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On the Conditions of a Center of the Liénard Equation*

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1. INTRODUCTION

In this paper, we discuss the conditions of a center for the system

$$\dot{x} = v, \qquad \dot{v} = -f(x)\,v - g(x) \tag{1.1}$$

or

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x) \tag{1.2}$$

which are both equivalence systems of the Liénard equation

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0,$$

where $F(x) = \int_0^x f(u)\,du$.

We assume throughout this paper the functions in the systems (1.1) and (1.2) are continuous and satisfy the conditions of unique solution to the initial value problem for the systems (1.1) and (1.2).

This problem has been discussed by many authors [1–8, 11]. They give lots of sufficient conditions which guarantee the origin is a local or a global center. For example, suppose $f(-x) = -f(x)$, $g(-x) = -g(x)$, and $g(x) > 0$ for $x > 0$. Then the origin is a local center of (1.1) or (1.2), if one of the following conditions is satisfied:

(1) From E. A. McHarg [3], $f(x) > 0$ for $x > 0$ and there exist $k > 0$ and $a > 0$ such that

$$f(x) < kg(x) \qquad \text{for} \quad 0 < x < a.$$

(2) From J. G. Wendel [4], there exist $k > 0$ and $a > 0$ such that

$$0 < f(x) < kg(x) \qquad \text{for} \quad 0 < x < a.$$

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(3) From A. F. Filippov [2, Theorem 34], $f(x) > 0$ for $x > 0$ and there exists $a > 0$ such that

$$g(x) \geqslant (1/4 + \varepsilon) f(x) F(x) \qquad (\varepsilon > 0) \quad \text{for} \quad 0 < x < a.$$

(4) From A. F. Filippov [5] (or see [1, Chap. 6, Sect. 4]) there exists $a > 0$ such that

(1) $G(\infty) = \infty$ where $G(x) = \int_0^x g(u)\, du$

(2) $F^2(x) \leqslant (8 - \varepsilon) G(x)$ $(\varepsilon \geqslant 0)$ for $0 < x < a$.

(5) From Z. Opial [6], there exists $a > 0$ such that

$$\int_0^x g(u)/|F(u)|\, du \geqslant (1/4 + \varepsilon)\, |F(x)| \qquad (\varepsilon > 0) \quad \text{for} \quad 0 < x < a.$$

(6) From T. Hara and T. Yoneyama [7], there exists $a > 0$ such that

$$|F(x)| > 0, \qquad \frac{1}{F(x)} \int_0^x \frac{g(u)}{F(u)} \geqslant \alpha > 1/4 \quad \text{for} \quad 0 < x < a.$$

(7) From Yu Shuxiang [8], there exist $a > 0$, $k_1 > 0$, and $k_2 < 0$ such that

$$k_2 \leqslant f(x)/g(x) \leqslant k_1 \qquad \text{for} \quad 0 < x < a.$$

In Sections 2 and 3 of our paper, by using a new method, we give some sufficient conditions of a local center for system (1.1) or (1.2). The criteria obtained by our method permit a function $r(x)$ which is able to be chosen. By choosing different functions $r(x)$, we can obtain various interesting criteria. In particular, the above conditions (1)–(7) can be decuded by choosing a special function $r(x)$. Therefore, the above results of Refs. [1–8] are all generalized in our paper, and some of their restrictions are omitted, for example the condition $f(x) > 0$ in Refs. [2–4] and $G(\infty) = \infty$ in [5]. Moreover, by taking the other function $r(x)$, we obtain a new result with weak conditions.

In Section 4, we give some sufficient and necessary conditions of a global center of (1.2). This problem has not been discussed until now.

2. The Conditions of a Local Center of System (1.1)

First, we give several auxiliary theorems.

Let $x = x(t)$, $v = v(t)$ be the equation of trajectory L of (1.1) passing through a point $(x_0, 0)$ at $t = t_0$, where $x_0 > 0$. L^+ and L^- are the positive and negative half-trajectory of L, respectively. Then we have

LEMMA 2.1. *If $xg(x)>0$ for $x\neq 0$, then for any $x_0>0$*

(i) *L^+ either intersects the negative v-axis or tends to the origin as $t\to\infty$,*

(ii) *L^- either intersects the positive v-axis or tends to the origin as $t\to-\infty$.*

Proof. We will only prove conclusion (i). (Conclusion (ii) is similar to prove).

