Commutative algebras of rational function matrices as endomorphisms of Kronecker modules II

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Abstract

This paper is the sequel to the paper “Commutative algebras of rational function matrices as endomorphisms of Kronecker modules I”. Here we focus on those Kronecker modules that are rank-2 extensions of finite-dimensional, torsion-free modules by the unique generic module K. We show that if such an extension is indecomposable, then its endomorphism algebra is trivial. This resolves the missing case from the previous paper, where the effects of height functions on the endomorphism algebras of extensions are studied.

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1. Introduction

We pick up from our previous paper [6] and use the same notation. Take the field of rational functions K(X) in X over an algebraically closed field K, and a K-linear map α : K(X) → K whose value at any rational function r is written as ⟨α, r⟩. Such maps α will be called functionals. For each θ in K, we let Xθ = (X − θ)−1, for the
sake of brevity. The values of a functional $\alpha$ on the standard basis of $K(X)$ over $K$ are called the coefficients of $\alpha$ and denoted as follows:

$$\alpha^n_\theta = \langle \alpha, X^n \rangle, \alpha^0_\theta = \langle \alpha, X^{n+1}_\theta \rangle \quad \text{for } \theta \in K \text{ and } n = 0, 1, 2, \ldots \quad (1)$$

### 1.1. Functionals and their deriver

Given a functional $\alpha$ and a rational function $r$, it is shown in [2, Proposition 3.4] that there is a unique rational function $\partial\alpha(r)$ so that $\partial\alpha(r)(\theta)$ is defined whenever $r(\theta)$ is defined, i.e. whenever $\theta$ is not a pole of $r$, and for all such $\theta$

$$\partial\alpha(r)(\theta) = \langle \alpha, \frac{r - r(\theta)}{X - \theta} \rangle = \langle \alpha, (r - r(\theta))X\theta \rangle.$$

(2)

From this it is easy to see that the mapping $(\alpha, r) \mapsto \partial\alpha(r)$ is $K$-linear in both $\alpha$ and $r$. We have referred to the $K$-linear map $\partial\alpha : K(X) \to K(X)$ as a deriver. For a functional $\alpha$ and a rational function $r$, the functional given by $t \mapsto \langle \alpha, rt \rangle$ is denoted by $\alpha^* r$. The name deriver is motivated by the following derivation-like property which is easy to deduce from (2):

$$\partial\alpha(st) = s\partial\alpha(t) + \partial\alpha^*(s)$$

(3)

From (2), or more easily by induction using (3), the detailed formula in [6] for $\partial\alpha$ on the standard basis can be recovered:

$$\partial\alpha(1) = 0,$$

$$\partial\alpha(X^n) = \alpha^n_\theta X^{n-1} + \alpha^\theta_1 X^{n-2} + \cdots + \alpha^\theta_{n-2} X + \alpha^\theta_{n-1}, \quad (4)$$

$$\partial\alpha(X^n_\theta) = - (\alpha^{\theta}_0 X^n_\theta + \alpha^{\theta}_1 X^{n-1}_\theta + \cdots + \alpha^{\theta}_{n-2} X^2_\theta + \alpha^{\theta}_{n-1} X_\theta),$$

for all $n \geq 1$ and all $\theta$ in $K$. These formulae make it clear that the linear correspondence $\alpha \leftrightarrow \partial\alpha$ between functionals and derivers is bijective. From (4) it can be seen that for each rational function $r$:

$$\text{ord}_\theta(\partial\alpha(r)) \leq \max\{0, \text{ord}_\theta(r)\} \quad \text{for all } \theta \text{ in } K,$$

(5)

and

$$\text{ord}_\infty(\partial\alpha(r)) < \max\{0, \text{ord}_\infty(r)\}.$$

(6)

In particular the poles of $\partial\alpha(r)$ form a subset of the poles of $r$.

### 1.2. The modules $\mathcal{R}_\alpha$

The modules $\mathcal{V}(m, h, \alpha)$ defined in [6] when specialized to $m = 1$ and to the height function $h$ that is constantly $\infty$ on all of $K \cup \{\infty\}$ will be denoted here as $\mathcal{R}_\alpha$. To review them briefly let

$$V_\alpha = \left\{ \begin{pmatrix} r \\ s \end{pmatrix} \in K(X)^2 : s + \partial\alpha(r) \in K \right\}.$$
and let

\[ V'_{\alpha} = \left\{ \left( \frac{r}{s} \right) \in K(X)^2 : s + \partial_\alpha(r) = 0 \right\}. \]

Then \( R_{\alpha} \) is the Kronecker module

\[ V'_{\alpha} \xrightarrow{a} V_{\alpha} \xrightarrow{b} V_{\alpha}, \]

where \( a : \left( \frac{r}{s} \right) \mapsto \left( \frac{r}{s} \right) \) and \( b : \left( \frac{r}{s} \right) \mapsto X \left( \frac{r}{s} \right) \).

The purpose of this paper is to prove the following theorem.

**Theorem 1.1.** If \( \text{char } K \neq 2 \) and \( \mathcal{A}_{\alpha} \) is indecomposable, then \( \text{End} \mathcal{A}_{\alpha} \cong K \).

In [6, Theorem 3.2] it is shown that \( \text{End} \mathcal{V}(m, h, \alpha) \cong K \) also for those indecomposable modules \( \mathcal{V}(m, h, \alpha) \) in which the height function \( h \) satisfies the following

\[ h < \infty \text{ on all of } K \cup \{ \infty \} \text{ and } h \leq 1 \text{ on all but a finite subset of } K \cup \{ \infty \}. \]

For all other height functions we construct in [6] classes of indecomposable modules \( V(m, h, \alpha) \) which admit non-trivial endomorphisms. One should note that the height functions of Theorem 1.1 and of [6, Theorem 3.2] are at opposite extremes. It seems curious that the largest height function shares Theorem 1.1 exclusively with the class of smallest height functions. A discussion of partial orders on equivalence classes of height functions can be found in [1].

As noted above, \( \mathcal{A}_{\alpha} \) is the same module as \( \mathcal{V}(1, h, \alpha) \) in which \( m = 1 \) and \( h = \infty \) on all of \( K \cup \{ \infty \} \). It follows from [5, Proposition 4.9] that for such \( h \) the algebra \( \mathcal{V}(m, h, \alpha) \) does not depend on \( m \). Thus Theorem 1.1 holds also for all modules \( \mathcal{V}(m, h, \alpha) \) where \( h = \infty \) on all of \( K \cup \{ \infty \} \) and \( m \) is any positive integer.

In the representation theory of tame, hereditary, artin algebras there is a generic module \( \mathcal{A} \) that stands out as the unique, indecomposable, torsion-free, divisible module, see e.g. [3,7]. It plays the same role that the module \( K(X) \) plays in the \( K[X] \)-module theory, see [1]. The modules \( \mathcal{A}_{\alpha} \) of this paper are the extensions of the unique, simple, projective Kronecker module by the Kronecker module \( \mathcal{A} \). One may ask if Theorem 1.1 holds for all such extensions in the general tame, hereditary theory.

### 2. Derivators and Laurent expansions

For each \( \theta \) in \( K \) let \( K[[X - \theta]] \) denote the algebra of formal series in \( X - \theta \). Also let \( K[[X^{-1}]] \) be the algebra of formal series in \( X^{-1} \). Given a functional \( \alpha \) and \( \theta \) in \( K \), use its coefficients \( \alpha_n^\theta \) to define the \( \theta \)-expansion of \( \alpha \):

\[ \alpha_\theta = \alpha_0^\theta + \alpha_1^\theta (X - \theta) + \cdots + \alpha_n^\theta (X - \theta)^n + \cdots, \text{ in } K[[X - \theta]]. \]  

(7)

Also define the \( \infty \)-expansion of \( \alpha \):

\[ \alpha_\infty = \alpha_0^\infty + \alpha_1^\infty X^{-1} + \cdots + \alpha_n^\infty X^{-n} + \cdots, \text{ in } K[[X^{-1}]]. \]

(8)
These expansions can be used in reverse as well to provide a way of uniquely building a functional by making one choice of power series in each \( K[[X - \theta]] \) and one choice in \( K[[X^{-1}]] \). The advantage accrued by representing functionals in terms of the series (7) and (8) will come from the multiplication within the algebras \( K[[X - \theta]] \) and \( K[[X^{-1}]] \).

2.1. Laurent representations of functions

For each \( \theta \in K \) the algebra \( K[[X - \theta]] \) lies in the field \( K((X - \theta)) \) of Laurent series in \( X - \theta \). As noted in [4, Chapter 2] this field may be viewed as the completion of \( K(X) \) with respect to the valuation \( \text{ord}_\theta \) on \( K(X) \). There is a unique field embedding of \( K(X) \) into \( K((X - \theta)) \) that fixes \( K \) and maps \( X - \theta \) in \( K(X) \) to itself in \( K((X - \theta)) \). If \( K \) is the complex numbers, this embedding gives the classical Laurent expansion of a rational function in powers of \( X - \theta \). For general fields we now outline this familiar embedding.

