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Algebraic transformation of unary partial algebras II: Single-pushout approach

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Abstract

The *single-pushout* approach to graph transformation is extended to the algebraic transformation of partial many-sorted unary algebras. Such a generalization has been motivated by the need to model the transformation of structures which are richer and more complex than graphs and hypergraphs.

The main result presented in this article is an algebraic characterization of the single-pushout transformation in the categories of all conformisms, all closed quomorphisms, and all closed-domain closed quomorphisms of unary partial algebras over a given signature, together with a corresponding operational characterization that may serve as a basis for implementation.

Moreover, all three categories are shown to satisfy all of the HLR (*high-level replacement*) conditions for parallelism, taking as occurrences the total morphisms in each category. Another important result presented in this article is the definition of HLR conditions for amalgamation, which are also satisfied by the categories of partial homomorphisms considered here, taking again the corresponding total morphisms as occurrences.

Keywords: Graph grammars; Algebraic graph transformation; Partial algebras; High-level replacement systems; HLR conditions

1. Introduction

This paper is the second in a series initiated with [7]. The global goal of this series is to study in detail the algebraic transformation of partial unary algebras, both under

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the double and single-pushout approach, and to apply this study to algebraic graph transformation. See also [1], which surveys some of the main results contained in this series of papers (although the approach to single-pushout transformation taken therein is different from the one taken here), [18], which translates some of these results to the setting of categories of *spans*, and [17], where such categories of spans are applied to develop a single-pushout approach to hypergraph rewriting based on conformisms.

We devote this paper to single-pushout transformation systems of partial unary algebras, based on different types of partial homomorphisms. This paper should be viewed as forming a unity with [7], and we shall freely use the notations introduced therein, usually without any further notice. In particular, we assume the reader to be familiar with the basic language of partial algebras as introduced in Appendix A therein.

The aim of this article is to extend systematically the single-pushout algebraic transformation to partial many-sorted unary algebras. The motivation is shared by the first paper in this series, namely that structures more complex than graphs and hypergraphs are better understood as partial algebras rather than total algebras, and that even in the case of graphs and hypergraphs, the theory of partial algebras offers new concepts that can be effectively used to develop new approaches to algebraic transformation. Additional motivation comes from the increasing use of partial algebras in the field of algebraic specification; see, for instance, [3] or the forthcoming [10].

As a matter of fact, different representations of unary partial algebras as total algebras could be tried in order to simulate the transformation of partial algebras by means of that of total algebras, cf. Section 2.2, but none of them is best suited across the different notions of partial morphism used in this paper, and then the representation would have to be tailored to each specific application.

Let us focus now on the contents of the present paper. We first of all recall that in the single-pushout approach to transformation in a category $\mathscr{C}$ (of partial morphisms, whatever it means), a *(single-pushout) production rule* is taken to be simply a morphism $r : L \to R$ in that category, and then the application of such a rule to an object G through an *occurrence* (a total morphism) $m : L \to G$ consists in computing (when possible) the pushout of r and m in $\mathscr{C}$

$$\begin{array}{ccc} L & \xrightarrow{r} & R \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ G & \xrightarrow[r']{} & H \end{array}$$

In this case the object H is said to be *derived* from G by the application of rule $r : L \to R$ through morphism m, or that H is obtained by means of a *direct derivation* from G by r, and it is denoted in general by $G \overset{r}{\Longrightarrow} H$.

Therefore, if $\mathscr{C}$ is a category closed under pushouts then every production rule can be applied to every occurrence, obtaining always a derived object unique up to isomorphism. See [12, 15] for more information on single-pushout transformation.

Let now $\Gamma = (S, \Omega, \eta)$ be a graph structure (a signature with all its operations unary). In this paper we lay the basis for a single-pushout approach to partial Γ-algebras transformation, using three different types of partial homomorphisms: *conformisms*, *closed quomorphisms* (c-quomorphisms, for short) and *closed-domain closed quomorphisms* (cdc-quomorphisms, for short). They are, respectively, closed homomorphisms from a weak, a relative and a closed subalgebra of the source partial algebra, and can be informally described as follows: conformisms are those partial homomorphisms that reflect the algebraic structure of the target algebra; c-quomorphisms are those conformisms that, moreover, preserve the algebraic structure of their domain; and cdc-quomorphisms are those c-quomorphisms whose domain is a closed subset of the source algebra. See Section 2.1 for details. These categories are known to be cocomplete [2, 9]. There is another popular type of partial homomorphism of partial algebras, the (plain) *quomorphisms* (plain homomorphisms from a relative subalgebra of their source; informally, those partial homomorphisms that preserve the algebraic structure of their domain), but in general the pushout of two quomorphisms does not exist, even for partial unary algebras. The case of quomorphisms is studied in [27], where the authors give, among other results, a necessary and sufficient condition on a pair of quomorphisms to have a pushout; we recall it in Section 2.4.

