ON REGULAR SEMIGROUPS II: AN EMBEDDING

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Any regular semigroup $S$ is shown to be embeddable as a full subsemigroup of a regular semigroup $T$ with certain properties, enabling us to solve seven problems of the form "characterize the idempotent-generated subsemigroups of bisimple regular semigroups" posed in a previous paper.

1. Introduction and summary

For any regular semigroup $S$ we denote by $E(S)$ the set of idempotents of $S$ and we call $\langle E(S) \rangle$, the subsemigroup of $S$ generated by $E(S)$, the core of $S$ (a word suggested by Mario Petrich).

Ever since the elegant determination by Douglas Munn (published in 1966) of the semilattices of bisimple inverse semigroups [14], an obvious problem has been to determine the cores of bisimple regular semigroups. The bands of bisimple orthodox semigroups, and then the fundamental cores of bisimple regular semigroups, were determined soon after by the author in [3] and [6]. From an amalgamation result of the author (Result 4) we are now able to solve the above problem. We also determine the cores of 0-bisimple, simple and 0-simple regular semigroups and we determine when an idempotent-generated regular semigroup $B$ is such that every regular semigroup $S$ with core $B$ is completely semisimple, a union of groups, and $\mathcal{H}$-compatible respectively. These results answer seven problems raised over ten years ago at the end of Section 6 of [6].

Our solutions to these seven problems are all applications of Theorem 1, an embedding result for regular semigroups, stating that any regular semigroup $S$ is embeddable in a regular semigroup $T$ such that $S$ is full in $T$ (that is, $E(S) = E(T)$) and such that for all $e, f \in E(S)$, if the cores of $eSe$ and $fSf$ are isomorphic, then $e$ and $f$ are $\mathcal{D}$-related in $T$. In Section 7 seven implications involving a core and its maximum fundamental image are given, while examples show that the seven reverse implications are false. A further application of our embedding theorem is made in Section 8, to locally inverse semigroups. Apart from those in Sections 7 and 8, the results of this paper were announced in [8].
2. Preliminaries

Take any semigroup $S$. For any element $a \in S$, we put

$$V(a) = \{x \in S: axa = a \text{ and } xax = x\},$$

the set of inverses of $a$ in $S$. For any $e, f \in E(S)$ we have that $e \equiv f$ if and only if there exist $a \in S$ and $a' \in V(a)$ such that $aa' = e$ and $a'a = f$ [11, Proposition II.3.6].

**Result 1** (W.D. Munn [15, Lemma 1]). Take any semigroup $S$, any $\equiv$-related $e, f \in E(S)$, and any $a \in S$, $a' \in V(a)$ such that $aa' = e$, $a'a = f$. Define mappings $\theta_{a', a}: eSe \to fSf$ and $\theta_{a, a'}: fSf \to eSe$ by $x\theta_{a', a} = a'xa$ for each $x \in eSe$, and $y\theta_{a, a'} = aya'$ for each $y \in fSf$. Then $\theta_{a', a}$ is a $\equiv$-class preserving isomorphism of $eSe$ onto $fSf$ and $\theta_{a, a'}^{-1} = \theta_{a', a}$.

**Result 2** [1, Section 8.4, Exercise 3]. A regular semigroup $S$ is $[0]$-simple if and only if for any [nonzero] idempotents $e, f$ there exists an idempotent $g$ such that $f \equiv g$ and $g \leq e$.

**Result 3** (the author [6, Theorem 5]). The maximum congruence contained in $\mathcal{H}$ on any regular semigroup $S$, $\mu = \mu(S)$ say, is given by

$$\mu = \{(a, b) \in \mathcal{H}: \text{ for some [all] } \mathcal{H}\text{-related inverses } a'\text{ of } a \text{ and } b'\text{ of } b, a'ea = b'eb \text{ for each idempotent } e \leq aa'\},$$

and equivalently by

$$\mu = \{(a, b) \in S \times S: \text{ for some inverses } a'\text{ of } a \text{ and } b'\text{ of } b, aa' = bb', a'a = b'b \text{ and } a'ea = b'eb \text{ for each idempotent } e \leq aa'\}.$$

A result due to Fitz-Gerald [2], of basic importance, is that if $S$ is a regular semigroup then the core of $S, \langle E(S) \rangle$, is also regular. The following two results each play a major role in our proof of the main theorem.

**Result 4** (the author [6, Theorem 14]). Let $E$ be any set of idempotents of any semigroup $S$.

(i) There is a regular subsemigroup of $S$ with $E$ as its set of idempotents if and only if $\langle E \rangle$, the subsemigroup generated by $E$, is such a semigroup, that is, a regular semigroup with $E$ as its set of idempotents.

(ii) If $\langle E \rangle$ is a regular subsemigroup with $E$ as its set of idempotents, then

$$E^C = \{a \in S: \text{ for some } a' \in V(a), aa', a'a, a'ea, afa' \in E \text{ for all } e, f \in E \text{ such that } e \leq aa', f \leq a'a\}$$

is the maximum regular subsemigroup of $S$ with $E$ as its set of idempotents.
By an \textit{amalgam of semigroups} we mean a list \((S_i, i \in I; U)\) of semigroups with \(U\) being a subsemigroup of each of the semigroups \(S_i\). This amalgam is said to be \textit{strongly embeddable} if there exist a semigroup \(W\) and monomorphisms \(\psi_i: S_i \rightarrow W (i \in I)\) agreeing on \(U\) (that is \(\psi_i|U = \psi_j|U\) for all \(i, j \in I\)) and satisfying \((S_i \psi_i) \cap (S_j \psi_j) = U \psi_i\) for all distinct \(i, j \in I\).