If \( r \in K(X) \), let \( P_\theta(r) \) be the component of the partial fraction expansion of \( r \) that involves the positive powers of \( X - \theta \). Clearly \( P_\theta(r) \) is simultaneously in \( K(X) \) and in \( K((X - \theta)) \). If \( P_\theta(r) = 0 \), i.e. \( r \) has no pole at \( \theta \), then the value \( r(\theta) \) in \( K \) is defined. In that case let

\[
\frac{r - r(\theta)}{X - \theta} = (r - r(\theta))X_\theta.
\]

The notation \( r^* \) does not reveal its dependency on \( \theta \), but this should not cause confusion. The rational function \( r^* \) also has no pole at \( \theta \), so that it makes sense to speak of \( r^{**}, r^{***}, \ldots \) and in general the \( n \)th iterate \( r^{*n} = r^{**} \cdots \). If \( r \) is any rational function, we can write

\[
r = P_\theta(r) + s,
\]

where \( s \) has no pole at \( \theta \), and then we can define

\[
r^0 = P_\theta(r) + s(\theta) + s^*(\theta)(X - \theta) + \cdots + s^{*n}(\theta)(X - \theta)^n + \cdots \tag{10}
\]

The field embedding of \( K(X) \) into \( K((X - \theta)) \) is the mapping given by \( r \mapsto r^0 \) as described in (10).

The image \( r^0 \) in \( K((X - \theta)) \) will be called the \( \theta \)-expansion of \( r \). If \( r \neq 0 \), we can write its \( \theta \)-expansion as

\[
r^0 = \sum_{n=m}^{\infty} r^0_n(X - \theta)^n \quad \text{where } m \text{ is an integer and } r^0_m \neq 0.
\]

The integer \( \text{ord}_\theta(r) \) is recaptured inside \( r^0 \), namely \( \text{ord}_\theta(r) = -m \). Thus the valuation \( \text{ord}_\theta \) can be extended to \( K((X - \theta)) \). Indeed, given \( \sigma \) in \( K((X - \theta)) \), write

\[
\sigma = \sum_{n=m}^{\infty} \sigma_n(X - \theta)^n \quad \text{where } m \text{ is an integer, } \sigma_n \in K \text{ and } \sigma_m \neq 0.
\]

Then put \( \text{ord}_\theta(\sigma) = -m \). It makes sense to say that \( \sigma \) has a pole at \( \theta \) when \( \text{ord}_\theta(\sigma) > 0 \), in other words when negative powers of \( X - \theta \) appear in the expansion for \( \sigma \). The
component of this expansion that involves only the negative powers of $X - \theta$ (i.e. positive powers of $X_\theta$) will be called the \textit{principal $\theta$-part} of $\sigma$ and denoted by $P_\theta(\sigma)$. That is

$$P_\theta(\sigma) = \sum_{n=m}^{-1} \sigma_n (X - \theta)^n = \sum_{n=m}^{-1} \sigma_n X_\theta^n.$$  

The principal part is a rational function that lies in $X_\theta K[X_\theta]$, a space that sits naturally inside $K((X - \theta))$. Of course if $m \geq 0$, then $P_\theta(\sigma) = 0$ and $\sigma$ has no pole at $\theta$. If $r \in K(X)$, we see that $P_\theta(r) = P_\theta(r^\theta)$ by definition (10) of $r^\theta$.

At this point it is meaningful to speak of the product $r^\theta \alpha^\theta$ for each rational function $r$ and each functional $\alpha$. If $r \in X_\theta K[X_\theta]$, then $r \in K((X - \theta))$ as well and the product $r \alpha^\theta$ makes sense. In particular $P_\theta(r r^\theta)$ makes sense for all $r \in K(X)$, and it is straightforward to see that

$$P_\theta(P_\theta(r) \alpha^\theta) = P_\theta(P_\theta(r^\theta) \alpha^\theta).$$  \hfill (11)

\textbf{Proposition 2.1.} If $r \in K(X)$ and $\theta \in K$, then $P_\theta(\partial_\alpha(r)) = -P_\theta(r r^\theta)$, and the expansion $r^\theta \alpha^\theta + \partial_\alpha(r)^\theta$ has no pole at $\theta$.

\textbf{Proof.} Formula (4) for $\partial_\alpha(X_n^\theta)$ can be rephrased as

$$\partial_\alpha(X_n^\theta) = -P_\theta((X - \theta)^{-n} \alpha^\theta) = -P_\theta(X_n^\theta \alpha^\theta) \quad \text{for all } n \geq 1.$$  

Using linearity on $K$-linear combinations of positive powers of $X_\theta$ deduce

$$\partial_\alpha(r) = -P_\theta(r \alpha^\theta) \quad \text{for all } r \in X_\theta K[X_\theta].$$

If $r$ is any rational function, then $P_\theta(r) \in X_\theta K[X_\theta]$, and thus using (11)

$$\partial_\alpha(P_\theta(r)) = -P_\theta(P_\theta(r \alpha^\theta)) = -P_\theta(r^\theta \alpha^\theta).$$

Because of (4) it is clear that $P_\theta(\partial_\alpha(r)) = \partial_\alpha(P_\theta(r))$, and the first claim follows. Furthermore

$$P_\theta(r^\theta \alpha^\theta + \partial_\alpha(r)^\theta) = P_\theta(r^\theta \alpha^\theta) + P_\theta(\partial_\alpha(r)^\theta) = P_\theta(r^\theta \alpha^\theta) + P_\theta(\partial_\alpha(r)) = 0,$$

thereby establishing the second claim. \hfill \Box

The algebra $K[[X^{-1}]]$ lies in the field $K((X^{-1}))$ of Laurent series in $X^{-1}$. Such field can be used as the completion of $K(X)$ using the valuation $ord_\infty$. We now go over the unique field embedding of $K(X)$ into $K((X^{-1}))$ that fixes $K$ and maps $X^{-1}$ in $K(X)$ to itself in $K((X^{-1}))$. If $r \in K(X)$, let $P_\infty(r)$ be the component of the partial fraction expansion of $r$ that involves non-negative powers of $X$. In other words $P_\infty(r)$ is the polynomial part of $r$. The function $r$ has no pole at $\infty$ if and only if $P_\infty(r) \in K$. In that case define

$$r(\infty) = P_\infty(r).$$

If $r(\infty) = 0$, i.e. if the polynomial part of $r$ vanishes, we say that $\infty$ is a \textit{zero} of $r$. Letting

$$r^* = X(r - r(\infty)),$$
we get another function with no pole at $\infty$, the same way that $r^\star$ had no pole at $\theta$.
Subsequently, for $r$ with no pole at $\infty$ we can form the $n$th iterate $r^{\star n} = r^{\star \cdots \star}$. If $r$ is
now any rational function, write

$$r = P_\infty(r) + s,$$  \hspace{1cm} (12)

where $s$ has no pole at $\infty$. In fact $s$ has a zero at $\infty$. Then we can define

$$r^\infty = P_\infty(r) + s^\star(\infty)X^{-1} + s^{\star \star}(\infty)X^{-2} + \cdots + s^{\star n}(\infty)X^{-n} + \cdots$$  \hspace{1cm} (13)

The field embedding of $K(X)$ into $K((X^{-1}))$ is the mapping given by $r \mapsto r^\infty$ as
described in (13). The image $r^\infty$ in $K((X^{-1}))$ will be called the $\infty$-expansion of $r$.
If $r \neq 0$, we can write its $\infty$-expansion as

$$r^\infty = \sum_{n=m}^{\infty} r_n X^{-n} \text{ where } m \text{ is an integer and } r_m \neq 0.$$  

The integer $\text{ord}_\infty(r)$ is recaptured inside $r^\infty$, namely $\text{ord}_\infty(r) = -m$. Thus the valuation $\text{ord}_\infty$ can be extended to $K((X^{-1}))$. Indeed, given $\sigma$ in $K((X^{-1}))$, write

$$\sigma = \sum_{n=m}^{\infty} \sigma_n X^{-n} \text{ where } m \text{ is an integer , } \sigma_n \in K \text{ and } \sigma_m \neq 0.$$  

Then put $\text{ord}_\infty(\sigma) = -m$. It makes sense to say that $\sigma$ has a pole at $\infty$ when $\text{ord}_\infty(\sigma) > 0$, in other words when positive powers of $X$ appear in the expansion for $\sigma$. The component of this expansion that involves only the non-negative powers of $X$ is called the polynomial part of $\sigma$ and denoted by $P_\infty(\sigma)$. That is

$$P_\infty(\sigma) = \sum_{n=m}^{0} \sigma_n X^{-n} = \sum_{n=0}^{-m} \sigma_n X^n.$$  

Clearly $P_\infty(r) = P_\infty(r^\infty)$ for any rational function $r$.
At this point it makes sense to speak of the product $r^\infty \sigma^\infty$ for every rational function $r$ and every functional $\sigma$. If $r$ is a polynomial, then $r$ is naturally in $K((X^{-1}))$ and the product $r^\infty \sigma^\infty$ makes sense too. For instance, $P_\infty(r)\sigma^\infty$ as well as $X^{-1}P_\infty(r)\sigma^\infty$ are defined for every rational function $r$. It is straightforward to see that $P_\infty(r^\infty \sigma^\infty) = P_\infty(P_\infty(r)\sigma^\infty)$ and consequently

$$P_\infty(X^{-1}r^\infty \sigma^\infty) = P_\infty(X^{-1}P_\infty(r)\sigma^\infty).$$  \hspace{1cm} (14)

**Proposition 2.2.** If $r$ is a rational function, then $P_\infty(\partial_\sigma(r)) = P_\infty(X^{-1}r^\infty \sigma^\infty)$ and $r^\infty \sigma^\infty = X \partial_\sigma(r)$ has no pole at $\infty$.