Under the conditions of the lemma, the origin is a unique singular point of the system (1.1). From (1.1), it is obvious that L^+ must go into the region $x>0$, $v<0$ when t increases from t_0. If L^+ does not tend to the origin as $t\to\infty$ and does not intersect the negative v-axis, then it follows from $\dot{x}=v<0$ that $\lim x(t)=a\geqslant 0$ and

$$\limsup_{t\to\infty} v(t)=0. \tag{2.1}$$

If (2.1) does not hold, that is, there exists $\alpha>0$ such that $\dot{x}=v(t)<-\alpha$ $(t>t_0)$, then

$$x(t)-x_0<-\alpha(t-t_0)\to-\infty \qquad (t\to\infty).$$

So L^+ must intersect the negative v-axis. This contradicts the above assumption. Therefore (2.1) holds. If $\liminf_{t\to\infty} v(t)=0$ then $(a,0)$ is the unique ω-limit point of (1.1). So, it must be a singular point of (1.1). It is impossible, because if $a=0$, then L^+ tends to the origin as $t\to\infty$; if $a>0$, then the singular point is not unique. Thus,

$$\liminf_{t\to\infty} v(t)=b<0 \tag{2.2}$$

(b may be $-\infty$). From (2.1) and (2.2) we know that L^+ must move up and down, and tends to the line $x=a$ as $t\to\infty$. Hence there exist $t_n\to\infty$ $(n\to\infty)$ and $b_1\in(b,0)$ such that

$$v(t_n)=b_1, \qquad \lim_{n\to\infty}\left[\frac{dv(t)}{dt}\right]_{t=t_n}<0, \qquad \lim_{n\to\infty}\left[\frac{dv(t)}{dt}\right]_{t=t_{n+1}}>0.$$

But it follows from (1.1) that the above two limits are both equal to $-f(a)\,b_1-g(a)$. Thus we obtain a contradiction. Lemma 2.1 is proved.

LEMMA 2.2. *Suppose*

(1) *$xg(x)>0$ for $x\neq 0$.*

(2) *There exist a continuously differentable function $r(x)$, $r(0)=0$, and positive numbers a and k such that for $0<x<a$*

(1) $r'(x) \geqslant 0$ $(\not\equiv 0)$,

(2) $|F(x)| \leqslant kr(x)$,

(3) $r'(x)\, g(x) \geqslant r(x)[|f(x)| + \varepsilon r'(x)]^2/4$,

where $\varepsilon > 0$ *and is sufficiently small.* "$r'(x) \not\equiv 0$" *means* $r'(x) \not\equiv 0$ *in any subinterval of* $(0, a)$; *that is,* $r'(x)$ *only has isolated zero points in* $(0, a)$. *Then for any* $x_0 \in (0, a)$

(i) L^+ *must intersect the negative* v*-axis,*

(ii) L^- *must intersect the positive* v*-axis.*

Proof. From Lemma 2.1, we only need to prove L^+ does not tend to the origin as $t \to \infty$. If not, assume L^+ stays in the region $x > 0$, $v < 0$ and tends to the origin as $t \to \infty$, then $dx(t)/dt = v(t) < 0$. So there exists the inverse function $t = t(x)$ of $x = x(t)$. Let $\bar{v}(x) = v(t(x))$, then $\bar{v}(x) < 0$.

Consider the derivate of $\bar{v}(x)/r(x)$ along L^+

$$\frac{d}{dx}\left[\frac{\bar{v}(x)}{r(x)}\right] = \frac{\bar{v}'(x)\, r(x) - r'(x)\, \bar{v}(x)}{[r(x)]^2}$$

$$= \frac{[-f(x)\, \bar{v}(x) - g(x)]\, r(x)/\bar{v}(x) - r'(x)\, \bar{v}(x)}{[r(x)]^2}$$

$$= \frac{-[r'(x)\, \bar{v}^2(x) + r(x)\, f(x)\, \bar{v}(x) + r(x)\, g(x)]}{[r(x)]^2\, \bar{v}(x)}.$$

If $r'(x_1) = 0$, $x_1 \in (0, a)$, then it follows from condition (2), Part (3), that $f(x_1) = 0$ and

$$\frac{d}{dx}\left[\frac{\bar{v}(x)}{r(x)}\right]\Bigg|_{x = x_1} = -\frac{g(x_1)}{r(x_1)\, \bar{v}(x_1)} > 0.$$

If $r'(x_2) > 0$, $x_2 \in (0, a)$, then it follows from condition (2), Part (3), that

$$\frac{d}{dx}\left[\frac{\bar{v}(x)}{v(x)}\right]\Bigg|_{x = x_2} = -\frac{r'(x_2)}{r^2(x_2)\, \bar{v}(x_2)} \left\{\left[\bar{v}(x_2) + \frac{r(x_2)\, f(x_2)}{2r'(x_2)}\right]^2 + \frac{r(x_2)}{[r'(x_2)]^2}\left[r'(x_2)\, g(x_2) - \frac{r(x_2)}{4} f^2(x_2)\right]\right\} > 0.$$

