositions. For AGM paradigm, the selection mechanism is set over maximal deductively closed subsets of the belief set, while for the base-generation representation, it is set over the sub-theories of the belief set that are generated by subsets of the base. In the literature, the selection mechanism is depicted in two different manners: choice function and preferential relation, and the former is more general than the latter from the mathematical point of view.

Recently, a general notion of an *epistemic state*, which subsumes both the AGM and base-generation paradigm, is introduced and studied by Bochman in [2,3,4,5,6]. In accordance with the opinion that [1] "a rational choice should be a relational choice", in Bochman's framework, the selection mechanism is characterized by a structure consisting of some sub-theories of the belief set, ordered by a preferential relation. So, the essential and common feature of AGM and base-generation representation is explicitly embodied in the notion of the epistemic state. In this sense, the notion of the *epistemic state* may be regarded as a generalization of common representations suggested for belief change.

However, in order to present a theory of belief change, it needs not only to provide an appropriate framework to represent agent's belief state but also to describe change operations based on this framework. For the moment, the operation of contraction over epistemic states has been studied in [2,3,5,6], however, for two other operations (i.e., expansion and revision) over epistemic states, just some initial steps are made in [2,4,5]. "*It turns out that the task of describing corresponding operations on epistemic states is far from being trivial or unequivocal*" [6]. It is worth stressing that, the shift in the level of representation results in some drastic changes in the character of the corresponding operations, in particular, Bochman's classification of operations on epistemic states does not correspond exactly to the distinction between contractions and expansions of belief sets. [2] However, both AGM theory and the theory of base change can be reconstructed in Bochman's framework by defining appropriate operations on the associated epistemic states (see [5]), [3] in this sense, we think that, Bochman's framework provides a general theory of belief change. Of course, as pointed out by Bochman himself, further work is needed (see [6]).

[1] Rott also holds the similar opinion, see [16].

[2] Quoted from [5, p. 335]

[3] In fact, some alternative approaches, such as Levi's coherentist theory or Fagin's flock-generated change, also can be naturally represented in Bochman's framework (see [6]).

It is well known that there exist intimate connections between belief change and nonmonotonic reasoning, the following slogan due to Gärdenfors and Makinson reflects this viewpoint (see [9,13]):

"*Belief change and nonmonotonic inference are actually two sides of the same coin.*"

In AGM paradigm, both contraction and revision operations on belief sets can correspond to nonmonotonic inferences with rules of the form "If α, then normally β" (see [9]). In Bochman's framework, similar work has been done also. Guided by the above slogan, Bochman introduces a new kind of nonmonotonic inference relation so-called *contraction relation* with contraction rules of the form "*If* α *is not believed*, *then normally* β", and shows that contraction relations correspond to contraction operations on epistemic states (see [3]). However, according to the intended meanings of contraction rules, it seems that, unlike the connection between the nonmonotonic inference and the traditional AGM theory, there is no natural and direct connection between contraction relations and revision operations on epistemic states. In other words, contraction relations can serve as counterparts to contraction operations on epistemic states, but not to revision operations. Fortunately, Bochman has shown that, another kind of inference relations called *skeptical inference relations* can directly correspond to revision operations on epistemic states, technical details may be found in [5].

In this paper, we concern ourselves with contraction relations. Bochman studies contraction relations and related postulates systematically, and provides semantic characterizations for various relations including AGM contraction and contractions that do not satisfy the postulate *Recovery* (see [3]). However, his results leave open questions of representation theorems for some contraction relations. This paper will deal with an important representation theorem for a special kind of contraction relations, in effect, our main result generalizes the one obtained by Hans Rott in [16]. Since the further explanation for the aim of this paper depends on some antecedent knowledge in this field, we leave it for the next section.

The rest of the paper is organized as follows: In Section 2, we recall some basic definitions and results related to this paper. A notion of an image structure is introduced in Section 3. In Section 4, we establish a representation theorem for a special kind of contraction relations in terms of *F-standard* epistemic AGM states. Finally, in Section 5, we compare our work with related ones appeared in the literature.

2. Preliminaries

In this section, we will recall some basic definitions and postulates from [3], which will be used in this paper. The motivations behind these definitions and postulates may be found in [3].

2.1. Contraction inference relations

We consider formulae of classical propositional calculus built over a set of atomic formulae denoted L plus two constants t and f (the formulae *true* and *false* respectively). We denote the set of all well formed formulae by $Form(L)$. As usual, $\models\alpha$ means that α is a tautology, $\models$ is the classical entailment relation, maximal consistent sets of formulae are called worlds, and sets of propositions closed with respect to $\models$ are called deductively closed. We use lower case letters of the Greek alphabet to denote formulae, and the letters v, v_1, $v_2, \ldots n, m, \ldots$ to denote worlds.

A contraction inference relation has rules of the form $\alpha \dashv \beta$ which are called contraction rules. The intended meaning of such rules is 'β should be believed in the absence of α' or 'If α is not believed then normally β'. Formally, a *contraction inference relation* is a binary relation over the set $Form(L)$ which satisfies the following postulates:

(C1) *Tautology*: $\alpha \dashv t$
(C2) *And*: If $\alpha \dashv \beta$ and $\alpha \dashv \gamma$ then $\alpha \dashv \beta \wedge \gamma$
(C3) *Right Weakening*: If $\alpha \models \beta$ and $\gamma \dashv \alpha$ then $\gamma \dashv \beta$
(C4) *Extensionality*: If $\models \alpha \leftrightarrow \beta$ and $\alpha \dashv \gamma$ then $\beta \dashv \gamma$
(CF) *Failure*: If $\alpha \dashv \alpha$ then $\alpha \dashv \beta$ if and only if $f \dashv \beta$
(C5) *Partial Antitony*: If $\alpha \wedge \beta \dashv \alpha$ then $\alpha \wedge \beta \wedge \gamma \dashv \alpha$
(C6) *Cautious Antitony*: If $\alpha \wedge \beta \dashv \alpha$ and $\beta \dashv \gamma$ then $\alpha \wedge \beta \dashv \gamma$
(C7) *Distributivity*: If $\alpha \dashv \beta$ and $\gamma \dashv \beta$ then $\alpha \wedge \gamma \dashv \beta$
(C8) *Cautious Monotony*: If $\alpha \wedge \beta \dashv \alpha \wedge \gamma$ then $\beta \dashv \gamma$

It turns out that preferential consequence relations introduced in [12] can be seen as a special case of contraction relations. More precisely, the former can be identified with a special kind of the latter satisfying the rule $\alpha \dashv \neg\alpha$, for any proposition α (see [3]).

In addition to the above postulates, some other interesting postulates are introduced in [3], among them, the following postulates are concerned in this paper.

(CI) *Inclusion*: If $\alpha \dashv \beta$, then $f \dashv \beta$.
(CV) *Vacuity*: If $f \not\dashv \alpha$ and $f \dashv \beta$, then $\alpha \dashv \beta$.
(CR) *Recovery*: If $f \dashv \beta$, then $\alpha \dashv \alpha \to \beta$.

A contraction relation will be called *coherent* if it satisfies *Inclusion* and *Vacuity*. A contraction relation will be called *recovering* contraction if it is *coherent* and satisfies *Recovery*.

Given a contraction relation $\dashv$, in the following, we will denote the set $\{\beta : \alpha \dashv \beta\}$ by $\bar{\alpha}$. A proposition α will be said to be *known* with respect to $\dashv$, if $\alpha \dashv \alpha$ holds; otherwise, it will be called *contingent*. The set of all propositions that are known with respect to $\dashv$ will be called the *knowledge set* and denoted by $K_{\dashv}$. A proposition α will be said to be *believed* with respect to $\dashv$, if $f \dashv \alpha$ holds. We will denote the set $\{\alpha : f \dashv \alpha\}$ by $B_{\dashv}$ and call it *belief set*.

Lemma 2.1 (Bochman [3]). *Given a contraction relation* $\dashv$, *we have*
(i) *If* $\alpha \wedge \beta \dashv \alpha \leftrightarrow \beta$ *then* $\bar{\alpha} = \bar{\beta}$.
(ii) *If* $\alpha \wedge \beta \dashv \alpha \to \beta$ *and* $\beta \wedge \gamma \dashv \beta \to \gamma$ *then* $\alpha \dashv \beta \to \gamma$.