In this paper we give, for conformisms, c-quomorphisms and cdc-quomorphisms, a detailed description of the pushout of two such partial morphisms as the pushout of two total closed homomorphisms, following the spirit of [14] or [15]. In all three cases, and contrary to the description of these pushouts given (without any detail) in the introduction [1] to this series of papers, the descriptions given here allow a Remove-Add description of the construction of the pushout objects, and elucidate the relation between single- and double-pushout transformations. The study we make of this relation sheds some light on the Open Problem 4.5.1 in [12], which asks for formal comparisons of double-pushout and single-pushout transformation systems.

Other, not so popular, categories of partial homomorphisms of partial algebras are considered in [2, 8, 13, 19], which we do not consider here. In most cases, the reason for not considering them is the lack of pushouts, even in the unary case. Actually, and to our knowledge, there are only three other types of partial homomorphisms of partial algebras having all pushouts in some non-trivial cases: the (uninteresting) partial mappings, which have nothing to do with the structure of the partial algebras involved; the cdc-quomorphisms with domain an initial segment of the source algebra, which have all pushouts only in the unary case and these pushouts are given by a construction similar to the one for cdc-quomorphisms explained in Section 5 below; and the c-quomorphisms with domain an initial segment of the source algebra, which have *always* all pushouts but these are given in general by a very involved construction, and which are the object of a separate study elsewhere [20].

The contents of this paper are briefly described as follows. For the convenience of the reader, in Section 2 we collect some preliminaries. Namely, on the one hand we recall the main concepts and results on partial homomorphisms of partial algebras used in this paper, followed by a detailed analysis of possible representations of partial algebras as total algebras, and on the other hand we give a detailed description of the pushout of two partial mappings of S-sets as it shall be used in the remaining sections. We also recall the construction of the pushout of quomorphisms given in [27], expressing it in a language similar to the one used in our constructions.

Then, we devote a section to each type of partial homomorphism considered in this paper. The sections dealing with the different types of partial homomorphisms are ordered from the more general to the more particular case: first conformisms, then c-quomorphisms and finally cdc-quomorphisms.

In the last section we show that the HLR conditions for parallelism introduced in [12] are satisfied for conformisms, c-quomorphisms and cdc-quomorphisms, taking as occurrences in any case the totally defined such morphisms. This gives three more examples of single-pushout transformation systems satisfying them, examples that are asked for in [12].

Moreover, we define HLR conditions for amalgamation and show that they are satisfied for conformisms, c-quomorphisms and cdc-quomorphisms, again taking the corresponding total morphisms as occurrences. This allows to decompose non parallel-independent derivations by rules sharing a common subrule into a common derivation (by that subrule) followed by parallel-independent derivations, where such common derivations induce a global synchronization mechanism.

2. Preliminaries

2.1. *Partial homomorphisms of partial algebras*

As we have already mentioned in the introduction, we assume the reader familiar with the basic language of partial algebras as presented in Appendix A in [7]. This section complements that Appendix, by introducing the most popular notions of partial homomorphisms of partial algebras, for which we use the names given in [9] and its "addendum" [2].

Nevertheless, for the convenience of the reader, we recall first of all the definitions of the different types of subalgebras and (total) homomorphisms of partial algebras underlying the different notions of partial homomorphisms.

Let $\Sigma = (S, \Omega, \eta)$ be a signature, where $\eta(\varphi) = (w(\varphi), \sigma(\varphi)) \in S^* \times S$ for every operation symbol $\varphi \in \Omega$. A signature is called a *graph structure*, denoted by Γ, when all its operations are unary, that is, when $w(\varphi) \subseteq S$ for all $\varphi \in \Omega$, and a graph structure Γ is called *monounary* when it has a single sort (which we shall omit as a subscript in practice) and a single unary operation φ.

Given an S-set $A = (A_s)_{s\in S}$ and a string $w \in S^*$, A^w denotes a singleton if $w = \lambda$, the empty word, and $A_{s_1} \times \cdots \times A_{s_n}$ if $w = s_1 \ldots s_n$.

Let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ be two partial Σ-algebras, with $B \subseteq A$ (that is, with $B_s \subseteq A_s$ for every $s \in S$).

- $\mathbf{B}$ is a *weak subalgebra* of $\mathbf{A}$ (*supported on* B) when it satisfies the following condition for every $\varphi \in \Omega$: if $\underline{b} \in \operatorname{dom}\varphi^{\mathbf{B}}$ then $\underline{b} \in \operatorname{dom}\varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(\underline{b}) = \varphi^{\mathbf{B}}(\underline{b})$.