**Proof.** Formula (4) for $\partial_\sigma(X^n)$, including $\partial_\sigma(1) = 0$, can be rephrased as:

$$\partial_\sigma(X^n) = P_\infty(X^{-1} X^n \sigma^\infty) \text{ for all } n \geq 0.$$  

By linearity deduce that

$$\partial_\sigma(r) = P_\infty(X^{-1} r \sigma^\infty) \text{ for all polynomials } r.$$
If \( r \) is any rational function, then \( P_\infty(r) \) is a polynomial and therefore
\[
\partial_\alpha(P_\infty(r)) = P_\infty(X^{-1}P_\infty(r)\alpha^\infty)
\]
for all rational functions \( r \).

From (4) we see that \( P_\infty(\partial_\alpha(r)) = \partial_\alpha(P_\infty(r)) \) and from (14) deduce that
\[
P_\infty(\partial_\alpha(r)) = \partial_\alpha(P_\infty(r)) = P_\infty(X^{-1}P_\infty(r)\alpha^\infty) = P_\infty(X^{-1}r\alpha^\infty).
\]
Since \( P_\infty(\partial_\alpha(r)) = P_\infty(\partial_\alpha(r)\alpha^\infty) \) we see from above that the polynomial part of \( X^{-1}r\alpha^\infty - \partial_\alpha(r)^\infty \) vanishes. Equivalently \( r\alpha^\infty - X\partial_\alpha(r)^\infty \) has no pole at \( \infty \).

The partial fraction expansion of a rational function \( t \) reveals that \( t \) is completely specified by its principal parts \( P_\theta(t) \) and its polynomial part \( P_\infty(t) \). Hence Propositions 2.1 and 2.2, taken together, give a complete specification of any derived function \( \partial_\alpha(r) \) in terms of multiplications in the fields \( K((X - \theta)) \) and \( K((X^{-1})) \).

2.2. The algebra of derivers

We check that the composite of two derivers, as \( K \)-linear operators on \( K(X) \), is again a deriver.

**Proposition 2.3.** If \( \alpha, \beta \) are functionals, then \( \partial_\alpha \circ \partial_\beta \) is the deriver coming from the functional \( \gamma \) defined by
\[
\gamma^\theta = -\alpha^\theta \beta^\theta \quad \text{for all } \theta \text{ in } K, \quad \text{and} \quad \gamma^\infty = X\alpha^\infty \beta^\infty.
\]

**Proof.** It suffices to show that for any rational function \( r \):
\[
P_\theta(\partial_\alpha(\partial_\beta(r))) = P_\theta(\partial_\gamma(r)) \quad \text{and} \quad P_\infty(\partial_\alpha(\partial_\beta(r))) = P_\infty(\partial_\gamma(r)).
\]

For the first equality we have
\[
P_\theta(\partial_\alpha(\partial_\beta(r)))
\begin{align*}
&= -P_\theta(\partial_\beta(r)^\theta \alpha^\theta) \quad \text{by Proposition 2.1} \\
&= -P_\theta(P_\theta(\partial_\beta(r))\alpha^\theta) \quad \text{by (11)} \\
&= -P_\theta(-r^\theta \beta^\theta \alpha^\theta) \quad \text{by Proposition 2.1} \\
&= -P_\theta(r^\theta \gamma^\theta) \quad \text{by definition of } \gamma^\theta \\
&= P_\theta(\partial_\gamma(r)) \quad \text{by Proposition 2.1}.
\end{align*}
\]

For the second equality we similarly have
\[
P_\infty(\partial_\alpha(\partial_\beta(r)))
\begin{align*}
&= P_\infty(X^{-1}\partial_\beta(r)^\infty \alpha^\infty) \quad \text{by Proposition 2.2} \\
&= P_\infty(X^{-1}P_\infty(\partial_\beta(r))\alpha^\infty) \quad \text{by (14)} \\
&= P_\infty(X^{-1}X^{-1}r \beta^\infty \alpha^\infty) \quad \text{by Proposition 2.2} \\
&= P_\infty(X^{-1}r \gamma^\infty) \quad \text{by definition of } \gamma^\infty \\
&= P_\infty(X^{-1}r \gamma^\infty) \quad \text{by Proposition 2.2}. \quad \square
\end{align*}
\]
Any $K$-linear combination of divisors is clearly another divisor, corresponding to the same linear combination in the original functionals. Thus Proposition 2.3 shows that the set of all divisors is a $K$-algebra. The formula for the functional $\gamma$ that gives the divisor $\partial_\alpha \circ \partial_\beta$ is symmetrical in $\alpha$ and $\beta$. Thus the algebra of divisors is commutative. The formula for $\gamma$ in Proposition 2.3 also reveals that this algebra has no unit, nor does it have non-zero nilpotents.

2.3. The coefficients of the functional $\alpha \ast r$

If $r \in K(X)$ and $\alpha$ is a functional, the functional given by $t \mapsto \langle \alpha, rt \rangle$, is denoted as $\alpha \ast r$. The coefficients of this functional have an expression in terms of the coefficients of $\alpha$. To find the expression we need to observe that divisors commute with the $/H_22813$-operation defined for $\theta$ in $K$, and almost commute with the $\cdot$-operation defined for $\infty$. Lemma 2.4.

Let $\alpha$ be a functional, $r \in K(X)$ and $\theta \in K$. If $\theta$ is not a pole of $r$, then

$$\partial_\alpha(r^{n+1}) = \partial_\alpha(r)^{n+1} + \langle \alpha, r^n \rangle - r^{n}(\infty) \langle \alpha, 1 \rangle$$

for all $n \geq 0$.

Proof. For the commutativity involving $\ast$, as defined using $\theta$, it suffices to check that $\partial_\alpha(r^\ast) = \partial_\alpha(r)^\ast$. This checks routinely:

$$\partial_\alpha(r^\ast) = \partial_\alpha(X_\theta (r - r(\theta)))$$

$$= X_\theta \partial_\alpha(r - r(\theta)) + \partial_\alpha \ast (r - r(\theta))(X_\theta)$$

by (3)

$$= X_\theta \partial_\alpha(r)^\ast + \partial_\alpha \ast (r(\theta)) (X_\theta)$$

since divisors kill scalars

$$= X_\theta \partial_\alpha(r) - (\alpha, X_\theta (r - r(\theta)))$$

using (4)

$$= X_\theta (\partial_\alpha(r) - \alpha (r(\theta)))$$

apply (2).

$$= \partial_\alpha(r)^\ast$$

For the quasi-commutativity involving the $\cdot$-operation, use induction on $n$. When $n = 0$ we require $\partial_\alpha(r^\ast) = \partial_\alpha(r)^\ast + \langle \alpha, r \rangle - r(\infty) \langle \alpha, 1 \rangle$. It is useful to observe that since $\infty$ is not a pole of $r$, then $\infty$ is a zero of $\partial_\alpha(r)$ using (4). Working from the left we get

$$\partial_\alpha(r^\ast) = \partial_\alpha(X (r - r(\infty)))$$

$$= X \partial_\alpha(r - r(\infty)) + \partial_\alpha \ast (r - r(\infty))(X)$$

using (3)

$$= X \partial_\alpha(r) + \partial_\alpha \ast (\alpha, r - r(\infty))(X)$$

derivatives kill scalars

$$= X \partial_\alpha(r) + (\alpha, r - r(\infty))$$

from (4)

$$= \partial_\alpha(r)^\ast + \langle \alpha, r \rangle - r(\infty) \langle \alpha, 1 \rangle$$

since $\partial_\alpha(r)(\infty) = 0$. 
Assume
\[ \partial_\alpha (r \cdot n) = \partial_\alpha (r) \cdot n + \langle \alpha, r \cdot (n - 1) \rangle - r \cdot (n - 1) (\infty) \langle \alpha, 1 \rangle. \]
Then
\[ \partial_\alpha (r \cdot (n + 1)) = \partial_\alpha (r \cdot n) \cdot + \langle \alpha, r \cdot n \rangle - r \cdot n (\infty) \langle \alpha, 1 \rangle, \]
by the part just done applied to \( r \cdot n \). From the assumption we come to
\[ \partial_\alpha (r \cdot (n + 1)) = \partial_\alpha (r) \cdot (n + 1) + \langle \alpha, r \cdot n \rangle - r \cdot n (\infty) (\alpha, 1). \]
Since \( \langle \alpha, r \cdot (n - 1) \rangle - r \cdot (n - 1) (\infty) (\alpha, 1) \) is a scalar, we obtain
\[ \langle \alpha, r \cdot (n - 1) \rangle - r \cdot (n - 1) (\infty) (\alpha, 1) \cdot = 0. \]
Hence \( \partial_\alpha (r \cdot (n + 1)) = \partial_\alpha (r \cdot (n + 1)) + \langle \alpha, r \cdot n \rangle - r \cdot n (\infty) (\alpha, 1) \), as desired. □

Lemma 2.5. If \( r \in K(X) \) and \( r \) has no pole at \( \infty \) and \( n \geq 0 \), then
\[ \langle \alpha, r \cdot n \rangle = r \cdot n (\infty) (\alpha, 1) - \partial_\alpha (r) \cdot (n + 1) (\infty). \]

Proof. Take the identity for \( \partial_\alpha (r \cdot (n + 1)) \) from Lemma 2.4 and evaluate it at \( \infty \).
The scalars \( \langle \alpha, r \cdot n \rangle \) and \( r \cdot n (\infty) (\alpha, 1) \) evaluate to themselves. Upon recalling that
\( \partial_\alpha (r \cdot (n + 1)) \) has a zero at \( \infty \), the desired formula comes out. □

The final result of this section will bear fruit in Section 4. For any \( \sigma \) in \( K((X - \theta)) \) the subscript notation \( \sigma_n \) will stand for the coefficient of \((X - \theta)^n\) in its expansion. If \( \sigma \) is in \( K((X^{-1})) \), then \( \sigma_n \) stands for the coefficient of \( X^{-n} \). For instance, when \( r \) is a rational function, then \( r_0 \) is the coefficient of \((X - \theta)^0\) in the expansion \( r \).