The connections between the above postulates and the ones for belief contraction function appeared in the literature may be found in [3]. Here, for a future reference, we list AGM postulates for contraction function presented in [1] as follows:

(k-1) $Cn(K \div \alpha) = K \div \alpha$	(closure)
(k-2) $K \div \alpha \subseteq K$	(inclusion)
(k-3) If $\alpha \notin K$, then $K \div \alpha = K$	(vacuity)
(k-4) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin K \div \alpha$	(success)
(k-5) If $Cn(\alpha) = Cn(\beta)$, then $K \div \alpha = K \div \beta$	(extensionality)
(k-6) $K \subseteq Cn((K \div \alpha) \cup \{\alpha\})$	(recovery)
(k-7) $(K \div \alpha) \cap (K \div \beta) \subseteq K \div (\alpha \wedge \beta)$	(conjunction1)
(k-8) If $\alpha \notin K \div (\alpha \wedge \beta)$, then $K \div (\alpha \wedge \beta) \subseteq K \div \alpha$	(conjunction2)

2.2. *Epistemic states*

An epistemic state E in a language L is a triple $(B, l, \prec)$, where B is a set of objects called *admissible belief states*, l is a function assigning a deductively closed theory in L to every state from B, and $\prec$ is a strict partial order and called preferential relation on B. Clearly, the notion of the epistemic state is very similar to the preferential or cumulative model defined in [12].

Following Bochman, if $s \prec t$, we will say that the state t is preferred to the state s. Notice that this notation reverses the direction of preference as compared with common representations in the literature on nonmonotonic reasoning.

Given an epistemic state $E = (B, l, \prec)$, for any $A \subseteq B$, we will use $max(A)$ to denote the set of all maximal elements of A with respect to $\prec$, i.e., $max(A) = \{s : s \in A$ and $\neg\exists t(t \in A$ and $s \prec t)\}$. For a proposition α, we will denote the set of all admissible belief states from E that do not satisfy α by $]\alpha[_E$, i.e., $]\alpha[_E = \{s : s \in B$ and $\alpha \notin l(s)\}$. We will omit the subscript in $]\alpha[_E$ when there is no ambiguity. The range of l will be denoted by $rang(l)$, i.e., $rang(l) = \{v : \exists s(s \in B$ and $l(s) = v)\}$; For any $X \subseteq B$, the set $\{v : \exists s(s \in X$ and $l(s) = v)\}$ will be denoted by $l(X)$.

A proposition will be said to be *known* in E if it holds in all admissible belief states from E; otherwise, it will be called *contingent* in E. A proposition will be said to be *believed* in E if it holds in all maximally preferred admissible belief states from E. The set of all propositions believed in E will be called a *belief set* of E and will be denoted by B_E.

Various conditions imposed on epistemic states may give rise to notions of special epistemic states. In the following, we recall some notions concerned in this paper.

Definition 2.1. An epistemic state will be called *standard* if the labeling function is injective, that is, no two admissible belief states are labeled with the same deductively closed theory.

Clearly, for standard epistemic states, admissible belief states can be identified with their associated theories. So, a standard epistemic state can be alternately described as a pair $(T, \prec)$, where T is a set of deductively closed theories and $\prec$ a preference relation on T.

Definition 2.2. An epistemic state E will be called *coherent* if it contains an admissible belief state k such that, for any other admissible state t, $t \prec k$ and $l(t) \subseteq l(k)$.

Let K be a deductively closed theory. A theory $u \subseteq K$ will be called *K-maximal* if it is a maximal proper subtheory of K. The set of all *K-maximal* theories, coupled with K itself, will be denoted by $K_{\perp}$.

Definition 2.3. A coherent epistemic state E with a belief set K will be called an *epistemic AGM state* if $l(s) \in K_{\perp}$ for any admissible belief state s from E.

Definition 2.4. An epistemic state will be called (*negatively*) *smooth* if, for any contingent proposition α and $s \in]\alpha[$, either s is maximal preferred in $]\alpha[$, or there exists a more preferred state t that is maximal preferred in $]\alpha[$.

Definition 2.5. A contraction rule $\alpha \dashv \beta$ will be said to be valid with respect to an epistemic state E if either α is contingent in E and β holds in all preferred states from $]\alpha[$, or α is known in E and β is believed in E. The set of contraction rules that are valid in E will be denoted by $\dashv_E$.

Bochman establishes a number of representation theorems in terms of epistemic states for various contraction relations in [3]. Among them, he obtains the following theorem (representation theorem for recovering contraction inferences):

A contraction relation $\dashv$ is recovering if and only if there exists an epistemic AGM state E such that $\dashv$ coincides with $\dashv_E$.

As Bochman has pointed out, the above representation result refers to arbitrary epistemic AGM states, not only to standard ones. In [16], Rott has showed that, for relational AGM contractions, in order to have a representation in terms of standard AGM states we need to add some further conditions, Rott gives a sufficient condition called *weak conjunctive inclusion* (*wci*) [4] as follows, where K is a belief set, $\div$ is a contraction operator over K:

$$(wci) \quad K \div (\alpha \wedge \beta) \subseteq Cn(K \div \alpha \cup K \div \beta)$$

For contraction relations, the above condition *wci* may be represented as follows:

$$(wci) \quad \overline{\alpha \wedge \beta} \subseteq Cn(\bar{\alpha} \cup \bar{\beta}), \text{ where } \alpha \text{ and } \beta \text{ are contingent propositions.}$$

In the finite framework, the condition *wci* turns out to be also a necessary condition for a representation in terms of standard AGM states. For the moment, we do not know whether it is necessary in the infinite case. Anyway, just as pointed by Bochman, it seems to be a difficult task to exactly characterize contraction relations which admit of a representation in terms of standard epistemic AGM states.

This paper will introduce a notion of an *F-standard epistemic AGM state*, which is similar to the *standard model* introduced by Freund in [7], and establish a representation theorem for *recovering* contraction relations satisfying the condition *wci* in terms of *F-standard* epistemic AGM states. Since *recovering* contraction relations can be characterized as contraction relations satisfying the basic AGM postulates (k-1)–

[4] In [16], the postulate *wci* is called k-8r. This paper follows Hansson (see [11]), and calls it *weak conjunctive inclusion*.

(k-6) and the supplementary postulate (k-7) and Rott's (k-8c)[5] (see [3]), our result generalizes the one obtained in [16], where, in the finite case, Rott establishes a representation theorem for contraction functions satisfying postulates (k-1)–(k-7), (wci) and (k-8c) in terms of negatively transitively relational partial meet contraction functions (see Corollary 2 in [16]).

2.3. Canonical epistemic states

In this subsection, we will outline the construction of *canonical epistemic state* presented by Bochman, which plays a central role in establishing representation theorems for contraction relations in [3]. It will be easy to see that, the canonical epistemic state coincides, in effect, with the corresponding construction introduced by Kraus et al. in [12], however, the definitions in Bochman's framework are rather different from the ones in [12].

Definition 2.6. Given a contraction relation $\dashv$, for any formulas α and β, $\alpha \leqslant \beta$ if and only if $\alpha \wedge \beta \dashv \alpha \to \beta$.

Definition 2.7. Given a contraction relation $\dashv$, a deductively closed theory u will be called *normal* for a contingent proposition α if $\alpha \notin u$ and, for any β such that $\alpha \leqslant \beta$, either $\beta \in u$ or $\bar{\beta} \subseteq u$. A deductively closed theory u will be called *rec-normal* for a contingent proposition α if u coincides with $v \cap B_{\dashv}$, where v is a normal world for α. For any deductively closed theory u, we will denote the set $\{\alpha : u$ is normal (or, rec-normal) for $\alpha\}$ by Ω_u (respectively, Ω_u^{rec}).

The following lemma lists some properties obtained in [3], which will be used in this paper.

Lemma 2.2. *If $\dashv$ is a contraction relation, then*
(i) *If $\alpha \leqslant \beta$ and $\alpha \leqslant \gamma$ then $\alpha \leqslant \beta \wedge \gamma$.*
(ii) *A world u is normal for α if and only if it includes both $\neg\alpha$ and $\bar{\alpha}$.*
(iii) *If $\alpha \leqslant \beta$, u is a normal theory for α and $\beta \notin u$, then u is normal for β.*

In order to establish representation theorem for contraction relations, Bochman introduces the notion of the canonical epistemic state. Given a contraction relation $\dashv$, the canonical epistemic state $(B_{\mathrm{c}}, l_{\mathrm{c}}, \prec_{\mathrm{c}})$ for $\dashv$ is defined as follows:
(i) $B_{\mathrm{c}} = \{(u, \alpha) : \alpha$ is contingent and u is normal for $\alpha\}$,
(ii) $(u, \alpha) \prec_{\mathrm{c}} (v, \beta)$ if and only if $\beta \leqslant \alpha$ and $\alpha \in v$, and
(iii) $l_{\mathrm{c}}((u, \alpha)) = u$.