\textbf{Result 5} (the author \cite[Theorem 8]{7}). Let \((S_i, i \in I; U)\) be any amalgam of regular semigroups such that \(E(U) = E(S_i)\) for all \(i \in I\). Then the amalgam is strongly embeddable in a regular semigroup \(W\) such that \(E(W) = E(U)\). If \(I\) and each \(S_i\) are finite, then \(W\) can be taken to be finite also.

We will also consider amalgams in which \(U\) is not a subsemigroup of each \(S_i\) but is merely isomorphic to a subsemigroup of each \(S_i\) (the difference of course is trivial); thus by an \textit{amalgam} of semigroups we shall also mean a list of the form \((S_i, i \in I; U; \phi_i, i \in I)\) where \(S_i (i \in I)\) and \(U\) are semigroups and \(\phi_i\) is a monomorphism of \(U\) into \(S_i\). This amalgam is said to be \textit{strongly embeddable} if there exist a semigroup \(W\) and monomorphisms \(\phi_i: S_i \rightarrow W (i \in I)\) such that \(\phi_i \psi_i = \phi_j \psi_j\) and \((S_i \psi_i) \cap (S_j \psi_j) = U \phi_i \psi_i\) for all distinct \(i, j \in I\).

\section{3. The embedding theorem}

Let \(S\) be any regular semigroup. For each \(e \in E(S)\), as in \cite{6} we put
\[\langle e \rangle = \langle E(eSe) \rangle = \langle x : x \in E(S), x \leq e \rangle,\]
the core of the regular subsemigroup \(eSe\).

Now let \(e, f\) be any \(\mathcal{D}\)-related idempotents in \(S\) and take any \(a \in S\) and \(a' \in V(a)\) such that \(aa' = e, a'a = f\). We call the map \(\theta_{a', a}: eSe \rightarrow fSf\) of Result 1 the \textit{inner partial automorphism of} \(S\) induced by \(a\) and \(a'\). Of course \(\theta_{a', a}|\langle e \rangle\), the restriction of \(\theta_{a', a}\) to \(\langle e \rangle\), is an isomorphism from \(\langle e \rangle\) onto \(\langle f \rangle\).

The following theorem answers affirmatively Question H3 of the Proceedings of the 1979 DeKalb Conference. For inverse semigroups the result is due to Norman Reilly (verbal communication at Monash, 1971, and \cite{18}).

\textbf{Theorem 1.} Any regular semigroup \(S\) is embeddable in a regular semigroup \(T\) such that

\begin{itemize}
  \item[(i)] \(S\) is full in \(T\), that is, \(E(S) = E(T)\), and
  \item[(ii)] for all idempotents \(e, f \in S\), we have \(e \mathcal{D} f \text{ in } T\) if (and only if) \(\langle e \rangle\) is isomorphic to \(\langle f \rangle\).
\end{itemize}

Moreover, \(T\) can be taken such that further, for all \(e, f \in E(S)\) with \(\langle e \rangle\) isomorphic to \(\langle f \rangle\), each isomorphism from \(\langle e \rangle\) onto \(\langle f \rangle\) is the restriction to \(\langle e \rangle\) of some inner partial automorphism of \(T\) with domain \(eTe\) and range \(fTf\). If \(S\) is finite, then \(T\) can be taken to be finite also.
Proof. Take any regular semigroup \( S \), and any idempotent \( e \in S \). We form the semigroup \( W \) which is the 0-direct union of \( S \) and \( eSe \). [1, Section 6.3] each with an adjoined zero if necessary; more formally we take any isomorphism \( \phi : eSe \to R \) from the regular semigroup \( eSe \) onto a semigroup \( R \) such that \( S^0 \cap R^0 = \{0\} \) and then define \( W \) to be the 0-direct union of \( S^0 \) and \( R^0 \) (that is, \( W = S^0 \cup R^0 \), \( S \) and \( R \) are subsemigroups of \( W \) and \( SR = RS = \{0\} \)).

We show first a very special case of the embedding theorem, namely that \( W \) can be embedded in a semigroup \( U \) such that \( E(W) = E(U) \) and \( e \not\in \theta \) in \( U \).

First we put \( M = \mathcal{A}(S; 2, 2; A) \), the Rees matrix semigroup of \( 2 \times 2 \) matrices over \( S^0 \) with at most one nonzero entry, and with sandwich matrix equal to the \( 2 \times 2 \) identity matrix. The subsemigroup \( M_{11} \) of \( M \), defined by \( M_{11} = \{(s, 1, 1) \in M : s \in S^0 \} \), is of course isomorphic to \( S^0 \) (as usual \((s, i, j)\) denotes the \( 2 \times 2 \) matrix with \( s \) as the \((i, j)\)-entry and with other entries 0), as is \( M_{22} = \{(s, 2, 2) \in M : s \in S^0 \} \), and of course \( M_{11} \cup M_{22} \) is precisely the 0-direct union of \( M_{11} \) and \( M_{22} \). Thus, our semigroup \( W \) is isomorphic to \( W' = M_{11} \cup \{(s, 2, 2) \in M : s \in eS^0 e\} \). We define

\[
U' = W' \cup \{(s, 1, 2) \in M : s \in Se\} \cup \{(s, 2, 1) \in M : s \in eS\},
\]

a subsemigroup of \( M \) such that \( E(W') = E(U') \). Further \((e, 1, 2) \not\in (e, 2, 2) \) in \( U' \) since \((e, 1, 2)\) and \((e, 2, 1)\) are mutually inverse in \( U' \) and \((e, 1, 2)(e, 2, 1) = (e, 1, 1), (e, 2, 1)(e, 1, 2) = (e, 2, 2)\). Since the obvious isomorphism from \( W \) onto \( W' \) carries \( e \) and \( e\phi \) to \((e, 1, 1)\) and \((e, 2, 2)\) respectively, we have our first required result, that there exists a semigroup \( U \) (isomorphic to \( U' \) of course) containing \( W \) such that \( E(U) = E(W) \) and \( e \not\in \theta \) in \( U \). Let \( \theta : U' \to U \) be an isomorphism such that \((s, 1, 1) \theta = s \) (for each \( s \in S^0 \)) and \((s, 2, 2) \theta = s\phi \in R \) (for each \( s \in Se \)).