Therefore, in the whole interval $(0, a)$

$$\frac{d}{dx}\left[\frac{\bar{v}(x)}{r(x)}\right] > 0 \tag{2.3}$$

so that $\bar{v}(x)/r(x)$ monotonically decreases as x monotonically decreases and tends to zero. Thus we have

$$\lim_{x\to 0}\frac{\bar{v}(x)}{r(x)} = -M < 0 \tag{2.4}$$

or

$$\lim_{x\to 0}\frac{\bar{v}(x)}{r(x)} = -\infty. \tag{2.5}$$

First, we discuss case (2.4). In this case, there exists $a_1 \in (0, a)$ such that

$$-M \leqslant \bar{v}(x)/r(x) < -M + \varepsilon < 0 \qquad \text{for} \quad 0 < x < a_1.$$

Now, integrating (1.1), we have

$$\begin{aligned}\bar{v}(0) &= \bar{v}(x) + \int_0^x [-f(u) - g(u)/\bar{v}(u)]\, du \\ &< (-M+\varepsilon)\, r(x) + \int_0^x [f(u) - g(u)/(Mr(x))]\, du \\ &= \int_0^x \{[f(u) + \varepsilon r'(u)] - [Mr'(u) + g(u)/(Mr(u))]\}\, du.\end{aligned}$$

By the triangle inequality $\alpha + \beta \geqslant 2\sqrt{\alpha\beta}$ and condition (2), Part (3), we have

$$\bar{v}(0) < \int_0^x [f(u) + \varepsilon r'(u) - 2\sqrt{g(u)\, r'(u)/r(u)}]\, du \leqslant 0.$$

This contradicts that L^+ tends to the origin as $t \to \infty$.

Now we consider case (2.5). Obviously, it follows from (2.5) that there exists a sufficiently small $a_2 \in (0, a)$, such that

$$\bar{v}(x)/r(k) \leqslant -k - 1 \qquad \text{for} \quad 0 < x \leqslant a_2.$$

We have

$$\begin{aligned}\bar{v}(0) &= \bar{v}(x) + \int_0^x [f(u) + g(u)/\bar{v}(u)]\, du \\ &\leqslant -(k+1)\, r(x) + F(x) \leqslant -r(x) < 0.\end{aligned}$$

Thus we have a contradiction. So, conclusion (i) is true.

Conclusion (ii) can be proved in the same way.

From Lemma 2.2, we have

THEOREM 2.1. *Suppose*

(1) $f(-x) = -f(x)$, $g(-x) = -g(x)$, $g(x) > 0$ *for* $x > 0$.

(2) *There exist a continuously differentiable function* $r(x)$, $r(0) = 0$, *and positive numbers a and k such that for* $0 < x < a$

(1) $r'(x) \geqslant 0$ ($\not\equiv 0$)

(2) $|F(x)| \leqslant kr(x)$

(3) $r'(x)\, g(x) \geqslant r(x)[|f(x)| + \varepsilon r'(x)]^2/4$, $\varepsilon > 0$.

Then the origin is a local center of system (1.1).

In fact, by condition (1), if $(x(t), v(t))$ is a solution of (1.1), then $(-x(-t), v(-t))$ is also a solutions of (1.1); that is, the trajectories of (1.1) have mirror symmetry about the v-axis. Therefore, from Lemma 2.2, for any $x_0 \in (0, a)$, L is a closed trajectory surrounding the origin. This means that the origin is a center.

In Theorem 2.1, by taking different $r(x)$, we can give many interesting results. For example, take $r'(x) = |f(x)|$. We have

COROLLARY 2.1. *Assume condition* (1) *of Theorem* 2.1 *is satisfied and there exists* $a > 0$ *such that* $|f(x)| \geqslant 0$ ($\not\equiv 0$), *and*

$$g(x) \geqslant (1/4 + \sigma)\, |f(x)| \int_0^x |f(u)|\, du, \qquad \sigma > 0 \quad \textit{for} \quad 0 < x < a.$$

Then the origin is a local center of (1.1).

Remark 2.1. In Corollary 2.1, $f(x)$ may be positive as well as negative. In particular, if $f(x) > 0$ $(0 < x < a)$ then this corollary is the result in [2] of Filippov.

If we take $r'(x) = g(x)/\sqrt{G(x)}$, then $r(x) = \int_0^x g(u)/\sqrt{G(u)}\, du = 2\sqrt{G(x)}$. We may obtain the following new result different from Refs. [1–8].

COROLLARY 2.2. *If condition* (1) *of Theorem* 2.1 *is satisfied and there exists an* $a > 0$ *such that for* $0 < x < a$

$$\frac{|f(x)|}{g(x)} \leqslant \frac{1}{\sqrt{G(x)}} (\sqrt{2} - \varepsilon), \qquad \varepsilon > 0, \tag{2.6}$$

then the origin is a local center of system (1.1).

Remark 2.2. If there exist $k_1 > 0$, $k_2 < 0$, and $x_1 > 0$ such that

$$k_2 \leqslant f(x)/g(x) \leqslant k_1 \qquad \text{for} \quad 0 < x < x_1$$

then (2.6) is satisfied for sufficiently small $a < x_1$. Therefore the above results of Yu Shuxiang [8], McHarg [3], and Wendel [4] in Section 1 are all results of this corollary.