Based on the above construction, Bochman obtains the following result in [3]:

[5] (k-8c) If $\beta \in K \div (\alpha \wedge \beta)$, then $K \div (\alpha \wedge \beta) \subseteq K \div \alpha$.

Theorem 2.1. *Let* $\dashv$ *be a contraction relation. If* $E_c = (B_c, l_c, \prec_c)$ *is the canonical epistemic state for* $\dashv$, *then*
(i) $\prec_c$ *is a strict partial order*,
(ii) $]\alpha[$ *is smooth, for any contingent proposition* α, *and*
(iii) $\dashv = \dashv_{E_c}$.

As observed by Bochman, when dealing with the representation theorem for recovering contraction inferences, we can restrict our attention to rec-normal theories. In other words, for recovering contractions, the set B_c may be redefined as follows:

$$(\mathrm{i}')\quad B_c = \{(u,\alpha) : \alpha \text{ is contingent and } u \text{ is rec-normal for } \alpha\}.$$

In the following, such canonical epistemic states will be called canonical epistemic states for recovering contractions (canonical rec-epistemic state, for short). Here, the following two remarks should be noted.

First, since epistemic AGM states are coherent, any epistemic AGM state should contain an admissible belief state k such that, for any other admissible state t, $t \prec k$ and $l(t) \subseteq l(k)$. However, if a recovering contraction relation $\dashv$ satisfies $f \dashv f$, then the set B_c is empty,[6] so, its canonical epistemic state is not coherent at all. Hence, strictly speaking, the canonical rec-epistemic state is applicable just for the recovering contraction inference such that $f \not\dashv f$, and we must deal with the limited case in another manner.[7] In the next section, we will consider these two different cases.

Second, even if a recovering contraction $\dashv$ satisfies $f \not\dashv f$, the canonical rec-epistemic state for $\dashv$ is just 'almost' coherent in that all its preferred states are labeled by $B_\dashv$, and we will get a required epistemic AGM state by identifying all these states (see [3]).

For exploring the properties of the *image structure* induced by the canonical rec-epistemic state (defined in the next section), the following lemma is useful, which lists some properties of the canonical rec-epistemic state. Although many of them are implied in [3], for integrality, we give proofs of them here.

Lemma 2.3. *Let* $\dashv$ *be a recovering contraction relation. If* $E_c = (B_c, l_c, \prec_c)$ *is the canonical rec-epistemic state for* $\dashv$, *then*
(i) *If* $\alpha \leqslant \beta$, u *is rec-normal for* α *and* $\beta \notin u$, *then* u *is rec-normal for* β.
(ii) *If* $\alpha \leqslant \beta, \beta \leqslant \gamma$, u *is rec-normal for* α *and* $\beta \in u$, *then* $\gamma \in u$.
(iii) *The relation* $\prec_c$ *is a strict partial order.*
(iv) *Let* $\beta \in B_\dashv$ *and* u *be rec-normal for* β. *If* (u,β) *is maximal in* $]\alpha[$ *then* $\beta \leqslant \alpha$.
(v) *Let* u *be rec-normal for* β. *If* $\alpha \notin u$ *and* $\beta \leqslant \alpha$, *then* (u,β) *is maximal in* $]\alpha[$.
(vi) *If* $\beta \notin B_\dashv$ *and* u *is rec-normal for* β, *then* (u,β) *is a preferred state in* B_c.
(vii) *For any contingent proposition* α, *the set* $]\alpha[$ *is smooth.*

[6] By *Right Weakening*, *Extensionality* and $f \dashv f$, we get $\alpha \wedge f \dashv \beta \wedge f$ for any α and β. Further, by *Cautious Monotony*, $\alpha \dashv \beta$ holds, in particular, we have $\alpha \dashv \alpha$. So, for any formula α, α is not contingent.

[7] When dealing with the representation theorem for the contraction relation in terms of the epistemic state (i.e., Representation Theorem 1 in [3]), since the triple $\langle \emptyset, \emptyset, \emptyset \rangle$ is an epistemic state trivially, it is not needed to consider two different cases.

(viii) *If* (u,β) *is maximal in* $]\alpha[$ *then* u *is rec-normal for* α.

(ix) *If* α *is contingent, then* $\alpha \dashv \beta$ *iff* β *belongs to all theories that are rec-normal for* α.

(x) $\alpha \dashv \beta$ *iff* $\alpha \dashv_{E_c} \beta$.

Proof. (i) Since u is rec-normal for α, there exists a normal world k for α such that $u = k \cap B_\dashv$. We consider two cases.

First, suppose that $\beta \notin k$. Since k is normal for α, $\alpha \leqslant \beta$ and $\beta \notin k$, by (iii) from Lemma 2.2, k is normal for β. So, u (i.e., $k \cap B_\dashv$) is rec-normal for β.

Second, suppose that $\beta \in k$. From $\beta \notin k \cap B_\dashv$ and $\beta \in k$, we get $\beta \notin B_\dashv$. So, by *Inclusion* and *Vacuity*, we have $\bar{\beta} = B_\dashv$. Therefore, $B_\dashv$ is the only rec-normal theory for β. In the following, we will show $u = B_\dashv$. Since $\alpha \leqslant \beta$ and $\alpha \leqslant \alpha$, we obtain $\alpha \leqslant \alpha \wedge \beta \leqslant \alpha$. Thus, $\alpha \wedge \beta \dashv \alpha \wedge \beta \rightarrow \alpha$ and $\alpha \wedge \beta \dashv \alpha \rightarrow \alpha \wedge \beta$, and hence $\alpha \wedge \beta \dashv \alpha \leftrightarrow \alpha \wedge \beta$. Consequently, by (i) from Lemma 2.1, $\bar{\alpha} = \overline{\alpha \wedge \beta}$. Since $B_\dashv$ is deductively closed and $\beta \notin B_\dashv$, we get $\alpha \wedge \beta \notin B_\dashv$. Hence, by *Inclusion* and *Vacuity*, we have $\bar{\alpha} = \overline{\alpha \wedge \beta} = B_\dashv$. Immediately, $u = B_\dashv$ follows from $\bar{\alpha} \subseteq k$ and $u = k \cap B_\dashv$, as desired.

(ii) By (ii) from Lemma 2.1, from $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$, we get $\alpha \dashv \beta \rightarrow \gamma$. Since u is a rec-normal theory for α, by *Inclusion*, we obtain $\bar{\alpha} \subseteq u$. Furthermore, $\gamma \in u$ follows from $\beta \in u$ and u is deductively closed.

(iii) Immediately follows from the definition of $\prec_c$, the transitivity of $\leqslant$ and (ii) from this lemma.

(iv) Assume that (u,β) is maximal in $]\alpha[$, but $\beta \leqslant \alpha$ does not hold. Thus, there exists a world k including both $\overline{\alpha \wedge \beta}$ and $\beta \wedge \neg\alpha$. So, by (ii) from Lemma 2.2, k is normal for $\alpha \wedge \beta$. Hence, $k \cap B_\dashv$ is rec-normal for $\alpha \wedge \beta$. By $\beta \wedge \neg\alpha \in k$ and $\beta \in B_\dashv$, we get $\beta \in k \cap B_\dashv$. Hence, $(u,\beta) \prec_c (k \cap B_\dashv,\ \alpha \wedge \beta)$, which contradicts the maximality of (u,β) in $]\alpha[$.

(v) Assume that $\alpha \notin u$, $\beta \leqslant \alpha$ and (u,β) is not maximal in $]\alpha[$. So, there exists a state (v,γ) such that $(u,\beta) \prec_c (v,\gamma)$ and $\alpha \notin v$. Consequently, $\beta \in v$ and $\gamma \leqslant \beta$. Furthermore, by (ii) from this lemma, we get $\alpha \in v$, which contradicts $\alpha \notin v$.

(vi) Assume that $\beta \notin B_\dashv$, u is a rec-normal theory for β and (u,β) is not a preferred state in the canonical rec-epistemic state. So, there exists a state (v,γ) such that $(u,\beta) \prec_c (v,\gamma)$. Hence, $\beta \in v$. Since v is rec-normal for γ, we get $v \subseteq B_\dashv$, and hence $\beta \in B_\dashv$, a contradiction.