Proposition 2.6. If \( \alpha \) is a functional and \( r \in K(X) \), then
\[ (\alpha * r)^0 = r^0 \alpha^0 + \partial_\alpha (r)^0 \quad \text{for every } \theta \in K \] (15)
and
\[ (\alpha * r)^\infty = r^\infty \alpha^\infty - X \partial_\alpha (r)^\infty. \] (16)

Proof. In order for (15) and (16) to even make sense, the expansion \( r^0 \alpha^0 + \partial_\alpha (r)^0 \)
must not have a pole at \( \theta \) while \( r^\infty \alpha^\infty - X \partial_\alpha (r)^\infty \) must not have a pole at \( \infty \). Such
facts are confirmed in Propositions 2.1 and 2.2.

First do the case where \( \theta \in K \) and \( \theta \) is not a pole of \( r \). The definition of \( r \cdot \) for \( \theta \)
says that
\[ X_\theta r = r(\theta) X_\theta + r \cdot \] (17)
Multiply (17) by \(X_\theta\) to get
\[
X_\theta^2 r = r(\theta)X_\theta^2 + X_\theta r^*.
\]
Apply (17) to \(X_\theta r\) to arrive at
\[
X_\theta^2 = r(\theta)X_\theta^2 + r^*(\theta)X_\theta + r^{(n+1)}.
\]
Then multiply through by \(X_\theta\), apply (17) to \(X_\theta r\), repeat the argument inductively to obtain
\[
X_n+1 = r(\theta)X_n + r^*(\theta)X_\theta + r^{(n+1)}
\]
for all \(n \geq 0\).

After that use (10) to get
\[
X_n+1 = r(\theta)X_n + r^*(\theta)X_\theta + r^{(n+1)}
\]
for all \(n \geq 0\).

Apply \(\alpha\) to this identity to yield
\[
\langle \alpha, X_n+1 \rangle = r(\theta)\alpha + r^*(\theta)\alpha + r^{(n+1)}\alpha.
\]
Now
\[
\langle \alpha, r^{(n+1)} \rangle = \partial_\alpha (r^{(n)})(\theta) = \partial_\alpha(r^{(n)})(\theta) = \partial_n(r_{\theta}).
\]
by first using the definition (2) on \(\partial_n(r^{(n)})(\theta)\), then applying Lemma 2.4, and then using (10) for the \(\theta\)-expansion of \(\partial_n(r)\). Hence
\[
\langle \alpha, X_n^2 \rangle = \langle r(\theta)\alpha + r^*(\theta)\alpha + r^{(n+1)}\alpha \rangle = \langle \alpha, r^{(n+1)} \rangle.
\]
The right side of the above equality is the coefficient \((r(\theta)\alpha + r^*(\theta)\alpha + r^{(n+1)}\alpha)\) of \((X_\theta - \theta)n\) for \(r(\theta)\alpha + r^*(\theta)\alpha + r^{(n+1)}\alpha\). On the left we have
\[
\langle \alpha, X_\theta^2 \rangle = \langle \alpha * r, X_\theta^2 \rangle = \langle \alpha * r \rangle \theta
\]
which is same coefficient for \((\alpha * r)\theta\). Hence \((\alpha * r)\theta = r(\theta)\alpha + r^*(\theta)\alpha + r^{(n+1)}\alpha\), when \(\theta \in K\) but not a pole of \(r\).

For the case where \(\theta\) is a pole of \(r\) in \(K\), notice that both sides of (15) are linear in \(r\). Thus an inspection of (9) shows that it suffices to check (15) for the functions \(X_m\) where \(m \geq 1\). With \(r = X_m\) we obviously have \((X_m)^0 = X_m\), and \(\partial_n(X_m)^0 = \partial_n(X_m)\). Starting from the right side of (15) we move to the left:
\[
\frac{X_m}{\alpha^0} + \partial_n(X_m)\theta = X_m^\alpha - P_n(X_m^\alpha)\theta \quad \text{by (4)}
\]
\[
= \sum_{n=0}^{\infty} \alpha_n^m (X - \theta)^n \quad \text{by (7) and simplifying}
\]
\[
= \sum_{n=0}^{\infty} \langle \alpha, X_n^m \rangle (X - \theta)^n \quad \text{by (1)}
\]
\[
= \sum_{n=0}^{\infty} (\alpha * X_n^m)(X - \theta)^n \quad \text{by (7) for } \alpha * X_n^m
\]
Next work on formula (16). Start with the case where \(\infty\) is not a pole of \(r\). The definition of \(r^*\) says that
Multiply (18) by \( X \) to get
\[
X^2 r = r(\infty)X^2 + Xr^*.
\]
Apply (18) to \( Xr^* \) to arrive at
\[
X^2 r = r(\infty)X^2 + r^*(\infty)X + r^{**}.
\]
Then multiply through by \( X \), apply (18) to \( Xr^{**} \), repeat the argument inductively to obtain
\[
X^n r = r(\infty)X^n + r^*(\infty)X^{n-1} + \cdots + r^{*(n-1)}(\infty)X + r^n
\quad \text{for all } n \geq 1.
\]
Using (13) this becomes
\[
X^n r = r_0^\infty X^n + r_1^\infty X^{n-1} + \cdots + r_{n-1}^\infty X + r^n.
\]
Apply \( \alpha \) to this, yielding
\[
\langle \alpha, X^n r \rangle = r_0^\infty \alpha_0^\infty + r_1^\infty \alpha_1^\infty + \cdots + r_{n-1}^\infty \alpha_{n-1}^\infty + \langle \alpha, r^n \rangle
\quad \text{for } n \geq 1.
\]
In Lemma 2.5 the formula for \( \langle \alpha, r^* \rangle \) is the same as
\[
\langle \alpha, r^* \rangle = r_0^\infty \alpha_0^\infty - \partial_\alpha(r)^\infty.
\]
Substitute this up above and use \( \langle \alpha \ast r, X^n \rangle = \langle \alpha, X^n r \rangle \) to obtain
\[
\langle \alpha \ast r, X^n \rangle = \left( r_0^\infty \alpha_0^\infty + r_1^\infty \alpha_1^\infty + \cdots + r_{n-1}^\infty \alpha_{n-1}^\infty + r_n^\infty \alpha_n^\infty \right) - \partial_\alpha(r)^\infty.
\]
This formula holds for all \( n \geq 1 \), but also for \( n = 0 \) because then it becomes the statement of Lemma 2.5. An inspection shows that this formula gives what we want, namely \( r^\infty \alpha^\infty - X\partial_\alpha(r)^\infty = (\alpha \ast r)^\infty \), as long as \( r \) has no pole at \( \infty \).

We are left to deal with (16) when \( \infty \) is a pole of \( r \). Since both sides of (16) are linear in \( r \), it follows from (12) that it suffices to check (16) when \( r = X^m \) and \( m \geq 1 \). Obviously \( (X^m)^\infty = X^m \), and from (4), \( \partial_\alpha(X^m)^\infty = \partial_\alpha(X^m) \).

Starting from the right side of (16) we move to the left:
\[
X^m \alpha^\infty - X\partial_\alpha(X^m) = \sum_{n=0}^{\infty} \alpha_{m+n} X^{-n}
\quad \text{by (4)}
\]
\[
= \sum_{n=0}^{\infty} \alpha_{m+n} X^{-n}
\quad \text{by (8) and simplifying}
\]
\[
= \sum_{n=0}^{\infty} \langle \alpha, X^{m+n} \rangle X^{-n}
\quad \text{by (1)}
\]
\[
= \sum_{n=0}^{\infty} \langle \alpha \ast X^m, X^n \rangle X^{-n}
\quad \text{using (8) for } \alpha \ast X^m.
\]
\[
= (\alpha \ast X^m)^\infty
\quad \text{using (8) for } \alpha \ast X^m.
\]
3. Endomorphisms of $\mathcal{R}_\alpha$ and the regulator

As noted in [6, Proposition 2.2] the endomorphism algebra of $\mathcal{R}_\alpha$ is the same as the algebra of all $K(X)$-linear operators on $K(X)^2$ that leave invariant the $K$-linear subspace $V_\alpha$. Such maps will leave $V_\alpha^-$ also invariant. As in [6] we shall view endomorphisms of $\mathcal{R}_\alpha$ as $K(X)$-linear maps on $K(X)^2$ and chase down what it means for such maps to leave $V_\alpha$ invariant.

