An Application of a Theorem of De Bruijn, Tengbergen, and Kruyswijk

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ABSTRACT

Let \( \{a_i\} \) be an increasing sequence of positive integers containing no three distinct elements \( a_i, a_j, a_k \), for which the lowest common multiple of \( a_i, a_j \) is equal to \( a_k \). By using the theorem mentioned in the title, we prove that for sufficiently large \( n \),

\[
\sum_{a_i \leq n} \frac{1}{a_i} < \left(2 \sqrt{\frac{\log n}{\log \log n}} + \epsilon \right) \frac{\log n}{\sqrt{\log \log n}}.
\]

Erdős, Sárkösy, and Szemerédi [1] have proved the existence of an absolute constant \( C \) such that, if \( \{a_i\} \) is an increasing sequence of positive integers for which

\[
\sum_{a_i \leq n} \frac{1}{a_i} > C \frac{\log n}{\sqrt{\log \log n}},
\]

then there are distinct elements \( a_i, a_j, a_k \) such that\(^1\) \( [a_i, a_j] = a_k \). The proof depends on a striking combinatorial result due to Kleitman [2]. Kleitman's result is the square-free case of the lemma below, and is based on an old theorem of Sperner [3]. In the form relevant to this paper, Sperner's theorem asserts that a set of divisors of the square-free number

\[
m = \prod_{i=1}^{n} p_i,
\]

no one divisor dividing another, can contain at most \( \binom{n}{\lceil n/2 \rceil} \) elements. The generalization of this theorem to any \( m \), not necessarily square-free,

\(^1\) \([a, b]\) denotes the least common multiple of \( a \) and \( b \).
We define the degree \( \Omega(m) \) of \( m = \prod_i p_i^{a_i} \) to be \( \Omega(m) = \sum_i a_i \), and denote by \( s(m) \) the number of divisors of \( m \) of degree \( \frac{1}{2}\Omega(m) \). We say further that a set \( d_1, \ldots, d_h \) of divisors of \( m \) which satisfy the conditions

\[
\frac{d_{i+1}}{d_i} \text{ is a prime} \quad (i = 0, \ldots, h - 1),
\]

\[
\Omega(d_i) + \Omega(d_h) = \Omega(m),
\]

form a symmetric chain. The authors of [4] proved in a very simple way that the divisors of \( m \) can be placed in \( s(m) \) disjoint-symmetric chains. It follows immediately that a set of divisors of \( m \), none dividing another, can contain at most one element from each chain and hence at most \( s(m) \) elements.

In this note we show how this result can be applied to obtain a numerical upper bound for the best possible value of \( C \). We first combine it with Kleitman’s elegant method to obtain the following combinatorial result.

**Lemma.** Let \( A = \{a_1, \ldots, a_r\} \) be a set of divisors of \( m \) such that for all \( i, j, k, [a_i, a_j] \neq a_k \). Then

\[
r \leq \tau(m) \min_{m \in \mathbb{N}} \frac{s(u)}{\tau(u)} \leq \frac{s(v)}{\tau(v)}
\]

where \( \tau \) denotes the divisor function.

**Proof:** Consider a particular choice of \( u, v \) and place the divisors \( b_i \) of \( u \) in \( s(u) \) disjoint-symmetric chains. Let us denote the divisors of \( v \) by \( c_i \).

For each \( c_i \) consider the first \( b \) in each symmetric chain for which \( bc_i \in A \), and denote the set of all elements of \( A \) arising from \( c_i \) in this way by \( S_i \).

Then, since there are \( \tau(v) \) choices of \( c_i \), \( \bigcup_i S_i \) contains at most \( s(u)\tau(v) \) elements of \( A \). Let \( A' = A - \bigcup_i S_i \), and suppose, if possible, that there exist \( b_1, c_1, c_2 \) such that \( b_1c_1 \in A' \), \( b_1c_2 \in A' \), \( c_1 \mid c_2 \). Then there exists \( b_2 \), in the same chain as \( b_1 \), for which \( b_2b_1 \) and \( b_2c_2 \in A - A' \). We then have \( [b_1c_1, b_2c_2] = b_1c_2 \), which contradicts the hypothesis of the lemma.

It follows by the theorem in [4] that for a fixed \( b_i \) there are at most \( s(v) \) divisors \( c_j \) of \( v \) such that \( bc_j \in A' \). Thus \( A' \) contains at most \( \tau(v) s(v) \) elements and hence there are at most \( s(u)\tau(v) + s(v)\tau(u) \) elements in \( A \).