(vii) Suppose that $(u,\beta) \in]\alpha[$ and (u,β) is not maximal in $]\alpha[$. So, by (vi) from this lemma, we get $\beta \in B_\dashv$. Furthermore, by (v) from this lemma, we know that $\beta \leqslant \alpha$ does not hold. The construction in the proof of (iv) provides a state $(k \cap B_\dashv,\ \alpha \wedge \beta)$ such that $(u,\beta) \prec_c (k \cap B_\dashv, \alpha \wedge \beta)$ and $\alpha \notin k \cap B_\dashv$. Moreover, by (v) from this lemma, $(k \cap B_\dashv,\ \alpha \wedge \beta)$ is maximal in $]\alpha[$. Hence, the set $]\alpha[$ is smooth.

(viii) We consider two cases. First, suppose that $\beta \in B_\dashv$. By (i) and (iv) from this lemma, we know that u is rec-normal for α. Second, suppose that $\beta \notin B_\dashv$. So, by *Inclusion* and *Vacuity*, we get $u = B_\dashv$. Further, since (u,β) is maximal in $]\alpha[$, we get $\alpha \notin B_\dashv$. Hence, due to the deductive closure of $B_\dashv$, there exists a world k including both $\neg\alpha$ and $B_\dashv$. Obviously, k is normal for α and $B_\dashv \subseteq k$. Consequently, u (i.e., $k \cap B_\dashv$) is rec-normal for α.

(ix) The implication from left to right follows immediately from *Inclusion* and (ii) from Lemma 2.2. In the following, we deal with the other direction. Suppose that

β belongs to all theories that are rec-normal for α. So, by the definition of a rec-normal theory, we get $\beta \in B_{\dashv}$. Further, by *Recovery*, we obtain $\alpha \dashv \alpha \to \beta$. We want to show that $\alpha \dashv \beta$. Assume that $\alpha \not\dashv \beta$. Since $\alpha \dashv \alpha \to \beta$ and $\alpha \not\dashv \beta$, by *And* and *Right Weakening*, we have $\alpha \not\dashv \alpha \vee \beta$. Hence, the set $\bar{\alpha} \cup \{\neg\alpha, \neg\beta\}$ is consistent. So, there is a world k including $\bar{\alpha} \cup \{\neg\alpha, \neg\beta\}$. Clearly, $k \cap B_{\dashv}$ is rec-normal for α and $\beta \notin k \cap B_{\dashv}$, a contradiction.

(x) If $\dashv$ satisfies $f \dashv f$, $\dashv = \dashv_{E_c}$ holds trivially.[8] In the following, we suppose that $\dashv$ satisfies $f \not\dashv f$. So, f is contingent. Further, since the case when α is not contingent can be reduced to showing that $f \dashv \beta$ iff $f \dashv_{E_c} \beta$,[9] we may suppose that α is contingent. By (v) and (viii) from this lemma, the set $l_c(max(]\alpha[))$ coincides with the set of theories that are rec-normal for α. So, by (ix) from this lemma, we get $\{\beta : \alpha \dashv \beta\} = \{\beta : \alpha \dashv_{E_c} \beta\}$. $\square$

3. Image structures

This paper aims to establish a representation theorem for recovering contraction relations satisfying the postulate *wci* in terms of so-called *F-standard* epistemic AGM states introduced in the next section, thus, how to get a desired epistemic state for a given contraction relation is crucial. This section will introduce a notion of an *image structure* associated with the canonical epistemic state, a similar structure for the preferential model has appeared in [19,18]. As the technique in [19], in this paper, an image structure is regarded as a transforming of the epistemic state through which a desired epistemic state (i.e., *F-standard* epistemic AGM state) is obtained. This standpoint differs from Bochman's, he constructs a standard epistemic state directly based on a given contraction relation when dealing with the representation theorem for *rational* contractions (see [3]).

Definition 3.1. Let $E_c = \langle B_c, l_c, \prec_c \rangle$ be a canonical rec-epistemic state. The *image structure* associated with E_c is a triple $\langle B^*, id, \prec^* \rangle$, where

(i) $B^* = rang(l_c)$,
(ii) id is the identity function over B^*, and
(iii) $\prec^*$ is a binary relation over the set B^* such that for any $u, v \in B^*$,

$$u \prec^* v \text{ if and only if } \forall\alpha(\alpha \in \Omega_u^{\mathrm{rec}} \Rightarrow \exists\beta(\beta \in \Omega_v^{\mathrm{rec}} \text{ and } (u, \alpha) \prec_c (v, \beta))).$$

In the following, the image structure associated with E_c will be denoted by $\Delta(E_c)$. For any formula α, $]\alpha[_{\Delta(E_c)} =_{\mathrm{def}} \{u : u \in B^* \text{ such that } \alpha \notin u\}$ and $max(]\alpha[_{\Delta(E_c)}) =_{\mathrm{def}} \{u : u \in]\alpha[_{\Delta(E_c)}$ and there is no $v \in]\alpha[_{\Delta(E_c)}$ such that $u \prec^* v\}$.

The following two lemmas are trivial, but useful.

[8] See footnote 6.
[9] See the proof of Lemma 4.10 in [3].

Lemma 3.1. *If $\Delta(E_c)=\langle B^*, id, \prec^*\rangle$ is the image structure induced by the canonical rec-epistemic state $E_c=\langle B_c, l_c, \prec_c\rangle$, then $\prec^*$ is a strict partial order.*

Proof. By the transitivity of $\prec_c$, it is obvious that $\prec^*$ is transitive. In the following, we will show that $\prec^*$ is irreflexive. Suppose that there is a theory $u\in B^*$ such that $u\prec^* u$. Since $u\in B^*$, by the definition of canonical rec-epistemic state, there exists a contingent proposition α such that $\alpha\in\Omega_u^{\mathrm{rec}}$ and $(u,\alpha)\in B_c$. Hence, by (v) from Lemma 2.3, (u,α) is maximal in $]\alpha[_{E_c}$. On the other hand, by $u\prec^* u$, there exists $\beta\in\Omega_v^{\mathrm{rec}}$ such that $(u,\alpha)\prec_c(u,\beta)$. Since $\prec_c$ is irreflexive and $(u,\beta)\in]\alpha[_{E_c}, (u,\alpha)$ is not maximal in $]\alpha[_{E_c}$, a contradiction. Thus, the relation $\prec^*$ is irreflexive, as desired. □

Lemma 3.2. *If $\Delta(E_c)=\langle B^*, id, \prec^*\rangle$ is the image structure induced by the canonical rec-epistemic state $E_c=\langle B_c,\ l_c, \prec_c\rangle$, then $l_c(max(]\alpha[_{E_c}))\subseteq max(]\alpha[_{\Delta(E_c)})$.*

Proof. Suppose that $u\in l_c(max(]\alpha[_{E_c}))$. So, there exists a contingent proposition γ such that $(u,\gamma)\in B_c$ and $(u,\gamma)\in max(]\alpha[_{E_c})$. In the following, we will show $u\in max(]\alpha[_{\Delta(E_c)})$. Suppose not. Then, there is a theory $v\in B^*$ such that $u\prec^* v$ and $v\in]\alpha[_{\Delta(E_c)}$. Thus, there exists a contingent proposition β such that $(u,\gamma)\prec_c(v,\beta)$. Furthermore, by $(v,\beta)\in]\alpha[_{E_c}$, we get $(u,\gamma)\notin max(]\alpha[_{E_c})$, a contradiction. Hence, $u\in max(]\alpha[_{\Delta(E_c)})$, as desired. □

Obviously, in order to define the image structure induced by any canonical epistemic state, we need only to adopt Ω_u (Ω_v) instead of Ω_u^{rec} (respectively, Ω_v^{rec}) in Definition 3.1. Moreover, the above two lemmas also trivially hold for these image structures.

4. Representation theorem for recovering contraction relations satisfying *WCI*

In this section, based on the image structure, we will establish a representation theorem for recovering contraction relations satisfying the condition *wci* in terms of *F-standard* epistemic AGM states. The proof of the representation theorem given in this section will follow the general pattern of the proof given for preferential relations in [19]. However, the definitions involved in this paper are quite different from the ones in [19], and this will bring some drastic changes in the proof.