3.1. The ideal $\mathcal{J}$ inside $\text{End} \mathcal{R}_\alpha$

It will be practical to represent endomorphisms of $\mathcal{R}_\alpha$ as $2 \times 2$ matrices of rational functions that act on the columns of $K(X)^2$ in the usual way. Let $\varphi = \begin{bmatrix} u & v \\ w & z \end{bmatrix}$ be an endomorphism. Since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V^-_\alpha$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V_\alpha$ their images under $\varphi$ lie in $V^-_\alpha$ and $V_\alpha$ respectively. Thus every endomorphism can be represented in the form

$$\varphi = \begin{bmatrix} u & v \\ -\partial_\alpha(u) & \lambda - \partial_\alpha(v) \end{bmatrix}$$

where $u, v \in K(X)$ and $\lambda \in K$.

Any matrix $\varphi$ as above can be written as

$$\varphi = \begin{bmatrix} u + \lambda & v \\ -\partial_\alpha(u + \lambda) & \lambda - \partial_\alpha(v) \end{bmatrix} - \lambda I,$$

where $I$ is the identity matrix. Note that $\partial_\alpha(\lambda) = 0$. Replacing $u + \lambda$ by $u$, we find that any endomorphism of $\mathcal{R}_\alpha$ takes the form

$$\varphi = \begin{bmatrix} u & v \\ -\partial_\alpha(u) & -\partial_\alpha(v) \end{bmatrix} - \lambda I,$$

where $u, v \in K(X), \lambda \in K$.

The set of endomorphisms that have the more restricted form

$$\varphi = \begin{bmatrix} u & v \\ -\partial_\alpha(u) & -\partial_\alpha(v) \end{bmatrix}$$

will be called $\mathcal{J}$. Such endomorphisms are the ones that map $V_\alpha$ into $V^-_\alpha$. The set $\mathcal{J}$ is an ideal of $\text{End} \mathcal{R}_\alpha$. We have just seen that

$$\text{End} \mathcal{R}_\alpha = KI \oplus \mathcal{J}.$$

We should caution that not every matrix of the type (19) represents an endomorphism. It may be that $\mathcal{J} = (0)$, in which case $\text{End} \mathcal{R}_\alpha = KI$, the trivial algebra. It is important to discover those functionals that make the maximal ideal $\mathcal{J}$ non-trivial.

**Proposition 3.1.** A matrix $\varphi$, exactly as in (19), belongs to $\mathcal{J}$ if and only if

$$\partial_\alpha(\varphi(\varphi_{11} \varphi_{22} - \varphi_{12} \varphi_{21})) = \partial_\alpha(\varphi_{11} \varphi_{22} - \varphi_{12} \varphi_{21}).$$
Proof. The matrix $\varphi$, given by (19), maps \[
\begin{pmatrix}
0 \\
1
\end{pmatrix}
to \begin{pmatrix}
v \\
-\partial_{\alpha}(v)
\end{pmatrix}
which lies in $V^-$. Since $V_{\alpha} = K\begin{pmatrix}
0 \\
1
\end{pmatrix} \oplus V_{\alpha}^-$, it becomes evident that $\varphi \in J$ if and only if $\varphi$ maps $V_{\alpha}^-$ into $V_{\alpha}^-$. Thus we come to the following equivalences:

\[
\varphi \in J \iff \begin{pmatrix}
u \\
-\partial_{\alpha}(v)
\end{pmatrix} \in V_{\alpha}^- \text{ for all } z \in K(X)
\]
\[
\varphi \in J \iff \begin{pmatrix}
u z - v\partial_{\alpha}(z) \\
-\partial_{\alpha}(v)\partial_{\alpha}(z)
\end{pmatrix} \in V_{\alpha}^- \text{ for all } z \in K(X)
\]
\[
\varphi \in J \iff \partial_{\alpha}(uv) - \partial_{\alpha}(v)\partial_{\alpha}(u) + \partial_{\alpha}(v)\partial_{\alpha}(z) = 0 \text{ for all } z \in K(X)
\]
\[
\varphi \in J \iff \text{the operator } \partial_{\alpha} \circ u - \partial_{\alpha} \circ v = 0 \text{ on } K(X).
\]

Now substitute the operator identities

\[
\partial_{\alpha} \circ u = \partial_{\alpha u} + \partial_{\alpha}(u)
\]
and

\[
\partial_{\alpha} \circ v = \partial_{\alpha v} + \partial_{\alpha}(v)
\]
coming from (3). The equivalences above reduce to the statement

\[
\partial_{\alpha u} - \partial_{\alpha v} = 0
\]
that was expected. □

In order to examine condition (20) more effectively the following change of variables will help. Given a matrix $\varphi$ in the form (19), put

\[
r = -v, \quad s = u + \partial_{\alpha}(v), \quad t = w,
\]
or in reverse $v = -r, \quad u = s + \partial_{\alpha}(r), \quad w = t$.

Then the matrix $\varphi$ takes on the alternative form

\[
\varphi = \begin{pmatrix}
s + \partial_{\alpha}(r) \\
-\partial_{\alpha}(r)
\end{pmatrix}
\]
where $\partial_{\alpha}^2(r) + \partial_{\alpha}(s) + t = 0$.

Proposition 3.1 can be restated as follows.

Proposition 3.2. A $2 \times 2$ matrix $\varphi$ of rational functions lies in $J$ if and only if $\varphi$ takes the form

\[
\varphi = \begin{pmatrix}
s + \partial_{\alpha}(r) \\
-\partial_{\alpha}(r)
\end{pmatrix}
\]
where $r, s, t \in K(X)$ and

\[
\partial_{\alpha u} + \partial_{\alpha v} = 0 \quad \text{and} \quad \partial_{\alpha}^2(r) + \partial_{\alpha}(s) + t = 0.
\]

We will now move towards rephrasing condition (22) in terms of a polynomial, having coefficients in the field $K(X)$, that strongly regulates the nature of $\text{End}_{\mathcal{A}}$. 

\[\text{□} \]
3.2. The algebra generated by a deriver and by multipliers

Given a rational function \( u \) in \( K(X) \), the multiplication operator on \( K(X) \), given by \( v \mapsto \) \( uv \) for every \( v \) in \( K(X) \), will be called a multiplier and denoted by the symbol \( u \) itself. It will be necessary to work inside the algebra of \( K \)-linear operators on \( K(X) \) that is generated by derivers and multipliers. We shall say that operator \( \sigma \) in that algebra has finite rank when its image \( \sigma(K(X)) \) is finite-dimensional over \( K \).

**Proposition 3.3.** If \( \gamma \) is a functional and \( u \in K(X) \) and if the \( K \)-linear operator \( \partial \gamma + u \) on \( K(X) \) has finite rank over \( K \), then \( \gamma = 0 \) and \( u = 0 \).

**Proof.** First check that \( u \) must be a polynomial. If not, \( u \) must have a pole at some \( \theta \) in \( K \). Thus for each non-zero \( v \) in \( K[X_\theta] \) we have
\[
\text{ord}_\theta(\gamma v) = \text{ord}_\theta \gamma + \text{ord}_\theta v > \text{ord}_\theta v \geq \text{ord}_\theta \partial \gamma (v).
\]
From this it follows that
\[
\text{ord}_\theta(\partial \gamma + u)(v) = \text{ord}_\theta(\partial \gamma (v) + uv) = \text{ord}_\theta(uv) = \text{ord}_\theta u + \text{ord}_\theta v.
\]
This makes \( \partial \gamma + u \) injective on the infinite-dimensional space \( K[X_\theta] \), and thereby \( \partial \gamma + u \) is prevented from having finite rank. So \( u \) is a polynomial.

Next check that the polynomial \( u \) must be 0. If \( u / 0 \), then for every non-zero polynomial \( v \) we have
\[
\text{ord}_\infty(\gamma v) = \text{ord}_\infty u + \text{ord}_\infty v > \text{ord}_\infty v > \text{ord}_\infty \partial \gamma (v).
\]
Consequently,
\[
\text{ord}_\infty(\partial \gamma + u)(v) = \text{ord}_\infty(\partial \gamma (v) + uv) = \text{ord}_\infty(uv) = \text{ord}_\infty u + \text{ord}_\infty v.
\]
This implies that \( \partial \gamma + u \) is injective on the space \( K[X] \) of polynomials. As such, this operator cannot have finite rank. Thus \( u = 0 \).

It remains to prove that \( \gamma = 0 \). If \( \gamma \neq 0 \) then \( \partial \gamma \neq 0 \) either, and we will check that \( \partial \gamma \) cannot have finite rank. Since \( \partial \gamma \neq 0 \), it follows that \( \partial \gamma(t) \neq 0 \) for some \( t \) that is either a polynomial in \( K[X] \) or a function in \( X_\theta K[X_\theta] \) for some \( \theta \) in \( K \).