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\[ This \ theorem \ is \ related \ to \ a \ result \ of \ Dilworth \ on \ partially \ ordered \ sets (Annals \ of \ Math. \ 51, 161-166). \]
It is proved in [5] that
\[ s(m) \leq \frac{\tau(m)}{2^{\Omega(m)}} \left( \frac{\Omega(m)}{\left[ \frac{1}{2} \Omega(m) \right]^{2}} \right). \]

On applying Stirling's formula we obtain, for any given \( \epsilon > 0 \), an integer \( N = N(\epsilon) \) such that
\[ s(m) \leq \left( \frac{2}{\pi} + \epsilon \right) \frac{\tau(m)}{\sqrt{\Omega(m)}} \]

provided \( \Omega(m) > N(\epsilon) \). We thus have

**Corollary.** If \( m \) can be expressed as \( m = m_1m_2 \) where \( (m_1, m_2) = 1 \) and \( \Omega(m_i) \geq \left( \frac{1}{2} - \epsilon \right) \Omega(m) \) \((i = 1, 2)\), then if \( \Omega(m) > N(\epsilon) \),
\[ r \leq \left( \frac{4}{\sqrt{\pi}} + \epsilon \right) \frac{\tau(m)}{\sqrt{\Omega(m)}}. \]

We are now in a position to prove our main result.

**Theorem.** Let \( \{a_i\} \) be an increasing sequence of positive integers which does not contain distinct elements \( a_i, a_j, a_k \) such that \([a_i, a_j] = a_k\). Then for \( n > n(\epsilon) \),
\[ \sum_{a_i \leq n} \frac{1}{a_i} \leq \left( \frac{2}{\sqrt{\pi}} + \epsilon \right) \frac{\log n}{\sqrt{\log \log n}}. \]

**Proof:** Let \( n \) be a sufficiently large positive integer. We have
\[ n \sum_{a_i \leq n} \frac{1}{a_i} = \sum_{a_i \leq n} \left[ \frac{n}{a_i} \right] + O(n) = \sum_{m \leq n} r(m) + O(n) \]
where \( r(m) \) denotes the number of \( a_i \) dividing \( m \). Thus by the corollary to the lemma, with \( r = r(m) \),
\[ \sum_{a_i \leq n} \frac{1}{a_i} \leq \left( \frac{4}{\sqrt{\pi}} + \epsilon \right) \frac{1}{n} \sum_{m \leq n} \frac{\tau(m)}{\sqrt{\Omega(m)}} + \frac{1}{n} \sum_{m \leq n} \tau(m) + O(1), \]

where \( \sum' \) extends over all \( m \leq n \) satisfying the conditions of the above corollary, and \( \sum'' \) extends over all other \( m \).

Define the iterated logarithmic function \( l_i(n) \) by
\[ l_1(n) = \log n, \quad l_{i+1}(n) = l(l_i(n)) \quad (i = 1, 2, \ldots). \]
Using the fact that
\[ \sum' 1 = o(n) \quad \text{as} \quad n \to \infty, \]
where summation is over all \( m \leq n \) such that
\[ | \Omega(m) - l_g(n) | > (l_g(n))^{1+\delta}, \]
it is proved in [5] by an elementary argument that
\[ \sum_{m \leq n} \frac{\tau(m)}{\sqrt{\Omega(m)}} \sim \frac{nl_1(n)}{\sqrt{2l_g(n)}}, \quad n \to \infty. \]

Hence it follows that
\[ \sum_{a_i \leq n} \frac{1}{a_i} \leq \left( 2\sqrt{\frac{2}{\pi}} + \epsilon \right) \frac{l_1(n)}{\sqrt{l_g(n)}} + \frac{1}{n} \sum' \tau(m). \quad (3) \]

Further, since
\[ \sum_{\Omega(m) \leq l_g(n)} \tau(m) \leq \sum_{m \leq n} 2l_2(n) = o \left( \frac{nl_1(n)}{\sqrt{l_g(n)}} \right), \]
we need consider in \( \sum' \) only those \( m \) for which \( \Omega(m) > l_g(n) \). Then \( \sum' \) extends over all \( m \leq n \) with \( \Omega(m) > l_g(n) \), which cannot be written in the form \( m = m_1m_2 \), where \( (m_1, m_2) = 1 \) and
\[ \Omega(m_i) \geq (\frac{1}{2} - \epsilon)\Omega(m). \]

Now clearly all such \( m \) can be expressed in the form \( m = p^u \) where
\[ \alpha = \alpha(n) = \left[ \frac{2l_g(n)}{l_1(2)} \right] + 1 \]
(and where \( p \) may or may not divide \( u \)). Thus, since
\[ \tau(p^u) \leq \tau(p^2)\tau(u), \]
we have finally
\[ \sum' \tau(m) = O \left\{ \sum_{p^u \leq n} \tau(p) \tau(u) \right\} \]
\[ = O \left\{ l_g(n) \sum_{p^u \leq n} \tau(u) \right\}. \]
\[
= O \left\{ l_3(n) \sum_{p \leq n^{1/2}} \frac{n}{p^2} l_1 \left( \frac{n}{p^2} \right) \right\}
\]

\[
= O \left\{ n l_1(n) \frac{1}{2^n} \right\}
\]

The theorem now follows from (3).

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REFERENCES