Definition 4.1. A recovering contraction relation $\dashv$ will be called *wci-recovering* contraction relation if it satisfies the following condition *wci*:

$$\overline{\alpha\wedge\beta}\subseteq Cn(\bar{\alpha}\cup\bar{\beta}),\quad \text{where } \alpha \text{ and } \beta \text{ are contingent propositions.}$$

Moreover, a contraction relation $\dashv$ will be said to be *consistent*, if $f\not\dashv f$ holds; otherwise, it will be said to be *inconsistent*.

Firstly, we deal with the limited case in which the contraction relation $\dashv$ is assumed to be inconsistent.

Observation 4.1. A contraction relation $\dashv$ is inconsistent if and only if $\dashv = \dashv_E$, where $E = \langle \{Form(L)\}, id, \emptyset \rangle$.

Proof. The implication from the right to the left follows immediately from Definition 2.5. In the following, we deal with the other direction. Suppose that $\dashv$ is inconsistent. For any proposition α and β, by *Right Weakening* and *Extensionality*, $\alpha \wedge f \dashv \beta \wedge f$ follows from $f \dashv f$. Further, by *Cautious Monotony*, we have $\alpha \dashv \beta$. So, by Definition 2.5, $\dashv = \dashv_E$ holds. □

In the rest of this section, we concern ourselves with consistent recovering contraction relations, however, we do not assume that contraction relations are consistent, and the following results, except for the ones indicated explicitly, also hold for inconsistent relations trivially.

Definition 4.2. Given a contraction relation $\dashv$, for any deductively closed theory u, u will be called *rec-normal closed* under conjunction if, for any contingent propositions α and β, $\{\alpha, \beta\} \subseteq \Omega_u^{\mathrm{rec}}$ implies $\alpha \wedge \beta \in \Omega_u^{\mathrm{rec}}$.

Lemma 4.1. *Let* $\dashv$ *be a* wci-recovering *contraction relation and* $E_c = \langle B_c, l_c, \prec_c \rangle$ *the canonical rec-epistemic state for* $\dashv$. *Then, for any deductively closed theory* $v \in rang$ (l_c), v *is rec-normal closed under conjunction.*

Proof. Suppose that v is rec-normal for the contingent proposition β_i, $i = 1, 2$. We will show that v is rec-normal for $\beta_1 \wedge \beta_2$. Since v is rec-normal for β_i $(i = 1, 2)$, there exist worlds u_1 and u_2 such that u_i is normal for β_i $(i = 1, 2)$ and $v = u_1 \cap B_\dashv = u_2 \cap B_\dashv$. Applying *Inclusion*, we get $\overline{\beta_1} \subseteq v$ and $\overline{\beta_2} \subseteq v$, and hence $\overline{\beta_1 \wedge \beta_2} \subseteq u_1$ follows from $\overline{\beta_1 \wedge \beta_2} \subseteq Cn(\overline{\beta_1} \cup \overline{\beta_2})$. By $\beta_1 \notin u_1$, we get $\beta_1 \wedge \beta_2 \notin u_1$. Consequently, by (ii) from Lemma 2.2, u_1 is normal for $\beta_1 \wedge \beta_2$. So, the theory v (i.e., $u_1 \cap B_\dashv$) is rec-normal for $\beta_1 \wedge \beta_2$, as desired. □

Lemma 4.2. *Let* $\dashv$ *be a* wci-recovering *contraction relation and* $E_c = \langle B_c, l_c, \prec_c \rangle$ *the canonical rec-epistemic state for* $\dashv$. *Suppose that the theory* $v \in rang(l_c)$. *Then, for any contingent proposition* α *such that* $\alpha \notin v$ *and* $\alpha \notin \Omega_v^{\mathrm{rec}}$, *the set* $\bigcup \{\overline{\alpha \wedge \beta} : \beta \in \Omega_v^{\mathrm{rec}}\} \cup \{\neg\alpha\} \cup \Omega_v^{\mathrm{rec}}$ *is consistent.*

Proof. Suppose that the set $\bigcup \{\overline{\alpha \wedge \beta} : \beta \in \Omega_v^{\mathrm{rec}}\} \cup \{\neg\alpha\} \cup \Omega_v^{\mathrm{rec}}$ is inconsistent. Hence, by the compactness, there exist $\beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1}, \ldots, \beta_m \in \Omega_v^{\mathrm{rec}}$ such that $\overline{\alpha \wedge \beta_1} \cup \overline{\alpha \wedge \beta_2} \cup \cdots \cup \overline{\alpha \wedge \beta_n} \vdash \beta_{n+1} \wedge \cdots \wedge \beta_m \to \alpha$. In the following, we denote $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n \wedge \beta_{n+1} \wedge \cdots \wedge \beta_m$ by θ. By Lemma 4.1, v is rec-normal for θ. Thus, we obtain $(v, \theta) \in B_c$. Since $\alpha \notin v$, we get $(v, \theta) \in]\alpha[_{E_c}$. By $\alpha \notin \Omega_v^{\mathrm{rec}}$ and (viii) from Lemma 2.3, we have $(v, \theta) \notin max(]\alpha[_{E_c})$. Further, by (vii) from Lemma 2.3, there exists a theory u and a contingent proposition β such that $(v, \theta) \prec_c (u, \beta)$ and $(u, \beta) \in max(]\alpha[_{E_c})$. So, $\beta \leqslant \theta$ and $\theta \in u$. We consider two cases as follows.

First, suppose that $\beta \in B_\dashv$. Thus, by (iv) from Lemma 2.3, $\beta \leqslant \alpha$ holds. Furthermore, by $\beta \leqslant \theta$ and (i) from Lemma 2.2, we get $\beta \leqslant \alpha \wedge \theta$. For any β_i $(1 \leqslant i \leqslant m)$, from $\beta \leqslant \alpha\wedge\theta \leqslant \alpha\wedge\beta_i$ and $\overline{\alpha\wedge\beta_i} \notin u$, by (i) from Lemma 2.3, we obtain $\alpha\wedge\beta_i \in \Omega_u^{\mathrm{rec}}$. So, $\overline{\alpha\wedge\beta_1} \cup \overline{\alpha\wedge\beta_2} \cup \cdots \cup \overline{\alpha\wedge\beta_n} \subseteq u$. Further, since u is deductively closed, $\alpha \in u$ immediately follows from $\theta \in u$ and $\overline{\alpha\wedge\beta_1} \cup \overline{\alpha\wedge\beta_2} \cup \cdots \cup \overline{\alpha\wedge\beta_n} \vdash \beta_{n+1}\wedge\cdots\wedge\beta_m \to \alpha$, which contradicts $(u,\beta) \in max(]\alpha[_{E_c})$.

Second, suppose that $\beta \notin B_{\dashv}$. Hence, by *Inclusion* and *Vacuity*, we obtain $u = B_{\dashv}$. So, by *Inclusion*, $\overline{\alpha\wedge\beta_1} \cup \overline{\alpha\wedge\beta_2} \cup \cdots \cup \overline{\alpha\wedge\beta_n} \subseteq u$ holds. Analogously, we have $\alpha \in u$, which contradicts $(u,\beta) \in max(]\alpha[_{E_c})$.

Consequently, the set $\bigcup\{\overline{\alpha\wedge\beta} : \beta \in \Omega_v^{\mathrm{rec}}\} \cup \{\neg\alpha\} \cup \{\beta : \beta \in \Omega_v^{\mathrm{rec}}\}$ is consistent. $\square$

Lemma 4.3. *Let $\dashv$ be a* wci-recovering *contraction relation and $E_c = \langle B_c, l_c, \prec_c\rangle$ the canonical rec-epistemic state for $\dashv$. Suppose that $v \in rang(l_c)$ and there exists a contingent proposition α such that $\alpha \notin v$ and $\alpha \notin \Omega_v^{\mathrm{rec}}$, then $\Omega_v^{\mathrm{rec}} \subseteq B_{\dashv}$.*

Proof. Since v is not rec-normal for α, by (viii) from Lemma 2.3, $(v,\gamma) \notin max(]\alpha[)$ for any $\gamma \in \Omega_v^{\mathrm{rec}}$. Thus, for any $\gamma \in \Omega_v^{\mathrm{rec}}$, by the smoothness of $\prec_c$ and $(v,\gamma) \in]\alpha[$, there exists $(u_\gamma, \delta_\gamma) \in B_c$ such that $(v,\gamma) \prec_c (u_\gamma, \delta_\gamma)$. So, by the definition of the canonical epistemic state, we get $\gamma \in u_\gamma$, further, $\gamma \in B_{\dashv}$ immediately follows from $u_\gamma \subseteq B_{\dashv}$. Thus, $\Omega_v^{\mathrm{rec}} \subseteq B_{\dashv}$, as desired. $\square$