Say \( t \) is a polynomial. For every non-zero polynomial \( v \) we have
\[
\partial \gamma (tv) = v \partial \gamma (t) + \partial \gamma (tv).
\]
Since \( \partial \gamma(t) \) is a non-zero polynomial, while \( \text{ord}_\infty \partial \gamma (tv) < \text{ord}_\infty v \), it follows that
\[
\text{ord}_\infty(v \partial \gamma(t)) = \text{ord}_\infty v + \text{ord}_\infty \partial \gamma(t) \geq \text{ord}_\infty v > \text{ord}_\infty \partial \gamma (tv).
\]
Thus
\[
\text{ord}_\infty \partial \gamma (tv) = \text{ord}_\infty(v \partial \gamma(t)) = \text{ord}_\infty v + \text{ord}_\infty \partial \gamma(t),
\]
so that \( \partial \gamma (tv) \neq 0 \). This makes \( \partial \gamma \) injective on the infinite-dimensional space \( tK[X] \), and certainly not of finite rank.
Next say $t \in X_\theta K[X_\theta]$. For every non-zero function $v$ in $K[X_\theta]$ we have
\[ \partial_\gamma (tv) = v \partial_\gamma (t) + \partial_\gamma \ast (v). \]

Since derivers leave $X_\theta K[X_\theta]$ invariant and since $\partial_\gamma (t) \neq 0$, it follows that $\text{ord}_\theta \partial_\gamma (t) > 0$. Consequently
\[ \text{ord}_\theta v \partial_\gamma (t) = \text{ord}_\theta v + \text{ord}_\theta \partial_\gamma (t) > \text{ord}_\theta v \geq \text{ord}_\theta \partial_\gamma \ast (v). \]

Thus
\[ \text{ord}_\theta \partial_\gamma (tv) = \text{ord}_\theta (v \partial_\gamma (t)) = \text{ord}_\theta v + \text{ord}_\theta \partial_\gamma (t), \]
and this implies that $\partial_\gamma (tv) \neq 0$. This means that $\partial_\gamma$ is injective on the infinite-dimensional space $t K[X_\theta]$, and thereby not of finite rank. □

Given a functional $\alpha$ let $\mathcal{A}$ be the $K$-subalgebra of $\text{End}_K K(X)$ that is generated by $\partial \alpha$ and by all multipliers. Inside $\mathcal{A}$ there is the ideal $\mathcal{I} = \{ \sigma \in \mathcal{A} : \sigma \text{ has finite rank} \}$.

The following result can be found in [5, Lemma 2.2].

**Proposition 3.4.** The quotient algebra $\mathcal{A}/\mathcal{I}$ is a commutative $K(X)$-algebra.

**Proposition 3.5.** A triplet of rational functions $r, s, t$ satisfies (22) if and only if the operator
\[ r \circ \partial^2_\alpha + s \circ \partial_\alpha + t \]
on $K(X)$ has finite rank.

**Proof.** By Proposition 3.4 the statement that $r \circ \partial^2_\alpha + s \circ \partial_\alpha + t$ has finite rank is equivalent to the same statement for the operator $\tau$ defined by
\[ \tau = \partial \alpha \circ \partial_\alpha \circ r + \partial_\alpha \circ s + t. \]

Property (3) gives the following identity of operators on $K(X)$:
\[ \partial_\alpha \circ u = \partial_{\alpha \circ u} + \partial_\alpha (u) \quad \text{for any } u \in K(X). \]

Using this with $u = r$ then $u = s$ and then $u = \partial_\alpha (r)$ and using the commutativity of derivers we obtain:
\[ \tau = \partial_\alpha \circ \partial_\alpha \circ r + \partial_\alpha \circ s + t \]
\[ = \partial_\alpha \circ (\partial_{\alpha \circ r} + \partial_\alpha (r)) + \partial_{\alpha \circ s} + \partial_\alpha (s) + t \]
\[ = \partial_\alpha \circ \partial_{\alpha \circ r} + \partial_\alpha \circ \partial_\alpha (r) + \partial_{\alpha \circ s} + \partial_\alpha (s) + t \]
\[ = \partial_{\alpha \circ (\alpha \circ r)} + \partial_\alpha \circ \partial_{\alpha \circ r} + \partial^2_\alpha (r) + \partial_{\alpha \circ s} + \partial_\alpha (s) + t \]
\[ = \partial_{\alpha \circ (\alpha \circ r)} + \partial_\alpha \circ \partial_{\alpha \circ r} + \partial^2_\alpha (r) + \partial_\alpha (s) + t \]
Proposition 2.1 ensures that the operator \( \partial_{\alpha^*}(s + \partial_{\alpha}(r)) + \partial_{\alpha^*} \circ \partial_{\alpha} \) is a deriver, and \( \partial_{\alpha}^2(r) + \partial_{\alpha}(s) + t \) is certainly a multiplier. According to Proposition 3.3, \( \tau \) will have finite rank if and only if
\[ \partial_{\alpha^*}(s + \partial_{\alpha}(r)) + \partial_{\alpha^*} \circ \partial_{\alpha} = 0 \quad \text{and} \quad \partial_{\alpha}^2(r) + \partial_{\alpha}(s) + t = 0, \]
which is precisely condition (22). \( \square \)

The next result follows immediately from Propositions 3.2 and 3.5.

**Proposition 3.6.** A \( 2 \times 2 \) matrix of rational functions \( \varphi \) is an endomorphism in \( J \) if and only if \( \varphi \) takes the form (21) and \( r \circ \partial_{\alpha}^2 + s \circ \partial_{\alpha} + t \) has finite rank.

### 3.3. The regulator

According to [6, Section 2] every functional comes with its regulator. To summarize, the regulator of a functional \( \alpha \) is the unique polynomial \( f(Y) \) in \( K(X)[Y] \) such that
- \( f(Y) \) is monic or possibly zero,
- the operator \( f(\partial_{\alpha}) \) on \( K(X) \) has finite rank, and
- \( f(Y) \) divides any polynomial \( g(Y) \) in \( K(X)[Y] \) that causes \( g(\partial_{\alpha}) \) to have finite rank.

**Proposition 3.7.** If \( \alpha \) is a functional for which \( \text{End} R_\alpha \) is non-trivial, then the regulator \( f(Y) \) of \( \alpha \) is linear or quadratic in \( Y \).

**Proof.** Since \( \text{End} R_\alpha \) is non-trivial, the ideal \( J \) is non-zero. According to Proposition 3.6 there is a non-zero matrix \( \varphi \) of the form (21) such that the operator \( r \circ \partial_{\alpha}^2 + s \circ \partial_{\alpha} + t \) has finite rank. Consequently the regulator \( f(Y) \) of \( \alpha \) divides the non-zero polynomial \( rY^2 + sY + t \) in \( K(X)[Y] \). Therefore \( f(Y) \) has degree 0, 1 or 2, but the degree of \( f(Y) \) cannot be 0, for that would yield a non-zero multiplier of finite rank. \( \square \)

As a consequence of Proposition 3.7, only those functionals \( \alpha \) which have a linear or a quadratic regulator have the potential to make \( \text{End} R_\alpha \) non-trivial. It is shown in [6, Proposition 2.4] that if the regulator of \( \alpha \) is linear, then \( R_\alpha \) decomposes with a finite-dimensional summand. In light of this and Proposition 3.7, the proof of Theorem 1.1 reduces to a consideration of those \( \alpha \) that have a quadratic regulator.
3.4. When the regulator is quadratic

Suppose that \( p, q \in K(X) \) and that \( f(Y) = Y^2 + pY + q \) is the regulator of a functional \( \alpha \). The matrix \( D \) defined by

\[
D = \begin{bmatrix}
p & -1 \\
q & 0
\end{bmatrix}
\]

will be called the generic matrix for such \( \alpha \). Since \( D \) takes the form of (21) and since \( \partial_\alpha^2 + p \circ \partial_\alpha + q \) has finite rank, Proposition 3.6 ensures that \( D \) is an endomorphism in \( \mathcal{J} \). Thus \( \text{End}\, \mathcal{R}_\alpha \) is non-trivial when \( \alpha \) has quadratic regulator.

**Proposition 3.8.** If \( \alpha \) has quadratic regulator \( f(Y) = Y^2 + pY + q \) and generic matrix \( D \) as in (23), then \( \text{End}\, \mathcal{R}_\alpha \) consists of all matrices

\[
rD + (\partial_\alpha(r) + \lambda)I \quad \text{where} \quad r \in K(X) \quad \text{and} \quad \lambda \in K.
\]

**Proof.** Since \( \text{End}\, \mathcal{R}_\alpha = KI \oplus \mathcal{J} \), we simply require that the endomorphisms which are in \( \mathcal{J} \) be nothing but the matrices of the form

\[
rD + \partial_\alpha(r)I \quad \text{where} \quad r \in K(X).
\]

Since \( f(\partial_\alpha) \) has finite rank, the operator \( r \circ f(\partial_\alpha) = r \circ \partial_\alpha^2 + rp \circ \partial_\alpha + rq \) has finite rank for any \( r \in K(X) \). From Proposition 3.6 the matrix

\[
\begin{bmatrix}
rp + \partial_\alpha(r) & -r \\
rq & \partial_\alpha(r)
\end{bmatrix}
\]

must be an endomorphism inside \( \mathcal{J} \).