Remark 1. The author happily acknowledges that the above exposition was produced after seeing a construction due to Stuart Margolis (letter to Don McAlister in September, 1980). The author's original proof was equivalent but less elegant, and involved an appropriate subsemigroup of \( \mathcal{A}(S^0 \cup S^0) \), the semigroup of transformations of \( S^0 \cup S^0 \), the 0-direct union of \( S^0 \) and a copy, with \( S^0 \cup S^0 \) embedded by its right regular representation.

We return to the proof of Theorem 1. Our semigroup \( U' \) is regular, for it is easy to see that for any \( x \in eS \) and \( x' \in V(x) \), we have \( x' e \in V(x) \cap Se \), and then that \((x, 2, 1)\) has \((x', e, 1, 2)\) as an inverse in \( U' \); likewise any element \((x, 1, 2) \in U' \) is regular, and clearly \( W' \) is regular.

We denote the regular semigroup \( U \) by \( U_e \), to signify its dependence on the choice of \( e \) (similarly we denote \( \phi, W, W', U', \theta \) by \( \phi_e, W_e, W'_e, U'_e, \theta_e \)). We take any \( e, f \in E(S) \) such that \( \langle e \rangle = \langle f \rangle \). Then \( \langle E(U_e) \rangle = \langle E(U_f) \rangle \), since each is the 0-direct union of \( \langle E(S) \rangle^0 \) and a copy of \( \langle e \rangle^0 \). This essentially enables us to consider an amalgam of the form \( (U_e, U_f; \langle E(U_e) \rangle) \). More formally, we take any isomorphism \( \alpha \) of \( \langle e \rangle \) onto \( \langle f \rangle \) and denote by \( \hat{\alpha} \) the isomorphism, induced by \( \alpha \), from \( \langle E(U_e) \rangle \) onto \( \langle E(U_f) \rangle \); that is, \( \hat{\alpha} \) is defined by \( sa = s \) for all \( s \in \langle E(S^0) \rangle \), and for any element
in \( \langle E(S) \rangle \), say \( x\phi_e \) where \( x \in \langle e \rangle \), \( (x\phi_e)\hat{\alpha} = x\alpha\phi_f \). Our amalgam is then the list \((U_e, U_f; \langle E(U_e) \rangle; i, \hat{\alpha})\), where \( i \) denotes the insertion of \( \langle E(U_f) \rangle \) in \( U_e \).

By Result 5, this amalgam is embeddable in a regular semigroup \( V \) such that \( E(V) = E(U_e) \); in particular, there exist monomorphisms \( \psi_e : U_e \to V \) and \( \psi_f : U_f \to V \) such that \( iv_e = \hat{\alpha}\psi_f \). Further \( S\psi_e \) is a subsemigroup of \( V \) isomorphic to \( S \) and \( e\psi_e \not\in f\psi_e \) in \( V \), the latter since \( e \not\in \cap e\phi_e \) in \( U_e,f \not\in \cap f\phi_f \) in \( U_f \), giving that in \( V \) we have

\[
e\psi_e \not\in e\phi_e \psi_e = e\phi_e \hat{\alpha}\psi_f = e\alpha\phi_f \psi_f
\]

\[
= f\phi_f \psi_f \not\in f\psi_f = (f\hat{\alpha})\psi_f = f\psi_e = f\psi_e.
\]

Our next step is to remove the idempotents of \( V \) not in \( S\psi_e \); in the light of Result 4 it is natural to take \( E(S\psi_e)^C \), the maximum regular subsemigroup of \( V \) with \( E(S\psi_e) \) as its idempotents. Because of the special nature of \( \langle E(V) \rangle \), namely as the 0-direct union of \( \langle E(S^0\psi_e) \rangle \) and a copy of \( \langle e^0 \rangle \), it is easy to see that for each mutually inverse pair \( u,v \in V \) such that \( uv = e\psi_e \), \( v'u = f\psi_e \), we have \( v,u \in E(S\psi_e)^C = N \) say, from Result 4, whence \( e\psi_e \not\in f\psi_e \in N \); however we prove this latter fact in more detail below, on our way to proving the second part of Theorem 1. A semigroup \( T \) satisfying just conditions (i) and (ii) could easily be obtained at this stage.

Consider the mutually inverse elements \((e,1,2) \) and \((e,2,1) \) in \( U_e \), and their images in \( U_e \) under our isomorphism \( \theta_e \) above, say \( a \) and \( a' \) respectively. Then \( aa' = e, a'a = e\phi_e \).

Likewise consider the mutually inverse elements \((f,1,2) \) and \((f,2,1) \) in \( U_f \), and let their images in \( U_f \) under the isomorphism \( \theta_f \) be \( b \) and \( b' \) respectively. Again of course we have \( bb' = f, b'b = f\phi_f \).