The following example shows that the conditions of Corollary 2.2 are weaker than the conditions of Refs. [2–4, 8].

EXAMPLE 1. In the system (1.1), $g(x) = x^{5/3}$, $f(x) = x^{1/3}$, or $f(x) = -x^{1/3}$. It is easy to check that the conditions of Corollary 2.2 are satisfied. Thus, the origin is a local center, but if $f(x) = x^{1/3}$, then there does not exist k in Refs. [3, 4] and k_1 in [8]; if $f(x) = -x^{1/3}$, then there does not exist k_2 in [8], and Filippov's condition [2] is not satisfied because $f(x) < 0$ for $0 < x < a$.

COROLLARY 2.3. *Suppose condition* (1) *of Theorem* 2.1 *is satisfied and* $g'(0) > 0$. *Then the system* (1.1) *has a local center at the origin.*

Proof. Let $g'(0) = b > 0$ From $\lim_{x \to 0} g(x)/x = g'(0)$, it follows that there exists $a > 0$, such that

$$|g(x)/x - b| < b/2 \qquad \text{for} \quad 0 < x < a,$$

that is

$$bx/2 < g(x) < 3bx/2 \qquad \text{for} \quad 0 < x < a.$$

Therefore

$$\frac{g(x)}{\sqrt{G(x)}} \geqslant \frac{bx/2}{\sqrt{3bx^2/4}} = \sqrt{\frac{b}{3}} \qquad \text{for} \quad 0 < x < a.$$

Since $f(0) = 0$, (2.6) is satisfied if a is sufficiently small.

Remark 2.3. Corollary 2.3 is Theorem 2.1 of Ref. [8]. But our method of proof is new.

We give another conclusion with simple conditions as follows.

THEOREM 2.2. *Suppose condition* (1) *of Theorem* 2.1 *is satisfied and there exist* $k_1 > 0$, $k_2 < 0$, *and* $a > 0$ *such that*

$$k_2 \leqslant \frac{F(x)}{G(x)} \leqslant k_1 \qquad \textit{for} \quad 0 < x < a.$$

Then the origin is a local center of (1.1).

Proof. From Lemma 2.1 and the mirror symmetry of trajectories about the v-axis, we only need to prove L^+ and L^- can not tend to the origin for any $x_0 \in (0, a)$ in the right half-plane. Otherwise, assume $x(t) \to 0$, $v(t) \to 0$ as $t \to \infty$. Then there exists $t_1 > t_0$ such that $-1/(2k_1) \leqslant v(t) < 0$ for $t \geqslant t_1$.

Now we integrate (1.1) along L^+ from t_1 to t $(t > t_1)$. Because L^+ is in the region $x > 0$, $v < 0$ and $x(t)$ decreases as t increases, we have

$$\begin{aligned} v(t) &= v(t_1) - \int_{t_1}^{t} f(x(t))\, v(t)\, dt - \int_{t_1}^{t} g(x(t))\, dt \\ &\leqslant -\int_{x(t_1)}^{x(t)} f(u)\, du - \int_{x(t_1)}^{x(t)} g(u)/v(u)\, du \\ &\leqslant -F(x(t)) + F(x(t_1)) + \int_{x(t_1)}^{x(t)} 2k_1 g(u)\, du \\ &\leqslant -F(x(t)) + k_1 G(x(t_1)) + 2k_1 [G(x(t)) - G(x(t_1))] \\ &= -F(x(t)) + 2k_1 G(x(t)) - k_1 G(x(t_1)). \end{aligned}$$

Let $t \to \infty$ then we have $0 \leqslant -k_1 G(x(t_1)) < 0$. This is a contradiction. So, L^+ cannot tend to the origin as $t \to \infty$.

We can prove L^- cannot tend to the origin as $t \to \infty$ in the same way.

Remark 2.4. Obviously, the above result of Refs. [3, 4, 8] in Section 1 are corollaries of this theorem. But our proof is by far simpler than that in [3, 4, 8].

3. The Conditions of a Local Center of System (1.2)

Let L^+ and L^- be the positive and the negative half-trajectories of (1.2) passing through the point $(x_0, F(x_0))$ on the isocline Γ: $y = F(x)$ at $t = t_0$, respectively.

By using the method of proof of Lemma 2.1, we may prove the following lemma

Lemma 3.1. *If $xg(x) > 0$ for $x \neq 0$, then for any $x_0 > 0$, L^+ either tends to the origin as $t \to \infty$ or intersects the negative y-axis; L^- either tends to the origin as $t \to \infty$ or intersects the positive y-axis.*

From Lemma 3.1 and system (1.2), we have

Lemma 3.2. *If $xg(x) > 0$ for $x \neq 0$ and there exists a sequence $\{x_n\}$, $x_n > 0$, $\lim_{n \to \infty} x_n = 0$, $F(x_n) = 0$ then L^+ must intersect the negative y-axis and L^- must intersect the positive y-axis.*

It follows from Lemma 3.2 at once that

COROLLARY 3.1. *If the conditions in Lemma* 3.2 *are satisfied, moreover,* $F(x)$ *and* $g(x)$ *are even and odd functions, respectively, then the origin is a local center of* (1.2).