Conversely Proposition 3.6 tells us that any matrix in \( \mathcal{J} \) must take the form

\[
\begin{bmatrix}
s + \partial_\alpha(r) & -r \\
t & \partial_\alpha(r)
\end{bmatrix}, \quad \text{where} \quad r, s, t \in K(X),
\]

and \( r \circ \partial_\alpha^2 + s \circ \partial_\alpha + t \) has finite rank. Hence \( f(Y) \) divides \( rY^2 + sY + t \) in \( K(X)[Y] \).

This just says that \( s = pr \) and \( t = qr \). Thus a matrix in \( \mathcal{J} \) must take the form

\[
\begin{bmatrix}
pr + \partial_\alpha(r) & -r \\
tr & \partial_\alpha(r)
\end{bmatrix} = rD + \partial_\alpha(r)I
\]

where \( r \in K(X) \). \( \Box \)

4. The nature of quadratic regulators

Given a monic, quadratic polynomial in \( K(X)[Y] \) we ask what it takes for it to be the regulator of some functional \( \alpha \). A preliminary answer lies in Proposition 3.5 and its relation to (22).
Proposition 4.1. A monic quadratic polynomial \( Y^2 + pY + q \) in \( K(X)[Y] \) regulates a functional \( \alpha \) if and only if
\[
\alpha \neq 0 \quad \text{and} \quad \partial_{\alpha^*p} + \partial_\alpha^2 = 0 \quad \text{and} \quad \partial_\alpha(p) + q = 0. \tag{24}
\]

Proof. In addition to having \( \alpha \neq 0 \), condition (24) is a special case of (22) taking \( r = 1, s = p, t = q \).

If (24) holds, Proposition 3.5 gives that \( \partial_\alpha^2 + p \circ \partial_\alpha + q \) has finite rank. Then the regulator of \( \alpha \) must divide \( Y^2 + pY + q \) in \( K(X)[Y] \). The regulator cannot have degree 0 as can be seen from its definition. Since \( \alpha \neq 0 \), that regulator is not linear in \( Y \) either. Indeed, if the regulator were \( Y + u \) say, then Proposition 3.3 would yield \( \alpha = 0 \). Thus the regulator is a monic, quadratic divisor of \( Y^2 + pY + q \).

The regulator must be \( Y^2 + pY + q \).

Conversely if \( Y^2 + pY + q \) regulates \( \alpha \), then \( \alpha \neq 0 \) because only \( Y \) regulates the zero functional. Since \( \partial_\alpha^2 + p \circ \partial_\alpha + q \) has finite rank, Proposition 3.5 gives the special case of (22), namely \( \partial_\alpha^*p + \partial_\alpha^2 = 0 \) and \( \partial_\alpha(p) + q = 0. \) \( \square \)

Let us now examine the condition \( \partial_\alpha^2 + \partial_{\alpha^*p} = 0 \) in terms of the various expansions of \( \alpha \).

Proposition 4.2. If \( p \in K(X) \), the identity \( \partial_{\alpha^*p} + \partial_\alpha^2 = 0 \) is equivalent to having
\[
(\alpha^0)^2 - p^0 \alpha^0 - \partial_\alpha(p)^0 = 0 \quad \text{for all } \theta \in K \tag{25}
\]
and
\[
(\alpha^\infty)^2 + Xp^\infty \alpha^\infty - X^2 \partial_\alpha(p)^\infty = 0. \tag{26}
\]

Proof. Let \( \gamma \) be the functional that gives the deriver \( \partial_\alpha^2 \) as in Proposition 2.3. The condition \( \partial_{\alpha^*p} + \partial_\alpha^2 = 0 \) is equivalent to having \( \alpha + \gamma = 0 \). In turn this is equivalent to having
\[
(\alpha^0)^0 + \gamma^0 = 0 \quad \text{for all } \theta \in K \quad \text{and} \quad (\alpha^\infty)^\infty + \gamma^\infty = 0.
\]

From the formulas in Propositions 2.3 and 2.6 the above conditions respectively are equivalent to
\[
p^0 \alpha^0 + \partial_\alpha(p)^0 - (\alpha^0)^2 = 0 \quad \text{for all } \theta \in K
\]
and
\[
p^\infty \alpha^\infty - X\partial_\alpha(p)^\infty + X^{-1}(\alpha^\infty)^2 = 0.
\]

These can easily be rearranged to the conditions (25) and (26). \( \square \)

This brings us to an understanding that in order for a functional to be regulated by a given quadratic polynomial, the functional’s expansions must satisfy quadratic equations over the fields \( K((X - \theta)) \) and \( K((X^{-1})) \).
Proposition 4.3. A non-zero functional \( \alpha \) is regulated by \( Y^2 + pY + q \) if and only if

\[
(\alpha^0)^2 - p^0 \alpha^0 + q^0 = 0 \quad \text{for all } \theta \text{ in } K
\]  

and

\[
(a^\infty)^2 + Xp^\infty a^\infty + X^2q^\infty = 0.
\]  

Proof. Suppose \( Y^2 + pY + q \) regulates \( \alpha \). By Proposition 4.1 followed by 4.2 we deduce (25) and (26). Into those formulae substitute \( \partial_\alpha(p) = -q \), causing (27) and (28) to emerge.

Conversely, suppose that (27) and (28) hold. For \( \theta \) in \( K \) the expansion \( a^\theta \) has no pole at \( \theta \). Thus \( (a^\theta)^2 \) and consequently \( p^\theta a^\theta - q^\theta \) has no pole at \( \theta \). This means that \( P_\theta(p^\theta a^\theta) = P_\theta(q^\theta) \). By Proposition 2.1, \( P_\theta(p^\theta a^\theta) = -P_\theta(\partial_\alpha(p)) \). Hence

\[
P_\theta(q^\theta) = P_\theta(-\partial_\alpha(p)) \quad \text{for all } \theta \text{ in } K.
\]

As argued with \( \theta \) the expansion \( (a^\infty)^2 \) has no pole at \( \infty \), and consequently \( Xp^\infty a^\infty + X^2q^\infty \) has no pole at \( \infty \). This means that the polynomial part of \( Xp^\infty a^\infty + X^2q^\infty \) is a scalar. In other words \( \infty \) is a zero of \( p^\infty a^\infty + Xq^\infty \), which is the same as having \( P_\infty(p^\infty a^\infty + Xq^\infty) = 0 \). From this it follows that also \( P_\infty(X^{-1}p^\infty a^\infty + q^\infty) = 0 \). We have \( P_\infty(\partial_\alpha(p)) = P_\infty(X^{-1}p^\infty a^\infty) \) from Proposition 2.2. Putting the last two equations together we get

\[
P_\infty(q^\infty) = P_\infty(-\partial_\alpha(p)).
\]

Since \( q^\theta \) and \( -\partial_\alpha(p)^\theta \) agree in all of their principal \( \theta \)-parts and in their polynomial part, it follows that \( q \) and \( -\partial_\alpha(p) \) have identical partial fraction expansions. That is \( q = -\partial_\alpha(p) \).

Substitute \( -\partial_\alpha(p) \) for \( q \) into (27) and (28) causing (25) and (26) to emerge. By Proposition 4.2 deduce \( \partial_\alpha p + \partial_\alpha^2 = 0 \), and using Proposition 4.1 the polynomial \( Y^2 + pY + q \) becomes the regulator of \( \alpha \). \( \square \)

4.1. The roots of a quadratic regulator

It turns out that every quadratic regulator splits. This result hinges on a more general, local–global principle regarding roots of polynomials. A polynomial \( g(Y) \) with coefficients in the field \( K(X) \) may be seen as a polynomial with coefficients in the larger field \( K((X - \theta)) \) for every \( \theta \) in \( K \). One may ask the following.

If \( g(Y) \) splits in every \( K((X - \theta)) \), does \( g(Y) \) split in \( K(X) \)?

We show here that this is so when \( g(Y) \) is quadratic. Since the proof hinges on the discriminant of the quadratic, it seems necessary, unfortunately, to avoid fields of
characteristic 2. We are not aware if our question is answered for polynomials of arbitrary degree and in arbitrary characteristic.

**Proposition 4.4.** Suppose \( \text{char } K \neq 2 \). If \( rY^2 + sY + t \in K(X)[Y] \) and \( r^\theta Y^2 + s^\theta Y + t^\theta \) has a root in \( K((X - \theta)) \) for every \( \theta \) in \( K \), then \( rY^2 + sY + t \) has a root in \( K(X) \).

**Proof.** The assumption implies that for each \( \theta \) in \( K \) the discriminant \((s^\theta)^2 - 4r^\theta t^\theta\) is a perfect square in \( K((X - \theta)) \). Since \((s^\theta)^2 - 4r^\theta t^\theta = (s^2 - 4rt)^\theta\), the \( \theta \)-expansion of \( s^2 - 4rt \) is a perfect square in \( K((X - \theta)) \). Because the valuation \( \text{ord}_\theta \) on the field \( K((X - \theta)) \) is multiplicative, \( \text{ord}_\theta((s^2 - 4rt)^\theta) \) is an even integer. Since \( \text{ord}_\theta \) on \( K((X - \theta)) \) extends \( \text{ord}_\theta \) on \( K(X) \), deduce that \( \text{ord}_\theta(s^2 - 4rt) \) is even for all \( \theta \) in \( K \). Therefore the discriminant \( s^2 - 4rt \) of \( rY^2 + sY + t \) is a perfect square in \( K(X) \). Since the characteristic of \( K \) is not 2, the polynomial \( rY^2 + sY + t \) has a root in \( K(X) \). \( \square \)

According to (27) the polynomial \( Y^2 - pY + q \) has the root \( \alpha^\theta \) in every completion \( K((X - \theta)) \). This polynomial has a root in \( K(X) \) if and only if the same holds for the regulator \( Y^2 + pY + q \). Thus we immediately obtain what we want for regulators.