It is easy to check that in \( V \) the elements \( v = (a\psi_e)(b'\psi_f) \) and \( v' = (b\psi_f)(a'\psi_e) \) are mutually inverse and that their two products are given by \( vv' = e\psi_e, v'v = f\psi_e \). We now show that, for all \( s \in \langle e \rangle \), \( v'(S\psi_e)v \) is \((sa)\psi_e \). We have

\[
v'(s\psi_e)v = (b\psi_f)(a'\psi_e)(s\psi_e)(a\psi_e)(b'\psi_f) = (b\psi_f)((a'sa)\psi_e)(b'\psi_f).
\]

In \( U_e \) we have \((e,2,1)(s,1,1)(e,1,2) = (s,2,2) \), so in \( U_e \) we have \( a'sa = s\phi_e \). Continuing from above, we have

\[
v'(s\psi_e)v = (b\psi_f)(s\phi_e\psi_e)(b'\psi_f) = (b\psi_f)(s\phi_e\psi_e)(b'\psi_f) = (b\psi_f)(s\phi_e\psi_e)(b'\psi_f) = (b\psi_f)(sa\phi_f)(b'\psi_f) = (b\phi_f)(bsa)(b'\psi_f).
\]

In \( U_f \) we have \((f,2,1)(sa,2,2)(f,1,2) = (sa,1,1) \) so in \( U_f \) we have \( b(sa\phi_f)b' = sa \). Thus \( v'(s\psi_e)v = (sa)\psi_f = (sa)\hat{\alpha}\psi_f = (sa)\psi_e = (sa)\psi_e, \) since \( sa \in \langle E(S) \rangle \).

Identifying \( S \) with \( S\psi_e \), we see that for any \( e,f \in E(S) \) with \( \langle e \rangle \cong \langle f \rangle \) and for each isomorphism \( \alpha : \langle e \rangle \to \langle f \rangle \), we have embedded \( S \) in a regular semigroup \( N = N(e,f,\alpha) \) say, such that \( E(N) = E(S) \) and there exist mutually inverse elements
By Result 5 again, the amalgam \((N(e,f,a), (e,f,a)_{S})\), where \(I\) is the set of all triples \((e,f,a)\) such that \(e,f \in E(S), \langle e \rangle \leq \langle f \rangle\) and \(a\) is an isomorphism of \(\langle e \rangle\) onto \(\langle f \rangle\), is embeddable in a regular semigroup \(T\) such that \(E(T) = E(S)\). Clearly this second appeal to Result 5 can be replaced with a simple transfinite induction argument by well-ordering the triples in \(I\).

If \(S\) is finite, then at each step in our argument all semigroups can be made finite (note that finiteness can be preserved when applying Result 5). Thus \(T\) can be taken to be finite when \(S\) is finite, and so \(T\) has all the required properties. This completes the proof of Theorem 1.

For the statement and proof of the following corollary only, we assume familiarity with the representation \((\varrho, \lambda)\) and the semigroup \(T_{(E)}\) of [6]. The corollary answers a question posed by K.S.S. Nambooripad (private communication). We consider the representation \((\varrho, \lambda): T \rightarrow T_{(E)}\), defined as in [6] for any regular semigroup with core \(\langle E \rangle\).

**Corollary 2.** For \(T\) as in Theorem 1, \(T(\varrho, \lambda) = T_{(E)}\); in particular \(T/\mu(T) = T_{(E)}\).

**Proof.** Since the core of \(T\) is \(\langle E \rangle = \langle E(S) \rangle\) we have that \(T(\varrho, \lambda) \subseteq T_{(E)}\), from [6, Theorem 7(iii)]. Conversely, we take any \(e, f \in E\) with \(\langle e \rangle \equiv \langle f \rangle\) and any \(a \in T_{e,f}\), that is, any isomorphism \(\alpha: \langle e \rangle \rightarrow \langle f \rangle\). Take the elements \(v, v' \in N = N(e,f,a)\) as in the second last paragraph of the proof of Theorem 1. We can assume without loss that \(N\) is a full subsemigroup of \(T\). It is now routine to check that \((\varrho_0, \lambda_0) = \phi(\alpha)\).

4. 0-bisimple and 0-simple regular semigroups

Any idempotent-generated regular semigroup will be called simply a core. Given a core \(B\), we define an equivalence relation \(\psi\) on \(E = E(B)\) as in [6] by

\[\psi = \{(e, f) \in E \times E: \langle e \rangle \equiv \langle f \rangle\}.\]

We call \(B\) uniform if \(\psi = E \times E\). If \(B\) has a zero element 0, then we call \(B\) 0-uniform if

\[\psi = \{(0, 0)\} \cup ((E \setminus \{0\}) \times (E \setminus \{0\})).\]

**Theorem 3.** The cores of bisimple [0-bisimple] regular semigroups are precisely the uniform [0-uniform] cores.

**Proof.** That the cores of bisimple [0-bisimple] regular semigroups are uniform
[0-uniform] follows trivially from Result 1 (and is stated in the final paragraph of [6, Section 6]).

Conversely, if a core $B$ is uniform [0-uniform], then by putting $S = B$ we see that the semigroup $T$ of Theorem 1 is regular, bisimple [0-bisimple] and has core $B$, as required.

We define the core $B$ to be subuniform if for any $e, f \in E$ there exists $g \in E$ such that $f \not\approx g \leq e$. Likewise, if $B = B^0$, we define $B$ to be 0-subuniform if for any non-zero elements $e, f \in E$ there exists $g \in E$ such that $f \not\approx g \leq e$.

**Theorem 4.** The cores of simple [0-simple] regular semigroups are precisely the subuniform [0-subuniform] cores.

**Proof.** That the core of a simple [0-simple] regular semigroup is subuniform [0-subuniform] follows trivially from Results 1 and 2 (and is stated in the final paragraph of [6, Section 6]).

Conversely, if a core $B$ is subuniform [0-subuniform], then putting $S = B$ we see that the semigroup $T$ of Theorem 1 is regular, simple [0-simple] (from Result 2) and has core $B$, as required.

Theorems 3 and 4 answer affirmatively questions posed in [6, Section 6] and posed also as Questions H1 and H2 respectively in the Proceedings of the 1979 DeKalb Conference. For inverse semigroups the theorems are due to Munn [14] and [16] respectively and for orthodox semigroups they are due to the author [3] and [6]; for an exposition see Howie's book [11]. For fundamental cores (including bands), the results are due to the author [6, Theorems 9 and 10].