In fact, since $F(x)$ and $g(x)$ are even and odd functions, respectively, the trajectories of (1.2) have mirror symmetry about the y-axis. This fact is used many times in the following.

Henceforth, we only need to discuss the case of $|F(x)| > 0$ for $0 < x < a$.

LEMMA 3.3. *Suppose*

(1) $xg(x) > 0$ *for* $x \neq 0$.

(2) *There exist* $a > 0$ *and a continuous function* $r(x)$ *such that for* $0 < x < a$

(1) $r(x) \geqslant F(x) > 0$,

(2) $(1/r(x)) \int_0^x (g(u)/r(u))\, dx \geqslant \beta > 1/4$.

Then for any $x_0 \in (0, a)$, (i) L^- *must intersect the positive* y-axis; (ii) L^+ *must intersect the negative* y-*axis.*

Proof. It follows from (1.2), condition (2), Part (1), and Lemma 3.1 that conclusion (i) is true obviously.

Now we prove conclusion (ii). Otherwise, it follows from Lemma 3.1 that L^+ must tend to the origin as $t \to \infty$ and cannot intersect the x-axis from the second formula in (1.2). Let the equation of L^+ be $y = y(x)$, then $y(x) > 0$ and L^+ stays in the region

$$D = \{(x, y) \mid y < F(x), x \geqslant 0\}.$$

Therefore, if we let

$$\alpha_0 = \sup_{0 < x \leqslant x_0} \frac{F(x) - y(x)}{F(x)} \tag{3.1}$$

then $0 < \alpha_0 \leqslant 1$, and

$$0 < F(x) - y(x) \leqslant \alpha_0 F(x) \qquad \text{for} \quad 0 < x < x_0. \tag{3.2}$$

It follows from (1.2) and condition (2) that

$$\begin{aligned} y(x) &= \int_0^x \frac{g(u)}{F(u) - y(u)}\, du \geqslant \int_0^x \frac{g(u)}{\alpha_0 F(u)}\, du \geqslant \int_0^x \frac{g(u)}{\alpha_0 r(u)}\, du \\ &\geqslant \beta r(x)/\alpha_0 \geqslant \beta F(x)/\alpha_0, \qquad \text{for} \quad 0 < x < x_0. \end{aligned} \tag{3.3}$$

Thus for $0<x<x_0$

$$\frac{F(x)-y(x)}{F(x)}\leqslant\frac{\alpha_0-\beta}{\alpha_0}.$$

From (3.1) and (3.2), we are sure that $\alpha_0\leqslant(\alpha_0-\beta)/\alpha_0$, so

$$\beta\leqslant\alpha_0(1-\alpha_0)=-(\alpha_0-1/2)+1/4\leqslant 1/4.$$

This contradicts condition (2). The conclusion is proved.

Similarly, we can prove that

LEMMA 3.4. *Suppose*

(1) $xg(x)>0$, *for* $x\neq 0$.

(2) *There exist* $a>0$ *and a continuous function* $r(x)$ *such that for* $0<x<a$

(1) $r(x)\geqslant -F(x)>0$,

(2) $(1/r(x))\int_0^x (g(u)/r(u))\,du\geqslant\beta>1/4$.

Then for any $x_0\in(0,a)$, L^+ *must intersect the negative y-axis and* L^- *must intersect the positive y-axis.*

It follows from Lemmas 3.3, 3.4, and the symmetry of the trajectories that

THEOREM 3.1. *Suppose*

(1) $F(-x)=F(x)$, $g(-x)=-g(x)$, $g(x)>0$ for $x>0$.

(2) *There exist* $a>0$ *and a continuous function* $r(x)$ *such that for* $0<x<a$

(1) $r(x)\geqslant|F(x)|>0$,

(2) $(1/r(x))\int_0^x (g(u)/r(u))\,du\geqslant\beta>1/4$.

Then the origin is a local center of (1.2).

If we take $r(x)=|F(x)|$ then Theorem 3.1 gives the results of Opial [6] and Hara and Yoneyama [7] as follows

COROLLARY 3.2. *Suppose*

(1) $F(-x)=F(x)$, $g(-x)=-g(x)$, $g(x)>0$ *for* $x>0$.

(2) *There exists* $a>0$ *such that*

$$\frac{1}{F(x)}\int_0^x \frac{g(u)}{F(u)}\,du \geqslant \beta > \tfrac{1}{4} \qquad \textit{for} \quad 0 < x < a.$$

Then the origin is a local center of (1.2).