**Proposition 4.5.** Suppose that the characteristic of \( K \) is not 2. If a polynomial \( Y^2 + pY + q \) regulates a functional, then \( Y^2 + pY + q \) has a root in \( K(X) \).

The extent to which a regulator determines its functional now reveals itself.

**Proposition 4.6.** If \( Y^2 + pY + q \) regulates \( \alpha \) and has roots \( u, v \), then

\[ \alpha^\theta = -u^\theta \quad \text{or} \quad \alpha^\theta = -v^\theta \quad \text{for all } \theta \text{ in } K \]

and

\[ \alpha^\infty = Xu^\infty \quad \text{or} \quad \alpha^\infty = Xv^\infty. \]

**Proof.** From \( Y^2 + pY + q = (Y - u)(Y - v) \) we get \( Y^2 - pY + q = (Y + u)(Y + v) \). Formula (27) applies and it becomes

\[ (\alpha^\theta + u^\theta)(\alpha^\theta + v^\theta) = 0 \quad \text{in the field } K((X - \theta)). \]

Thus \( \alpha^\theta = -u^\theta \) or \( \alpha^\theta = -v^\theta \). Also \( Y^2 + XpY + X^2q = (Y - Xu)(Y - Xv) \). Formula (28) applies and it becomes

\[ (\alpha^\infty - Xu^\infty)(\alpha^\infty - Xv^\infty) = 0 \quad \text{in the field } K((X^{-1})). \]

Hence \( \alpha^\infty = Xu^\infty \) or \( \alpha^\infty = Xv^\infty \). \( \square \)
4.2. Constructing all possible quadratic regulators and their functionals

If \(u, v \in K(X)\), we say that these rational functions are detached provided

- \(u, v\) are distinct,
- no \(\theta\) in \(K\) is a pole of both \(u\) and \(v\),
- \(\infty\) is a zero of at least one of \(u\) or \(v\).

For instance, \(0, 1\) are detached, and so are \(X^2 + 1/X + 1, 1/(X - 1)\). Pairs of detached functions can be written at will.

**Proposition 4.7.** If the characteristic of \(K\) is not 2, and \(Y^2 + pY + q\) is a regulator, then its roots are detached.

**Proof.** As seen in Proposition 4.6 the regulated functional \(\alpha\) satisfies \(\alpha^\theta = -u^\theta\) or \(\alpha^\theta = -v^\theta\) for each \(\theta\) in \(K\). Since \(\alpha^\theta\) has no pole at \(\theta\), at least one of \(u^\theta\) or \(v^\theta\) cannot have a pole at \(\theta\). Consequently one of \(u\) or \(v\) has no pole at \(\theta\).

The functional \(\alpha\) also satisfies \(\alpha^\infty = Xu^\infty\) or \(\alpha^\infty = Xv^\infty\). Since \(\alpha^\infty\) has no pole at \(\infty\), at least one of \(Xu^\infty\) or \(Xv^\infty\) has no pole at \(\infty\). This is the same as saying that at least one of \(u\) or \(v\) has a zero at \(\infty\).

If the regulator has only one root, the above considerations show that such root has no pole anywhere and vanishes at \(\infty\). Thus that common root is 0. Consequently \(Y^2\) is the regulator, meaning that \(\partial Y^2\) has finite rank. By Proposition 2.3 \(\partial Y^2\) is a deriver. From Proposition 3.3 deduce that \(\partial Y^2 = 0\). Since the algebra of derivers has no nilpotents it follows that \(\partial = 0\), yielding \(Y\) as the regulator, not \(Y^2\). Thus the regulator has two roots.

Therefore \(u, v\) are detached. \(\Box\)

Next comes the converse of Proposition 4.6.

**Proposition 4.8.** Suppose \(Y^2 + pY + q \in K(X)[Y]\) with detached roots. Define a functional \(\alpha\) by its expansions as follows. For each \(\theta\) in \(K\) pick a root, say \(u\), of \(Y^2 + pY + q\) that has no pole at \(\theta\), and put \(\alpha^\theta = -u^\theta\). Then pick a root, again call it \(u\), that has a zero at \(\infty\), and put \(\alpha^\infty = Xu^\infty\). Then, as long as the functional \(\alpha\) thus specified is not zero, \(\alpha\) is regulated by \(Y^2 + pY + q\).

**Proof.** The choice of \(\alpha\) has been custom designed to satisfy the conditions (27) and (28) of Proposition 4.3 that will make \(Y^2 + pY + q\) the regulator of \(\alpha\). \(\Box\)

The danger of having picked \(\alpha = 0\) can easily be avoided, because all but finitely many \(\theta\) in \(K\) are non-poles of both roots of \(Y^2 + pY + q\). To avoid getting \(\alpha = 0\) make sure the root \(u\) that is picked for at least one \(\theta\) is a non-zero root.
4.3. Idempotents in $\text{End} R_{\alpha}$

For our concluding results we assume that the characteristic of $K$ is not 2, that $\alpha$ is a functional regulated by the polynomial $Y^2 + pY + q$ having detached roots $u, v$. Since $u \neq v$ the rational function $1/(u - v)$ is defined.

Lemma 4.9. $\partial_{\alpha}(1/(u - v)) \equiv u/(u - v) \mod K$.

Proof. For brevity let $w = 1/(u - v)$. It suffices to check that

$$P_\theta(\partial_{\alpha}(w)) = P_\theta(wu) \quad \text{for all } \theta \text{ in } K,$$

and

$$P_\infty(\partial_{\alpha}(w)) \equiv P_\infty(wu) \mod K.$$  \hfill (29)

For $\theta$ in $K$, Proposition 2.1 followed by Proposition 4.6 give

$$P_\theta(\partial_{\alpha}(w)) = -P_\theta(w^\theta u_{\theta}^\theta) = \begin{cases} P_\theta(w^\theta u_{\theta}^\theta) = P_\theta(wu), & \text{or} \\ P_\theta(w^\theta v_{\theta}^\theta) = P_\theta(wv). & \end{cases}$$

If the first option holds, that's just (29). If the second option holds, observe that $wu = 1 + wv$, so that $P_\theta(wu) = P_\theta(wv)$ and thereby return to the desired option (29).

Proposition 2.2 followed by Proposition 4.5 give

$$P_\infty(\partial_{\alpha}(w)) = P_\infty(X^{-1} w_\infty u_{\infty}^\infty) = \begin{cases} P_\infty(X^{-1} w_\infty X u_{\infty}^\infty) = P_\infty(wu), & \text{or} \\ P_\infty(X^{-1} w_\infty X v_{\infty}^\infty) = P_\infty(wv). & \end{cases}$$

If the first option holds, that's just $P_\infty(\partial_{\alpha}(w)) = P_\infty(wu)$, a strong version of (30). If the second option holds, observe once more that $wu = 1 + wv$, so that $P_\infty(wu) = 1 + P_\infty(wv)$. Thereby come to $P_\infty(\partial_{\alpha}(w)) = P_\infty(wu) - 1$ which is good enough to give (30). \hfill □

At this point observe that our final result will yield the proof of Theorem 1.1. Indeed take an indecomposable module $\mathcal{R}_{\alpha}$. If $\mathcal{R}_{\alpha}$ had a non-trivial endomorphism, then the regulator of $\alpha$ would be linear or quadratic by Proposition 3.7. The linear case is ruled out by [6, Proposition 2.4], and the quadratic case is ruled out by our concluding theorem.

Theorem 4.10. If $\alpha$ is regulated by $Y^2 + pY + q$, then $\text{End} \mathcal{R}_{\alpha}$ contains a proper idempotent.

Proof. With $u, v$ be the roots of the regulator, Lemma 4.9 gives a $\lambda$ in $K$ such that

$$\partial_{\alpha}(1/(u - v)) + \lambda = u/(u - v).$$

We have $Y^2 + pY + q = (Y - u)(Y - v)$ so that the generic matrix from (23) is

$$D = \begin{bmatrix} -u - v & -1 \\ u & 0 \end{bmatrix}.$$
The candidate for idempotency is
\[
\epsilon = \left(\frac{1}{u-v}\right)D + \left(\frac{1}{(u-v)} + \lambda\right)I \\
= \frac{1}{(u-v)}D + \frac{u}{(u-v)}I \\
= \frac{1}{u-v} \begin{bmatrix} -v & -1 \\ u & u \end{bmatrix}.
\]

By Proposition 3.8, \(\epsilon\) is an endomorphism of \(R_\alpha\). It is obviously non-trivial. To see that \(\epsilon\) is idempotent, just square it. □

References