5. Completely semisimple semigroups

Here we consider two questions: for which cores $B$ is it the case that every regular semigroup with core $B$ is completely semisimple [a union of groups]?

We define the relation $\not\approx$ on $B$ to be flat if $\not\approx$ contains no pair of distinct comparable idempotents, that is, if for all $(e, f) \in \not\approx$, $f \leq e$ implies $f = e$.

**Theorem 5.** Every regular semigroup with core $B$ is completely semisimple if and only if $\not\approx$ is flat.

**Proof.** (i) If $\not\approx$ is flat, then every regular semigroup with core $B$ contains no distinct comparable $\not\approx$-related idempotents and so is completely semisimple. (The 'if' statement also occurs in the final paragraph of [6, Section 6].)

(ii) Conversely, with $S = B$ and with $T$ as in Theorem 1, we see that $T$ being completely semisimple implies that $\not\approx$ on $B$ is flat.
We call $B$ antiuniform if $\mathcal{U} = \mathcal{D}(B) \cap (E \times E)$, where $\mathcal{D}(B)$ denotes Green's relation $\mathcal{D}$ on $B$.

**Theorem 6.** Every regular semigroup with core $B$ is a union of groups if and only if $B$ is antiuniform and a union of groups.

**Remark 2.** Recall from [6, Theorem 3] that $B = \langle E \rangle$ is a union of groups if and only if for all $e, f, g \in E$ such that $e \mathcal{L} f \mathcal{R} g$ in $B$ there exists $h \in E$ such that $e \mathcal{R} h \mathcal{L} g$.

**Proof of Theorem 6.** The 'if' statement occurs in the final paragraph of [6, Section 6]. Conversely, with $S = B$ and with $T$ as in Theorem 1, we have that if $T$ is a union of groups then $\langle E(T) \rangle = B$ is a union of groups (from Remark 2 above), and for all $(e, f) \in \mathcal{U}$, from $e \mathcal{D} f$ in $T$, a union of groups, we have $e \mathcal{R} ef \mathcal{L} f$ in $T$ and hence also in the regular subsemigroup $\langle E(T) \rangle = B$ [11, Proposition II.4.5], whence $e \mathcal{D} f$ in $B$; thus $\mathcal{U} \subseteq \mathcal{D}(B) \cap (E \times E)$. But always $\mathcal{U} \subseteq \mathcal{D}(B) \cap (E \times E)$ so $\mathcal{U} = \mathcal{D}(B) \cap (E \times E)$, which completes the proof.

For inverse semigroups Theorem 6 is due to Howie and Schein [10] and for orthodox semigroups it is due to the author [3]. For $B$ fundamental, Theorems 5 and 6 are due to the author [6, Theorems 11 and 12].

6. $\mathcal{H}$-compatible regular semigroups

A semigroup is called $\mathcal{H}$-compatible if Green's relation $\mathcal{H}$ is a congruence. We call any core $B$ taut if for each $e \in E(B)$ the only automorphism of $\langle e \rangle$ is the identity function on $\langle e \rangle$; of course this is equivalent to there being a unique isomorphism from $\langle e \rangle$ to $\langle f \rangle$ for all $(e, f) \in \mathcal{U}$.

**Theorem 7.** Every regular semigroup with core $B$ is $\mathcal{H}$-compatible if and only if $B$ is taut.

**Proof.** (i) Suppose $B$ is taut and take any $a, b \in S$ such that $a \not\mathcal{H} b$. Take any idempotents $e \in R_a$, $f \in L_a$ and take the inverses $a'$ and $b'$ of $a$ and $b$ respectively in $L_e \cap R_f$. Now there is only one isomorphism from $\langle e \rangle$ onto $\langle f \rangle$, so $\theta_{a', a}$ and $\theta_{b', b}$ agree on $\langle e \rangle$, in particular on $E(eSe)$. By Result 3 we have $(a, b) \in \mu$, whence $\mathcal{H} = \mu$, a congruence, as required. (The 'if' statement also occurs in the final paragraph of [6, Section 6].)

(ii) Conversely, put $S = B$ and suppose that $T$ as in Theorem 1 is $\mathcal{H}$-compatible. Suppose that $B$ is not taut, that is, that for some $e \in E$, $\langle e \rangle$ has two different automorphisms, say $\alpha$ and $\beta$. Then in $T$ there are elements $a, b$ with inverses $a', b'$ respectively such that $a, b, a', b' \in H_e$ and $\theta_{a', a} \langle e \rangle = \alpha$ and $\theta_{b', b} \langle e \rangle = \beta$, by Theorem 1. Since $\langle e \rangle = \langle E(eSe) \rangle$ we have that $\alpha$ and $\beta$ disagree on $E(eSe)$, whence
$\theta_{a',a}$ and $\theta_{b',b}$ disagree on $E(eSe)$. Again by Result 3 we have $(a,b) \notin \mu$, whence $\not\leqslant \mu$, the maximum congruence on $T$ contained in $\mathcal{H}'$, that is, $T$ is not $\mathcal{H}'$-compatible, a contradiction, as required.

The corresponding result for inverse semigroups was proved by W.D. Munn in the proof of [14, Theorem 3.2]. For $B$ fundamental (including $B$ a band) the result is due to the author [6, Theorem 13].

7. Counterexamples

The following theorem lists seven implications. We provide examples to show that the seven converse implications are false. There is a more conceptual proof of the theorem, and of the two lemmas, from Theorem 1; we prefer to give the following more elementary, though longer, proof.

**Theorem 8.** Take any core $B$.

(i) If $B$ is uniform [0-uniform, subuniform, 0-subuniform], then its maximum fundamental image $B/\mu$ is likewise.

(ii) If $B/\mu$ is antiuniform [taut], then $B$ is antiuniform [taut].

(iii) If $\Psi$ on $B/\mu$ is flat, then $\Psi$ on $B$ is flat.