Moreover, we also have

COROLLARY 3.3. *Suppose*

(1) $F(-x) = F(x)$, $g(-x) = -g(x)$, $g(x) > 0$ *for* $x > 0$.

(2) *There exist* $\alpha > 0$, $\gamma > 0$, *and* $a > 0$ *such that for* $0 < x < a$

$$|F(x)| \leqslant \alpha[G(x)]^{\gamma}, \tag{3.4}$$

where $1/2 < \gamma < 1$ *or* $\gamma = 1/2$, $\alpha < \sqrt{8}$.

Then the origin is a local center of (1.2).

Proof. Take $r(x) = \alpha[G(x)]^{\gamma}$, then condition (2), Part (1), in Theorem 3.1 is satisfied. Now we only need to prove that condition (2), Part (2), holds too.

If $1/2 < \gamma < 1$, $\alpha > 0$, then condition (2), Part (2), in theorem 3.1 becomes

$$1/(\alpha[G(x)]^{\gamma}) \int_0^x g(u)/(\alpha[G(x)]^{\gamma})\,du \geqslant \beta > 1/4,$$

that is,

$$1/(1-\gamma)[G(x)]^{-\gamma}\,[G(x)]^{1-\gamma} > \alpha^2/4. \tag{3.5}$$

From $G(0) = 0$, if a is sufficiently small, then (3.5) holds $0 < x < a$; that is, condition (2), Part (2), in Theorem 3.1 is satisfied.

If $\gamma = 1/2$ then (3.5) becomes $2 > \alpha^2/4$, so condition (2), Part (2), of Theorem 3.1 is satisfied when $0 < \alpha < \sqrt{8}$. The proof is completed.

Remark 3.1. If $\gamma = 1/2$, $0 < \alpha < \sqrt{8}$ then (3.4) is the condition of Filippov [5] (that is, condition (4) in Section 1). But this condition in Ref. [5] is obtained by using Filippov's transformation $z = \int_0^x g(u)\,du$ to (1.2). Here we do not use this transformation. Moreover, we omit the restriction $G(\infty) = \infty$.

EXAMPLE 2. In (1.2), we take $g(x) = x/(1 + x^4)$, $F(x) = \sqrt{\operatorname{arctg} x^2}$, $G(x) = \operatorname{arctg} x^2/2$. Then it is easy to prove that the conditions of Corollary 3.3 are satisfied, so that the origin is a local center of (1.2), but, where $G(\infty) < \infty$.

4. The Conditions of a Global Center

First, from the proof for results in Sections 2 and 3, it is easy to prove the following sufficient conditions of a global center for (1.1) or (1.2).

THEOREM 4.1. *If the conditions of the theorem or corollaries in Section* 3 *are satisfied and*

$$\limsup_{x \to \infty} F(x) = \infty, \qquad \liminf_{x \to \infty} F(x) = -\infty \tag{4.1}$$

then the origin is a global center of system (1.2).

In fact, it follows from the results in Section 3 that the origin is a local center. And from (4.1), the positive trajectory passing through an arbitrary point over the isocline Γ in the right half-plane must intersect Γ; the negative trajectory passing through an arbitrary point below Γ also must intersect Γ. Therefore, they must be closed trajectories according to Lemma 3.1 and the symmetry of the trajectories.

Remark 4.1. Reference [7] has a similar result (Theorem 4.4), but it needs an additional condition $G(\infty) = \infty$. However, this restriction is not in Theorem 4.1. This means that $G(\infty) = \infty$ is also not a necessary condition of a global center under (4.1).

EXAMPLE 3. In (1.2), we let

$$g(x) = \frac{x}{1 + x^4}, \qquad F(x) = x \sin x \sqrt{\operatorname{arctg} x^2}.$$

It is easy to check that the conditions of Theorem 4.1 are satisfied. Therefore, the origin is a global center, but, where $G(\pm\infty) < \infty$. Hence, the conditions of Theorem 4.4 in Ref. [7] are not satisfied.

Now, we discuss the case in which the conditions (4.1) are not satified and give some sufficient and necessary conditions of a global center. Presently, this problem has not been studied.

THEOREM 4.2. *Suppose the conditions of the theorem or corollary in Section* 3 *are satisfied. If there exists* $A > 0$ *such that for all* $x > 0$

$$|F(x)| \leqslant A, \tag{4.2}$$

then (1.2) *has a global center at the origin if and only if*

$$\int_0^{\infty} g(x)\, dx = \infty. \tag{4.3}$$

Proof. If $\int_0^\infty g(x)\,dx < \infty$, then system (1.2) must have unbounded solutions according to Ref. [9]. So, the origin cannot be a global center of (1.2).

Let (4.3) hold. First, we prove that the positive trajectory $L_1^+ : x = x(t)$, $y = y(t)$ passing through an abitrary point (x_0, y_0) $(x_0 \geqslant 0)$ over isocline Γ at $t = t_0$ must intersect Γ. Without loss of generality, we can assume $y_0 > A$. If L_1^+ does not intersect Γ, then

$$y(t) > F(x(t)) \geqslant -A(t > t_0), \qquad x(t) \to \infty \quad (t \to \infty) \tag{4.4}$$

because L_1^+ cannot tend to the origin which is a local center of (1.2).