Lemma 4.4. *Let $\dashv$ be a* wci-recovering *contraction relation and $E_c = \langle B_c, l_c, \prec_c\rangle$ the canonical rec-epistemic state for $\dashv$. Suppose that the theory $v \in rang(l_c)$. Then, for any contingent proposition α such that $\alpha \notin v$ and $\alpha \notin \Omega_v^{\mathrm{rec}}$, there exists a theory $u_0 \in rang(l_c)$ such that $v \prec^* u_0$ and u_0 is rec-normal for α.*

Proof. Suppose that α is a contingent proposition such that $\alpha \notin v$ and v is not rec-normal for α. By Lemma 4.2, the set $\bigcup\{\overline{\alpha\wedge\beta} : \beta \in \Omega_v^{\mathrm{rec}}\} \cup \{\neg\alpha\} \cup \Omega_v^{\mathrm{rec}}$ is consistent. Thus, there exists a world k including this set. Obviously, for any $\gamma \in \Omega_v^{\mathrm{rec}}$, by (ii) and (iii) from Lemma 2.2, k is normal for both α and $\alpha\wedge\gamma$, further, $k \cap B_{\dashv}$ is rec-normal for them. On the other hand, by Lemma 4.3, $\Omega_v^{\mathrm{rec}} \subseteq B_{\dashv}$. Consequently, $\Omega_v^{\mathrm{rec}} \subseteq k \cap B_{\dashv}$. So, for any $\gamma \in \Omega_v^{\mathrm{rec}}$, we have $(v,\gamma) \prec_c (k \cap B_{\dashv}, \alpha\wedge\gamma)$. Hence, $v \prec^* k \cap B_{\dashv}$ holds, as desired. $\square$

Lemma 4.5. *Let $\dashv$ be a* wci-recovering *contraction relation and $E_c = \langle B_c, l_c, \prec_c\rangle$ the canonical rec-epistemic state for $\dashv$. Then, the relation $\prec^*$ is smooth.*

Proof. Suppose that α is contingent, and u is a theory such that $u \in]\alpha[_{\Delta(E_c)}$ and $u \notin max(]\alpha[_{\Delta(E_c)})$. Since $u \in]\alpha[_{\Delta(E_c)}$ and $u \notin max(]\alpha[_{\Delta(E_c)})$, by Lemma 3.2 and (v) from Lemma 2.3, the theory u is not rec-normal for α. Hence, by Lemma 4.4, there exists a theory v such that $u \prec^* v$ and v is rec-normal for α. Moreover, by Lemma 3.2 and (v) from Lemma 2.3, $v \in max(]\alpha[_{\Delta(E_c)})$ holds. $\square$

Lemma 4.6. *Let $\dashv$ be a* wci-recovering *contraction relation and $E_c = \langle B_c, l_c, \prec_c\rangle$ the canonical rec-epistemic state for $\dashv$. Then, $l_c(max(]\alpha[)) = max(]\alpha[_{\Delta(E_c)})$ for any contingent proposition α.*

Proof. By Lemma 3.2, it is enough to show that $l_c(max(]\alpha[))\supseteq max(]\alpha[_{\Delta(E_c)})$. Suppose that $u\in max(]\alpha[_{\Delta(E_c)})$. By Lemma 4.4, u is rec-normal for α. So, by (v) from Lemma 2.3, the state (u,α) is maximal in $]\alpha[_{E_c}$. Hence, $u\in l_c(max(]\alpha[_{E_c}))$, as desired. □

As observed in [3], given a (consistent) recovering contraction, the canonical epistemic AGM state for it is just 'almost' coherent in that all its preferred states are labeled by $B_\dashv$, and we will get a desired epistemic AGM state by identifying all these states. However, for any *wci-recovering* contraction relation such that $f\not\dashv f$, the image structure induced by its canonical rec-epistemic AGM state is a genuine epistemic AGM state. Formally, we have the following lemma.

Lemma 4.7. *Let* $\dashv$ *be a* consistent wci-recovering *contraction relation and* $E_c=\langle B_c, l_c,\prec_c\rangle$ *the canonical rec-epistemic state for* $\dashv$. *Then*, $\Delta(E_c)=\langle B^*,id,\prec^*\rangle$ *is a standard epistemic AGM state.*

Proof. First, we show $B_\dashv\in B^*$. Since $f\not\dashv f$, there exists a contingent proposition γ such that $\gamma\notin B_\dashv$. By *Inclusion* and *Vacuity*, we get $\bar{\gamma}=B_\dashv$, further, $B_\dashv$ is rec-normal for γ. Hence, $B_\dashv\in B^*$, as desired.

Second, we show that $B_\dashv$ is the most preferred admissible belief state in $\Delta(E_c)$ and includes all other admissible states. Suppose that u is a state from $\Delta(E_c)$ such that $u\neq B_\dashv$. So, $u\subset B_\dashv$ immediately follows from the definition of the rec-normal theory. In the following, we will show $u\prec^* B_\dashv$. For any $\gamma\in\Omega_u^{\mathrm{rec}}$, since u is rec-normal for γ and $u\subset B_\dashv$, by *Vacuity*, we get $\gamma\in B_\dashv$. Since $f\not\dashv f$, there exists a contingent proposition α such that $\alpha\notin B_\dashv$. By the deductive closure of $B_\dashv$, $\alpha\wedge\gamma\notin B_\dashv$ holds. So, by *Inclusion* and *Vacuity*, $B_\dashv$ is rec-normal for $\alpha\wedge\gamma$. Obviously, $(u,\gamma)\prec_c(B_\dashv,\alpha\wedge\gamma)$. Since γ is taken to be an arbitrary proposition from Ω_u^{rec}, by the definition of the relation $\prec^*$, we obtain $u\prec^* B_\dashv$, as desired. Further, by Lemmas 3.1 and 4.5, $\Delta(E_c)$ is a coherent epistemic state.

Finally, we show that, for any $v\in B^*$, v is $B_\dashv$ itself or a maximal deductively closed subset of $B_\dashv$. Suppose that $v\in B^*$ and $v\neq B_\dashv$. By the definition of rec-normal theory, we know that v is a subtheory of $B_\dashv$. In the following, we will show that v is a maximal proper subtheory of $B_\dashv$. Suppose that there exists a deductively closed theory u such that $v\subset u\subset B_\dashv$. Since $v\in B^*$, there exists a contingent proposition α such that $\alpha\in\Omega_v^{\mathrm{rec}}$. By the definition of the rec-normal theory, there exists a world k such that $v=k\cap B_\dashv$ and k is normal for α. From $v\subset u\subset B_\dashv$ and *Recovery*, we get $\alpha\notin u$. Since $v\subset u$, there is a proposition β such that $\beta\in u-v$. Since $\beta\in B_\dashv$ and $\beta\notin v$, we get $\neg\beta\in k$. So, $\beta\to\alpha\in k$. From $v\subset B_\dashv$ and $\alpha\in\Omega_v^{\mathrm{rec}}$, by *Vacuity*, $\alpha\in B_\dashv$ holds. So, $\beta\to\alpha\in B_\dashv$. Thus, $\beta\to\alpha\in k\cap B_\dashv=v$. Further, by $v\subset u$, we obtain $\beta\to\alpha\in u$. So, a contradiction immediately follows from $\alpha\notin u$, $\beta\to\alpha\in u$, $\beta\in u$ and u is deductively closed.

Consequently, $\Delta(E_c)$ is a standard epistemic AGM state. □

For any inconsistent *wci-recovering* contraction relation $\dashv$, since $B_c=B^*=\emptyset$,[10] the above lemma does not hold.

[10] See footnote 6.

As a consequence of the above lemmas, we immediately obtain the following corollary.

Corollary 4.1. *Let* $\dashv$ *be a* wci-recovering *contraction relation and* $E_c = \langle B_c, l_c, \prec_c \rangle$ *the canonical rec-epistemic state for* $\dashv$. *Then,* $\dashv_{E_c} = \dashv_{\Delta(E_c)}$.

Proof. If $\dashv$ is inconsistent, then, $\dashv_{E_c} = \dashv_{\Delta(E_c)}$ immediately follows from $E_c = \Delta(E_c) = \langle \emptyset, \emptyset, \emptyset \rangle$. In the following, we suppose that $\dashv$ is consistent.