Theorem 8(i) follows directly from the following lemma (and Lallement's Lemma); we prove statements (ii) and (iii) after proving two lemmas.

For any regular semigroup $S$ we denote by $\Psi(S)$ the equivalence relation $\Psi$ defined above on $E = E(S)$. By $\Psi(S)\mu^h$ we mean the equivalence relation $\{(e\mu,f\mu) \in (S/\mu) \times (S/\mu) : (e,f) \in \Psi\}$ on $E(S/\mu) = E(S)\mu^h$.

**Lemma 9.** For $S$ any regular semigroup, $\Psi(S)\mu^h \subseteq \Psi(S/\mu)$.

**Proof.** Take any $e \in E(S)$. From [6, Corollary 6], or from Result 3, we have $\mu(\langle e \rangle) = \mu(S) \cap (\langle e \rangle \times \langle e \rangle)$, whence $\langle e \rangle / \mu(\langle e \rangle) \equiv \langle e \rangle \mu^h \subseteq S/\mu$. From Lallement's Lemma applied to the regular semigroup $eSe$ we have that the subsemigroup $\langle e\mu \rangle$ of $S/\mu$ equals $\langle e \rangle \mu^h$. Thus, for any $(e,f) \in \Psi(S)$ we have

$$\langle e\mu \rangle = \langle e \rangle \mu^h \equiv \langle e \rangle / \mu(\langle e \rangle) \equiv \langle f \rangle / \mu(\langle f \rangle) \equiv \langle f \rangle \mu^h = \langle f\mu \rangle,$$

which gives us that $\Psi(S)\mu^h \subseteq \Psi(S/\mu)$, as required.

Note that in the examples below we have $\Psi(S)\mu^h \neq \Psi(S/\mu)$.

**Lemma 10.** For any regular semigroup $S$ and any $(e,f) \in \Psi(S)$, each isomorphism $\alpha : \langle e \rangle \rightarrow \langle f \rangle$ induces an isomorphism of $\langle e\mu \rangle$ upon $\langle f\mu \rangle$, namely the unique morphism $\alpha^*$ satisfying $(x\mu)\alpha^* = (x\alpha)\mu$ for each $x \in E(eSe)$. 
Proof. Of course $(\mu(\langle e \rangle))\alpha = \mu(\langle f \rangle)$, where by $(\mu(\langle e \rangle))\alpha$ we mean $\{(a\alpha, b\alpha) \in \langle f \rangle \times \langle f \rangle : (a, b) \in \mu(\langle e \rangle)\}$, since $\alpha$ is an isomorphism and $\mu(\langle e \rangle), \mu(\langle f \rangle)$ are the maximum idempotent-separating congruences on $\langle e \rangle, \langle f \rangle$ respectively. Hence the map $\beta : \langle e \rangle/\mu(\langle e \rangle) \to \langle f \rangle/\mu(\langle f \rangle)$ given by $(a\mu(\langle e \rangle))\beta = (a\alpha)\mu(\langle f \rangle)$ for all $a \in \langle e \rangle$, is an isomorphism. By considering the obvious isomorphism from $\langle e \rangle/\mu(\langle e \rangle)$ to $\langle e\mu \rangle$ and the one from $\langle f \rangle/\mu(\langle f \rangle)$ to $\langle f\mu \rangle$, we see that the map $\alpha^* : \langle e\mu \rangle \to \langle f\mu \rangle$, given by $(a\mu)\alpha^* = (a\alpha)\mu$ for all $a \in \langle e \rangle$, is an isomorphism. Further, from Lallement's Lemma applied to $eSe$, we see that $\langle e\mu \rangle = \{\{x\mu : x \in E(eSe)\}\}$, so $\alpha^*$ is the unique morphism from $\langle e\mu \rangle$ to $\langle f\mu \rangle$ satisfying $(x\mu)\alpha^* = (x\alpha)\mu$, for all $x \in E(eSe)$.

We now prove statements (ii) and (iii) of Theorem 8. From [4, Theorem 10] we have that for any regular semigroup $S$, $\varphi(S/\mu) = \varphi(S)\mu^1 (= \{(a\mu, b\mu) : (a, b) \in \varphi(S)\})$ of course, so if a core $B$ is such that $B/\mu$ is antiuniform, then for any $(e, f) \in \varphi(B)$ we have from Lemma 9 that $(e\mu, f\mu) \in \varphi(B/\mu) \subseteq \varphi(B/\mu) = \varphi(B)\mu^1$ whence $(e, f) \in \varphi(B)$ and $B$ is antiuniform also.

Note that for any core $B$ and for any $(e, f) \in \varphi(B)$, two distinct isomorphisms $\alpha, \beta$ from $\langle e \rangle$ to $\langle f \rangle$ differ on $E(eSe)$ so $\alpha^*, \beta^*$ differ on $E((e\mu)/(S/\mu)(e\mu))$ (since $\mu$ is idempotent-separating) and hence $\alpha^*$ and $\beta^*$ are distinct. Thus, if $B/\mu$ is taut, then $B$ is taut also.

For any regular semigroup $S$, for any $e, f \in E(S)$, we have (from Lallement's Lemma applied to $eSe$) that $f\mu \leq e\mu$ in $S/\mu$ if and only if $f \leq e$ in $S$. Statement (iii) of Theorem 8 now follows from Lemma 9. This completes the proof of Theorem 8.