Now, integrating (1.2) along L_1^+, we have

$$y(t) = y_0 + \int_{x_0}^{x(t)} \frac{g(u)}{F(u) - y}\,du \leqslant y_0 - \int_{x_0}^{x(t)} \frac{g(u)}{y_0 + A}\,du \to -\infty \qquad (t \to \infty).$$

This contradicts (4.4). Similarly, we can prove that the negative trajectory passing through an arbitrary point below isocline Γ must intersect the Γ. Hence, from Lemma 3.1, the trajectory passing through an arbitrary point in the right half-plane must intersect the positive and negative y-axis, and it must be a closed trajectory according to the symmetry of the trajectories. So, the origin is a global center.

For the system (1.1), we can prove

THEOREM 4.3. *Suppose the conditions of the theorem or corollary in Section 2 and* (4.2) *are satisfied. Then* (1.1) *has a global center at the origin, if and only if the assumption* (4.3) *holds.*

Proof. If $\int_0^\infty g(x)\,dx < \infty$, then the system (1.1) must have unbounded solutions according to Ref. [10]. So, the origin is not a global center.

If (4.3) holds then the trajectory L_1 of (1.1) passing through an arbitrary point (x_0, y_0), $x_0 > 0$, must intersect the positive x-axis. In fact, if $y_0 > 0$ (the case $y_0 < 0$ is similar), and L_1 does not intersect the positive x-axis, then

$$y(t) > 0 \qquad \text{for} \quad t > 0; \quad x(t) \to \infty \quad \text{as} \quad t \to \infty.$$

Integrating (1.1), we have

$$\begin{aligned} y(t) &= y_0 - \int_{x_0}^{x(t)} f(x)\,dx - \int_{x_0}^{x(t)} g(x)/y\,dx \\ &\leqslant y_0 - F(x(t)) + F(x) \leqslant y_0 + 2A. \end{aligned}$$

Hence

$$y(t) \leqslant y_0 + 2A - \int_{x_0}^{x(t)} \frac{g(x)}{y_0 + 2A} dx \to -\infty, \qquad \text{as} \quad t \to \infty.$$

This contradicts $y(t) > 0$. So, L_1 must be closed trajectory since the origin is a local center. Theorem 4.4 is proved.

Now, we discuss cases in which conditions (4.1) and (4.2) are not satisfied.

THEOREM 4.4. *Suppose the conditions of the theorem or corollary in Section 3 are satisfied and*

$$\limsup_{x \to \infty} F(x) = k. \tag{4.5}$$

Then the origin is a global center of (1.2) *if and only if*

$$\int_M^{\infty} \frac{g(x)}{c - F(x)} dx = \infty, \tag{4.6}$$

where $c = |k| + 1$, M *is a sufficiently large positive number.*

Proof. First, we prove the sufficiency. It follows from (4.5) that there exists $M > 0$ such that $F(x) < c$ for $x \geqslant M$. Now we prove that a positive half-trajectory $L_1^+ : x = x(t),\ y = y(t)$ passing through any point (x_0, y_0) over the iscoline Γ must intersect Γ. Without loss of generality, we can suppose $M > x_0 \geqslant 0$, $y_0 \geqslant c$. If L_1^+ does not intersect Γ, then it must stay in a region $D = \{(x, y) \mid y > F(x), x > 0\}$ and $y(t)$ decreases, $x(t)$ increases as t increases, and $x(t) \to \infty$ $(t \to \infty)$. Therefore, L_1^+ must intersect line $x = M$. Now, integrating (1.2) along L_1^+, we have

$$\begin{aligned} y(t) &= y_0 + \int_{x_0}^{x(t)} \frac{g(x)}{F(x) - y} dx \\ &= y_0 + \int_M^{x(t)} \frac{g(x)}{F(x) - y} dx + \int_{x_0}^M \frac{g(x)}{F(x) - y} dx \\ &< y_0 + \int_{x_0}^M \frac{g(x)}{F(x) - y} dx + \int_M^{x(t)} \frac{g(x)}{F(x) - y_0} dx \end{aligned} \tag{4.7}$$

From (4.6), it is easy to prove

$$\int_M^{\infty} \frac{g(x)}{y_0 - F(x)} dx = \infty.$$

In fact, because $c - F(x) \geqslant 1/2$,

$$\begin{aligned} y_0 - F(x) &= c - F(x) + y_0 - c \leqslant (c - F(x)) + 2(y_0 - c)(c - F(x)) \\ &= (c - F(x))[2y_0 - 2c + 1]. \end{aligned}$$

Hence

$$\frac{g(x)}{c - F(x)} \leqslant \frac{g(x)}{y_0 - F(x)} [2y_0 - 2c + 1].$$

Therefore, in (4.7), let $t \to \infty$, then $y(t) \to -\infty$, but, from condition (4.5) and the definition of the region D, this is impossible. Therefore, L_1^+ must intersect Γ.