By Definition 3.1, we get $rang(l_c) = B^*$, so, for any formula α, α is contingent with respect to E_c if and only if it is contingent with respect to $\Delta(E_c)$. If α is contingent, then, by Lemma 4.6, we have, $\alpha \dashv_{E_c} \beta$ iff $\alpha \dashv_{\Delta(E_c)} \beta$, for any β. Now, suppose that α is known. By the proof of Lemma 4.7, $B_{\dashv}$ is the only most preferred admissible belief state in both $\Delta(E_c)$ and E_c, so, $\alpha \dashv_{E_c} \beta$ iff $\alpha \dashv_{\Delta(E_c)} \beta$, for any β. Consequently, $\dashv_{E_c} = \dashv_{\Delta(E_c)}$ holds, as desired. □

For any consistent *wci-recovering* contraction relation $\dashv$, by (x) from Lemma 2.3, we know that the canonical rec-epistemic state E_c for $\dashv$ satisfies $\dashv = \dashv_{E_c}$, furthermore, by Corollary 4.1 and Lemma 4.7, $\Delta(E_c)$ is a standard epistemic AGM state such that $\dashv = \dashv_{\Delta(E_c)}$. On the other hand, if $\dashv$ is inconsistent, by Observation 4.1, we have $\dashv = \dashv_E$, where $E = \langle \{Form(L)\}, id, \emptyset \rangle$ is a *standard* epistemic AGM state trivially. In conclusion, we have the following theorem:

Theorem 4.1. *Let* $\dashv$ *be a* wci-recovering *contraction relation, then there exists a standard epistemic AGM state E such that* $\dashv = \dashv_E$.

In order to establish the representation theorem for *wci-recovering* contraction relation in the infinite framework, we introduce a notion of an *F-standard epistemic AGM state* as follows, which is similar to the *standard model* introduced in [2].

Definition 4.3. A standard epistemic AGM state $E = \langle B, \prec \rangle$ will be said to be *F-standard* if it satisfies the following condition:

For any contingent proposition α and deductively closed theory u, $u \in max(]\alpha[)$ if and only if u is rec-normal for α with respect to $\dashv_E$.

Corollary 4.2. *Let* $\dashv$ *be a* wci-recovering *contraction relation, then there exists an F*-standard *epistemic AGM state E such that* $\dashv = \dashv_E$.

Proof. We consider two cases as follows.

First, suppose that $\dashv$ is consistent. Let E_c be the canonical rec-epistemic state induced by $\dashv$. By Lemma 4.7, we know that the image structure $\Delta(E_c)$ is a *standard* epistemic AGM state. Further, according to the construction of canonical rec-epistemic state, (v) and (viii) from Lemmas 2.3 and 4.6, $\Delta(E_c)$ is *F-standard*, moreover, by Corollary 4.1 and (x) from Lemma 2.3, we have $\dashv = \dashv_{\Delta(E_c)}$.

Second, suppose that $\dashv$ is inconsistent. Obviously, $E = \langle \{Form(L)\}, id, \emptyset \rangle$ is *F-standard* trivially, further, by Observation 4.1, $\dashv = \dashv_E$ holds. □

It is worth pointing out that, when the language is finite, *F-standard* epistemic AGM states coincide with *standard* ones, formally, we have the following observation. A similar observation for the preferential model is due to Lehmann [16, p. 236].

Observation 4.2. In the finite framework, $E=\langle B,\prec\rangle$ is a *standard* epistemic AGM state if and only if it is *F-standard*.

Proof. The implication from the right to the left follows immediately from Definition 4.3. In the following, we deal with the other direction.

First, suppose that $u\in max(]\alpha[)$. So, by the definition of the epistemic AGM state, $u\subseteq B_E$ holds. In the following, we will show that, u is rec-normal for α with respect to $\dashv_E$. From $u\in max(]\alpha[)$, we get $\alpha\notin u$ and $\bar{\alpha}\subseteq u$, where $\bar{\alpha}=\{\beta:\alpha\dashv_E\beta\}$. Since E is an epistemic AGM state, the relation $\dashv_E$ satisfies *Inclusion* and *Vacuity* (see [3]). We consider two cases.

Case 1, suppose that $\alpha\notin B_E$. By *Inclusion* and *Vacuity*, we get $\bar{\alpha}=B_E$, further, $u=B_E$ follows from $u\subseteq B_E$ and $\bar{\alpha}\subseteq u$. Since u is a consistent theory such that $\alpha\notin u$, there exists a world k including $u\cup\{\neg\alpha\}$. Clearly, k is a normal world for α such that $u=k\cap B_E$.

Case 2, suppose that $\alpha\in B_E$. So, $u\neq B_E$ follows from $\alpha\in B_E-u$. In the following, we will show that the set $u\cup\{\neg\beta:\beta\in B_E-u\}$ is consistent. Otherwise, there are $\beta_1,\beta_2,\ldots,\beta_n\in B_E-u$ such that $\theta\in u$, where $\theta=\beta_1\vee\beta_2\vee\cdots\vee\beta_n$. By the definition of the epistemic AGM state and $u\neq B_E$, u is a maximal proper subtheory of B_E, so, we have $\beta_i\to\alpha\in u$ for any β_i $(1\leqslant i\leqslant n)$, [11] further, $\theta\to\alpha\in u$. Hence, we get $\alpha\in u$, which contradicts $u\in max(]\alpha[)$. Consequently, the set $u\cup\{\neg\beta:\beta\in B_E-u\}$ is consistent, and there exists a world k including it. Obviously, by (ii) from Lemma 2.2, k is normal for α, and $u=k\cap B_E$ follows from $k\supseteq u\cup\{\neg\beta:\beta\in B_E-u\}$ and $u\subseteq B_E$.

Together case 1 with case 2, we know that, u is rec-normal for α with respect to $\dashv_E$.

Second, suppose that u is rec-normal for α with respect to $\dashv_E$. In the following, we want to show $u\in max(]\alpha[)$. Suppose not. Since E is an epistemic AGM state, B_E is the most preferred admissible belief state in E. So, $u\neq B_E$, and u is a maximal proper subtheory of B_E. [12] By *Vacuity* and $\bar{\alpha}\subseteq u\subset B_E$, $\alpha\in B_E$ holds. From the first part of this proof and the finiteness of the language, by the definition of the rec-normal theory, the set $max(]\alpha[)$ is finite. So, we may suppose that $max(]\alpha[)=\{v_1,v_2,\ldots,v_n\}$. Hence, for any v_i $(1\leqslant i\leqslant n)$, v_i is a maximal proper subtheory of B_E such that $v_i\neq u$. Further, by the maximality of v_i and $u\subseteq B_E$, there exists β_i such that $\beta_i\in v_i-u$ for any $1\leqslant i\leqslant n$. Consequently, $\theta\in\bar{\alpha}$, where $\theta=\beta_1\vee\beta_2\vee\cdots\vee\beta_n$. Since u is rec-normal for α, we get $\theta\in u$. On the other hand, for any β_i $(1\leqslant i\leqslant n)$, we have $\beta_i\to\alpha\in u$, [13] so, $\theta\to\alpha\in u$. Thus, $\alpha\in u$ follows from $\theta\to\alpha\in u$ and $\theta\in u$, which contradicts the assumption that u is rec-normal for α. □

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[11] Otherwise, suppose that $\alpha\notin Cn(u\cup\{\beta_i\})$. So, $Cn(u\cup\{\beta_i\})$ is a proper subtheory of B_E, and includes u properly, which contradicts the fact that u is a maximal subtheory of B_E.

[12] See the proof of Lemma 4.7 in this paper.

[13] See footnote 11.