Example 1. For $n = 1, 2, 3, ...$ put $M_n = \mathcal{R}(\langle a_n \rangle; 2, 2; P)$, a Rees matrix semigroup over a cyclic group with generator $a_n$ say, the sandwich matrix $P$ being given by

$$P = \begin{pmatrix} 1 & 1 \\ 1 & a_n \end{pmatrix}.$$ 

We assume without loss of generality that the semigroups $M_n$, $n = 1, 2, 3, ...$, are pairwise disjoint, we put $B = \bigcup \{M_n : n = 1, 2, 3, ... \}$ and we extend the binary operation of each $M_n$ to $B$ as follows: for any positive integers $m, n$ with $m < n$, for any $x \in M_m$, $y \in M_n$, we define $xy = yx = y$. It is easy to see that $B$ is a semigroup, in fact a core and a union of groups, and that $\mu(B) = \mathcal{R}(B)$. The band $B/\mu(B)$ is uniform (and is an example of an almost commutative band as in [5]); by choosing different orders for two of the cyclic groups, $\langle a_m \rangle$ and $\langle a_n \rangle$ say, where $2 \leq m < n$, we obtain a core $B$ which is not uniform. By making all the orders of the groups $\langle a_n \rangle$ distinct we obtain a core $B$ which is antiuniform, in particular not sub-uniform; further $\varphi(B)$ is flat while of course $\varphi(B/\mu) = (B/\mu) \times (B/\mu)$ is not flat; and finally the core $B^0$ is antiuniform, neither 0-uniform nor 0-subuniform, while $B^0/\mu$ is 0-uniform. These examples show that six of the seven reverse implications are false, namely all except the implication "$B$ is taut implies $B/\mu$ is taut", which we
show is false with our next example.

**Example 2.** Let $\langle a \rangle$ and $\langle b \rangle$ be disjoint cyclic groups, and put $S=\langle a \rangle; 2; 2; P$, $T=\langle b \rangle; 2; 2; Q$, where

$$P = \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 1 \\ 1 & b \end{pmatrix}.$$ 

Take new symbols $e, f$. Extend the multiplication of $S$ to one on $S \cup \{e\}$ by defining $e^2 = e$, $e(g; i, \lambda) = (g; 1, \lambda)$ and $(g; i, \lambda)e = (g; i, 1)$, for all $(g, i, \lambda) \in S$ (then $S \cup \{e\}$ is the ideal extension of $S$ by the two element semilattice $\{0, e\}$ with $0 < e$, determined by the partial homomorphism $e \mapsto (1; 1, 1) \in S$ [1, Theorem 4.11]).

Similarly, extend the multiplication of $T$ to one of $T \cup \{f\}$ by defining $f^2 = f$, $f(g; i, \lambda) = (g; 1, \lambda)$ and $(g; i, \lambda)f = (g; i, 1)$, for all $(g, i, \lambda) \in T$. Now add a common zero $0$ to both $S \cup \{e\}$ and $T \cup \{f\}$ and let $U$ be the $0$-direct union of the two semigroups thus obtained. Finally put $B = U^1$, the semigroup obtained by adjoining an identity element $1$ to $U$. Now $B$ is a core, $\mu(B) = \kappa(B)$ and the band $B/\mu$ has two automorphisms, so $B/\mu = (B/\mu)^1$ is not taut. By prescribing the orders for the groups $\langle a \rangle$ and $\langle b \rangle$ to be distinct (so that $S$ and $T$ are not isomorphic) we obtain a core $B$ which is taut: for note that while $S$ and $T$ have four automorphisms each, $S \cup \{e\}$ and $T \cup \{f\}$ have only one each (the trivial one), giving us that $B = B^1$ has only automorphism; of course the semigroup $\langle x \rangle$, for each $x \in E(B)$, is either $B$ or a semilattice which is a chain of one, two or three elements, and so has only one automorphism.

8. Embedding locally inverse semigroups

By a **locally inverse semigroup** we mean a regular semigroup $S$ such that for each idempotent $e \in S$, the subsemigroup $eSe$ is an inverse semigroup. Our final application of Theorem 1 is the following embedding theorem for locally inverse semigroups; comments on its significance follow immediately, in Remark 4. By an order ideal in a partially ordered set $(P, \preceq)$ is meant a subset $I$ of $P$ such that for all $i \in I$, $p \in P$, $p \preceq i$ implies $p \in I$.

**Theorem 11.** Any locally inverse semigroup $S$ is embeddable in a locally inverse semigroup $T$ with a maximum $\mathcal{J}$-class (that is, such that $T = TuT$ for some idempotent $u \in T$) and such that $E(S)$ is an order ideal of $E(T)$.

**Remark 4.** Following on from work of Pastijn [17], McAlister [12] showed that any locally inverse semigroup $T$ such that $T = TuT$, for some idempotent $u$, is a locally isomorphic image of a regular Rees matrix semigroup over an inverse semigroup. Theorem 11 enabled a generalization to be obtained for all locally inverse semigroups, and this is given in [13, Theorem 2.1], though with a new proof. For
$S = S^0$, further generalizations are given in [13, Theorem 4.1]. I am grateful to Don McAlister for telling me of the problem solved by Theorem 11, especially so, since considering this problem gave me ideas that led to the proof of Theorem 1.

Remark 5. Because of its simplicity, we give a proof of Theorem 11 in the special case where $S = S^0$; note that we would lose generality by assuming $S = S^0$ (this is emphasized by the difficulty of our general proof) since $E(S)$ is not an order ideal of $E(S^0)$ if $S \neq S^0$. Assuming $S = S^0$, we take $F$ to be the 0-direct union of the semilattices $\{E(Se) : e \in E(S), e \neq 0\}$ (we may take only one of each isomorphism type if we wish) and then we take $W$ to be the 0-direct union of $S$ and $F^u$, where $F^u$ denotes $F$ with an identity element $u$ adjoining. Embedding $W$ in a semigroup $T$ as in Theorem 1 gives us an embedding of $S$ in $T$ as required ($J_u$ is the maximum $\mathcal{J}$-class of $T$).

Proof of Theorem 11. We take a fixed idempotent, $i$ say, in $S$. For each $e \in E(S)$ we let $(ei)'$ be an inverse of $ei$ and we consider $i((ei)'e)$, which is also an inverse of $ei$ (and an idempotent; the earliest reference for this elementary but important result seems to be [9, Lemma 1.1]).