Moreover, we prove that the negative half-trajectory starting from an arbitrary point below isocline Γ must intersect Γ. If $\liminf_{x \to \infty} F(x) = -\infty$ then it is obvious. If $\liminf_{x \to \infty} F(x) > -\infty$, then there exists $A > 0$ such that $|F(x)| < A$ for all $x > 0$. On the other hand, it follows from (4.6) that $\int_c^\infty g(x)\, dx = \infty$. Hence, it is easy to see from the proof of Theorem 4.3 that this conclusion also holds. It follows that (1.2) has a global center at the origin.

Secondly, we prove the necessity. If (4.6) does not hold, then we have

$$\int_M^\infty \frac{g(x)}{c - F(x)}\, dx = \alpha < \infty, \qquad \alpha > 0.$$

Assume for $0 \leqslant x \leqslant M$

$$|F(x)| < N, \qquad \int_0^M \frac{g(x)}{N - F(x)}\, dx = \beta.$$

Now take $y_0 \geqslant c + \alpha + \beta + N$, and consider the positive half-trajectory $L_2^+ : x = x(t)$, $y = y(t)$ of the system (1.2) passing through the point $(0, y_0)$ at $t = t_0$. First we prove that L_2^+ can not intersect line $y = c$ and $y = N$ in $0 \leqslant x \leqslant M$. Otherwise, suppose there exists $t_1 > t_0$ such that $y(t_1) = c$ or $y(t_1) = N$, $0 \leqslant x(t_1) \leqslant M$. Then by integrating (1.2) we have

$$\begin{aligned} y(t_1) &= y_0 + \int_0^{x(t_1)} \frac{g(x)}{F(x) - y}\, dx > y_0 + \int_0^{x(t_1)} \frac{g(x)}{F(x) - N}\, dx \\ &> y_0 - \beta > N + c. \end{aligned}$$

This contradicts the hypothesis for t_1. It is because in $0 \leqslant x \leqslant M$, $|F(x)| < N$. This contracition shows that L cannot intersect isocline $y = F(x)$ in $0 \leqslant x \leqslant M$. Next, we prove that L_2^+ also cannot intersect the isocline in $x > M$. Here we only need to prove that L_2^+ cannot intersect the line $y = c$. If not, then there exists $t_2 > t_0$ such that $y(t_2) = c$, $x(t_2) > M$, and

$$
\begin{aligned}
y(t_2) &= y_0 + \int_0^M \frac{g(x)}{F(x)-y}\,dx + \int_M^{x(t_2)} \frac{g(x)}{F(x)-y}\,dx \\
&> y_0 + \int_0^M \frac{g(x)}{F(x)-N}\,dx + \int_M^{x(t_2)} \frac{g(x)}{F(x)-y}\,dx \\
&> y_0 - \beta + \int_M^{x(t_2)} \frac{g(x)}{F(x)-c}\,dx > y_0 - \beta + \int_M^{\infty} \frac{g(x)}{F(x)-c}\,dx \\
&> y_0 - \beta - \alpha > N + c.
\end{aligned}
$$

This contradicts the hypothesis for t_2. Therefore, L_2^+ must stay in the region D and $x(t) \to \infty$ $(t \to \infty)$, so (1.2) does not have a global center at the origin. The proof is completed.

Similarly, we may prove the following

THEOREM 4.5. *Suppose the conditions of the theorem or corollary in Section* 3 *are satisfied, and*

$$\liminf_{x \to \infty} F(x) = k. \tag{4.8}$$

Then (1.2) *has a global center at the origin if and only if*

$$\int_M^{\infty} \frac{g(x)}{c + F(x)}\,dx = \infty, \tag{4.9}$$

where $c = |k| + 1$, M *is a sufficiently large positive constant.*

Remark 4.2. It is easy to see from the proof of Theorem 4.4 that we do not need the conditions of the theorem or corollaries in Section 3 as the prerequisite hypotheses to guarantee that the origin is not a global center. Then we have

COROLLARY 4.1. *If* $xg(x) > 0$, *and if one of the following conditions is satisfied, then the origin is not a global center*:

$$(1)\quad \limsup_{x \to \infty} F(x) = k, \qquad \int_M^{\infty} \frac{g(x)}{c - F(x)}\,dx < \infty$$

$$(2)\quad \liminf_{x \to \infty} F(x) = k, \qquad \int_M^{\infty} \frac{g(x)}{c + F(x)}\,dx < \infty,$$

where $c = |k| + 1$.

In particular, if $|F(x)| < A$, $x > 0$, $G(+\infty) < \infty$, then (1.2) does not have a global center at the origin.

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