Lemma 4.8. *If a contraction relation $\dashv$ is represented by an F-standard epistemic AGM state $E=\langle B,\prec\rangle$, then $\dashv$ is a* wci-recovering *contraction relation.*

Proof. Since the contraction relation determined by an epistemic AGM state is a recovering contraction (see [3]), it is enough to show that $\dashv_E$ (i.e., $\dashv$) satisfies the condition *wci*. Let α and β be two contingent propositions. Suppose that $\gamma\in\overline{\alpha\wedge\beta}$. We want to show $\gamma\in Cn(\bar{\alpha}\cup\bar{\beta})$. Assume that $\gamma\notin Cn(\bar{\alpha}\cup\bar{\beta})$. Since $\gamma\in\overline{\alpha\wedge\beta}$, by *Inclusion*, we get $f\dashv\gamma$. By *Recovery*, $\alpha\dashv\alpha\to\gamma$ and $\beta\dashv\beta\to\gamma$ hold. By *Tautology*, we have $\alpha\dashv\gamma\to\gamma$. Further, since $\gamma\notin Cn(\bar{\alpha}\cup\bar{\beta})$ and $\{\alpha\to\gamma,\beta\to\gamma,\gamma\to\gamma\}\subseteq Cn(\bar{\alpha}\cup\bar{\beta})$, we obtain $\alpha\vee\beta\vee\gamma\notin Cn(\bar{\alpha}\cup\bar{\beta})$. Hence, the set $\bar{\alpha}\cup\bar{\beta}\cup\{\neg(\alpha\vee\beta\vee\gamma)\}$ is consistent. So, there exists a world k including $\bar{\alpha}\cup\bar{\beta}\cup\{\neg(\alpha\vee\beta\vee\gamma)\}$. Obviously, by (ii) from Lemma 2.2, $k\cap B_{\dashv}$ is rec-normal for both α and β. By the definition of the *F-standard* epistemic AGM state, $k\cap B_{\dashv}\in max(]\alpha[)\cap max(]\beta[)$. Since $max(]\alpha[)\cap max(]\beta[)\subseteq max(]\alpha\wedge\beta[)$, we get $k\cap B_{\dashv}\in max(]\alpha\wedge\beta[)$. A contradiction immediately follows from $\gamma\in\overline{\alpha\wedge\beta}$ and $\gamma\notin k\cap B_{\dashv}$. Consequently, $\gamma\in Cn(\bar{\alpha}\cup\bar{\beta})$ holds, as desired. □

Theorem 4.2 (Representation theorem). *$\dashv$ is a* wci-recovering *contraction relation if and only if $\dashv$ is represented by an F-standard epistemic AGM state.*

Proof. It immediately follows from Lemma 4.8 and Corollary 4.2. □

REMARK We have heard one comment about the above theorem that, since the notion of the *F-standard* epistemic state is defined in terms of the generated contraction relation, this notion is defined essentially with respect to the source contraction relation, so, the above theorem is circular and is not fully satisfactory. We tend to disagree with this comment. We think that, Definition 4.3 provides a nontrivial sufficient condition which guarantees the generated relation to be a *wci-recovering* relation (see Lemma 4.8). In fact, this definition only refers to the generated contraction relation but not the condition *wci*. Moreover, if someone does not tolerate that the notion of the *F-standard* epistemic state is defined in terms of the generated relation, we can redefine this notion equivalently as follows:

A standard epistemic AGM state $E=\langle B,\prec\rangle$ will be said to be *F-standard* if, for any contingent proposition α and deductively closed theory u, $u\in max(]\alpha[)$ if and only if there exists a world v satisfying the following conditions:

(i) $\neg\alpha\in v$,
(ii) $u=v\cap B_E$, and
(iii) $\bigcap max(]\alpha[)\subseteq v$.

Clearly, according to Definition 2.7 and (ii) from Lemma 2.2, the above definition coincides with Definition 4.3, and does not refer to the generated contraction relation at all. So, we do not think that Theorem 4.2 is circular. However, since the notion of an *F-standard* state is defined in the slightly unusual manner as compared with ones in the literature, a representation result for *wci-recovering* relations in terms of a species of epistemic states, which is depicted in a nice manner, will improve the above result remarkably.

Together the above theorem with Observation 4.2, the following corollary is obtained trivially. A similar result in the AGM tradition is due to Rott (see [16]).

Corollary 4.3. *In the finite framework*, $\dashv$ *is a* wci-recovering *contraction relation if and only if* $\dashv$ *is represented by a* standard *epistemic AGM state*.

5. Discussion

This paper establishes a representation theorem for recovering contraction relations satisfying *weak conjunctive inclusion* in terms of *F-standard* epistemic AGM states, in effect, this theorem generalizes a result obtained by Rott in the finite framework. Recently, Bochman explores the semantic character for a more wider category of contraction relations (i.e., so-called recovering contraction relations), and obtains a representation theorem for them based on the notion of the epistemic AGM state. As observed by Rott, the difference between epistemic AGM states and standard ones is essential, even in the finite framework, there exist contraction relations generated by epistemic AGM states, which are not representable by standard ones (see [3]). Theorem 4.1 in this paper indicates that, the condition *wci* is sufficient for recovering contraction relations to be represented by standard epistemic AGM states. Rott also obtains a similar result in the AGM tradition with the distinct technical strategy in [16]. Moreover, the result due to Rott in [12] [14] and Corollary 4.3 in this paper demonstrate that, in the finite case, the condition *wci* is also necessary for a standard representation in both the AGM tradition and Bochman's framework, respectively. However, in the infinite case, since the difference between standard epistemic AGM states and *F-standard* ones seemed to be essential, Theorem 4.2 in this paper suggests that, *wci* is not necessary for recovering contraction relations to be represented by standard ones. In other words, it seems that the condition *wci* is not valid in *standard* epistemic AGM states in the infinite framework. Incidentally, results in [16] and this paper indicate that, the condition *wci* is valid in both *standard* epistemic AGM states in the finite language and any *F-standard* ones, but is not valid in epistemic AGM states even in the finite case.

Anyway, for the moment, it seems to be a difficult task to exactly characterize contraction relations which admit a representation in terms of standard epistemic AGM states. The similar difficulty also appears in the nonmonotonic logic, for instance, Pino Pérez and Uzcátegui write in [15]:

> "*There two families of consequence relation seem so complex that we will not be surprised if there is no such a characterization* (*at least in terms of the type of postulates used so far to classify consequence relations*)".

In the above quote, two families point at preferential relations which are representable by injective preferential models and standard preferential models introduced by Freund in [7], respectively. In the finite framework, Freund provides a

[14] See Corollary 2 in [16].

representation theorem for preferential relations satisfying the condition *WDR*[15] in terms of injective preferential models (see [7]), which corresponds to the result due to Rott in [16] and Corollary 4.3 in this paper.

Inspired by Freund's work, in [19], we introduce a notion of a valuation structure which consists of worlds ordered by a binary relation defined in [18], and present a canonical approach to obtain an injective preferential model for any preferential relation satisfying the property *WDR*. Furthermore, we give uniform proofs of representation theorems for injective preferential relations appeared in the literature, in particular, we provide the semantic character for preferential inference relations satisfying *WDR* (see, Theorem 4.3 in [17]). This result may be regarded as a counterpart of Theorem 4.2 in this paper for nonmonotonic logics, in effect, which establishes the semantic character for *recovering* contraction relations satisfying both $\alpha \dashv \neg\alpha$ and *wci*. However, due to manifest differences between the definitions involved in [19] and the ones in this paper, the proofs in this paper are rather different from [19].

Recently, Pino Pérez and Uzcátegui also give a uniform and simple framework to prove the hard part of representation theorems for injective preferential relations based on *essential pre-structure* (see [15]). From the present results, strictly speaking, their method provides a uniform approach to construct injective preferential models only for any preferential relations satisfying *Disjunctive Rationality*[16] (see [19]). They show that, if a preferential inference relation satisfies *WDR*, then it may be generated by the *essential pre-structure* associated with it. Unfortunately, in the infinite framework, so far it is unknown whether this structure is an injective preferential model when a given inference relation satisfies *WDR* but not *Disjunctive Rationality*.[17,18] In addition, Pino Pérez and Uzcátegui provide an example which reveals that, in the infinite language, the condition *WDR* is not necessary for a representation in terms of injective preferential models. In a sense, this result also supports the conjecture that, in the infinite case, the postulate *wci* is not necessary for recovering contraction relations to be represented by standard epistemic AGM states.

A more detailed comparison about the work in [7,19] and [15] may be found in [19].

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[15] *WDR*: $C(\alpha \vee \beta) \subseteq Cn(C(\alpha) \cup C(\beta))$, where $C(\alpha) = \{\gamma : \alpha \mid\!\sim \gamma\}$.

[16] *Disjunctive Rationality*: If $\alpha \mid\!\not\sim \beta$ and $\gamma \mid\!\not\sim \beta$ then $\alpha \vee \gamma \mid\!\not\sim \beta$.

[17] For preferential relations, the condition *Disjunctive Rationality* is properly stronger than *WDR*.

[18] When the language is finite, it is an injective preferential model.

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