We put $e^* = (ei)(i((ei)'e)$ and $e'' = (i((ei)'e)(ei)$; then $e'$ and $e''$ are $\mathcal{D}$-related idempotents and $e' \leq e$, $e'' \leq i$. Since $e' \not\leq e''$ we have $e'Se'$ and $e''Se''$ are isomorphic, so the semilattices $\langle e' \rangle$ and $\langle e'' \rangle$ are isomorphic; we see next how to adjoin a copy of $\langle e' \rangle$ to $S$ amalgamating $\langle e'' \rangle$ and $\langle e'' \rangle$; specifically, denoting $\theta_{i((ei)'e)} \langle e'' \rangle$ by $\psi$, we embed the amalgam $(S, \langle e' \rangle; \langle e'' \rangle; \psi, i)$ in a special way into a locally inverse semigroup (here $i$ denotes the insertion of $\langle e'' \rangle$ in $\langle e' \rangle$).

Now $\langle e' \rangle$ is an ideal of $\langle e \rangle$ with identity element $e'$, so, as in [1, Theorem 4.19], the semilattice $\langle e \rangle$ is a semilattice $\langle e' \rangle$ determined by the partial homomorphism $\phi_e : \langle e' \rangle \rightarrow \langle e' \rangle$ defined by $\phi_e = e'$. Since $\psi : \langle e'' \rangle \rightarrow \langle e'' \rangle$ is an isomorphism of $\langle e' \rangle$ into $S$, from which we can construct an ideal extension of $S$ by $\langle e' \rangle \rightarrow \langle e' \rangle$ (assumed without loss to be disjoint from $S$) as in [1, Theorem 4.19]; this new semigroup is also locally inverse and has $E(S)$ as an order ideal of its set of idempotents. This idea is modified below to deal with all $\langle e' \rangle$ simultaneously after adjoining a common identity $u$.

Remark 6. There are always cases in which $e = e'$ (for example, when $e \leq i$). If $e = e'$, then $\psi$ is the empty function and the ideal extension of $S$ by $\langle e' \rangle$ determined by $\phi_e \psi$, also the empty function, is just $S$. The degenerate cases where $e = e'$ do no harm to the following proof. Alternatively, consideration of cases where $e = e'$ is easily avoided; one simply restricts the construction below to idempotents $e$ such that $J_e \leq J_i$, which exist unless $S$ has $J_i$ as its maximum $\mathcal{J}$-class. In either case, the constructions are actually identical.

We return to proving Theorem 11. For each $e \in E(S)$ let $T_e$ be an isomorphic copy of $\langle e' \rangle$ disjoint from $S$ and such that different $T_e$'s have in common
only their zero element, 0 say; also let \( \eta_e : T_e \rightarrow \langle e \rangle / \langle e' \rangle \) be an isomorphism. We denote by \( U \) the semilattice obtained by forming the 0-direct union of \( \{ T_e : e \in E(S) \} \) and then adjoining an identity element, \( u \) say. There is an obvious partial homomorphism of \( U \setminus \{ 0 \} \) into \( S \), say \( \phi \), defined by \( u\phi = i \) and for all \( e \in E(S) \), for all \( t \in T_e \setminus \{ 0 \} \), \( r \phi = \eta + \phi \eta \). We denote by \( V \) the semigroup obtained by taking the ideal extension of \( S \) by \( U \) determined by the partial homomorphism \( \phi \).

Some properties of \( V \) are as follows: \( E(V) = E(S) \cup (U \setminus \{ 0 \}) \) and \( E(S) \) is an order ideal of \( E(V) \); in \( V \), the semigroup \( \langle u \rangle \) is the ideal extension of \( \langle i \rangle \) by \( U \) determined by \( \phi \), so \( \langle u \rangle \) is a semilattice; \( V \) is locally inverse (\( V \) is regular since \( S \) and \( U \) are regular); for each \( e \in E(S) \), if \( e \neq e' \), then \( e\eta^{-1} e' \in T_e \setminus \{ 0 \} \subseteq U \setminus \{ 0 \} \) and within \( V \) the semilattices \( \langle e \rangle \) and \( \langle e\eta^{-1} e' \rangle \) are isomorphic (since \( \langle e \rangle \) is the ideal extension of \( \langle e' \rangle \) by \( \langle e\eta^{-1} e' \rangle \) determined by \( \phi \) and \( \eta^{-1} \) is the ideal extension of \( \langle e' \rangle \) by \( T_e \) determined by \( \eta \phi \) and \( \eta : T_e \rightarrow \langle e \rangle / \langle e' \rangle \) is an isomorphism).

Now we embed \( V \) in a regular semigroup \( T \) as in Theorem 1. Since \( E(V) = E(T) \), we have that \( T \) is locally inverse and that \( E(S) \) is an order ideal of \( E(T) \). Further, for any \( f \in E(T) \), either \( f \in U \) whence \( f \leq u \), or \( f \in S \). If \( f \in S \), then either \( f \neq f' \) whence \( \langle f \rangle \equiv \langle f\eta^{-1} \rangle \) in \( V \) so that \( f \not\bigtriangledown f\eta^{-1} \leq u \) in \( T \), or \( f = f' \not\bigtriangledown f'' \leq i \leq u \) (in \( V \) and in \( T \)). Thus in all cases we have \( J_f \leq J_u \), that is, \( T \) has a maximum \( J \)-class, \( J_u \); this completes the proof.

Remark 7. If \( S \) has minimal idempotents, equivalently if \( S \) has a least \( J \)-class which is a completely simple subsemigroup (for example, if \( S \) is finite), then by choosing \( i \) to be a minimal idempotent we can simplify the above proof.

References