- **B** is a *relative subalgebra* of **A** (*supported on* B) when it is a weak subalgebra of **A** satisfying the following further condition for every $\varphi \in \Omega$: if $\underline{b} \in B^{w(\varphi)} \cap \operatorname{dom} \varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(\underline{b}) \in B_{\sigma(\varphi)}$ then $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{B}}$.
- B is a *closed subset* of **A** when it satisfies the following condition for every $\varphi \in \Omega$: if $\underline{b} \in B^{w(\varphi)} \cap \operatorname{dom} \varphi^{\mathbf{A}}$ then $\varphi^{\mathbf{A}}(\underline{b}) \in B_{\sigma(\varphi)}$.
- **B** is a *closed subalgebra* of **A** (*supported on* B) when it is a relative subalgebra of **A** and B is a closed subset of **A**. That is, when it is a weak subalgebra of **A** satisfying the following further condition for every $\varphi \in \Omega$: if $\underline{b} \in B^{w(\varphi)} \cap \operatorname{dom} \varphi^{\mathbf{A}}$ then $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{B}}$.

Given a subset B of the universe A of a partial algebra **A**, there are in principle many weak subalgebras of **A** supported on B, but only one relative subalgebra of **A** supported on B. And there is a (unique) closed subalgebra of **A** supported on B iff B is closed in **A**.

Let now $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi \in \Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi \in \Omega})$ be two arbitrary partial Σ-algebras and let $f : A \to B$ be a mapping of S-sets (that is, a family of mappings $f_s : A_s \to B_s$, $s \in S$).

- f is a (*plain*) *homomorphism* from **A** to **B** when it satisfies the following condition for every $\varphi \in \Omega$: if $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$ then $f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}}$ and $\varphi^{\mathbf{B}}(f(\underline{a})) = f_{\sigma(\varphi)}(\varphi^{\mathbf{A}}(\underline{a}))$.[1]
- f is a *closed homomorphism* from **A** to **B** when it is a homomorphism from **A** to **B** and it satisfies the following further condition for every $\varphi \in \Omega$: for every $\underline{a} \in A^{w(\varphi)}$, if $f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}}$ then $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$.

We shall denote by Alg_Σ and $\mathrm{C\text{-}Alg}_\Sigma$ the categories whose objects are all partial Σ-algebras and whose morphisms are the plain homomorphisms and the closed homomorphisms, respectively.

Finally, let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi \in \Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi \in \Omega})$ be still two arbitrary partial Σ-algebras, and let now $f : A \rightharpoonup B$ be a *partial mapping* of S-sets (that is, a family of partial mappings $f_s : A_s \rightharpoonup B_s$, $s \in S$). Let $\operatorname{Dom} f_s$ denote the domain of f_s, for every $s \in S$, and let $\operatorname{Dom} f = (\operatorname{Dom} f_s)_{s \in S}$ be the domain of f.

- f is a *quomorphism* from **A** to **B** when it is a *plain* homomorphism from *the relative* subalgebra of **A** supported on $\operatorname{Dom} f$ to **B**.
- f is a *conformism*[2] from **A** to **B** when it is a *closed* homomorphism from *some weak* subalgebra of **A** supported on $\operatorname{Dom} f$ to **B**.
- f is a *closed quomorphism*[3], *c-quomorphism* for short, from **A** to **B** when it is a quomorphism and a conformism simultaneously; that is, when it is a *closed* homomorphism from *the relative* subalgebra of **A** supported on $\operatorname{Dom} f$ to **B**.
- f is a *closed-domain closed quomorphism*[4], *cdc-quomorphism* for short, from **A** to **B** when it is a closed quomorphism with $\operatorname{Dom} f$ a closed subset of **A**.

[1] If $\underline{a} = (a_{s_1}, \ldots, a_{s_n}) \in A^{s_1 \ldots s_n}$ then $f(\underline{a})$ stands for $(f_{s_1}(a_{s_1}), \ldots, {}_{s_n}(a_{s_n}))$.

[2] p-Morphism, in [22]

[3] Quomorphic conformism, in [1]

[4] Closed-domain quomorphic conformism, in [1]

In other words,

- f is a *quomorphism* from **A** to **B** when it satisfies the following condition:[5] for every $\varphi \in \Omega$ and $\underline{a} \in (\operatorname{Dom} f)^{w(\varphi)}$

$$\left\{\begin{array}{l} \underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}} \\ \varphi^{\mathbf{A}}(\underline{a}) \in (\operatorname{Dom} f)_{\sigma(\varphi)} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}} \\ \varphi^{\mathbf{B}}(f(\underline{a})) = f_{\sigma(\varphi)}(\varphi^{\mathbf{A}}(\underline{a})) \end{array}\right\}$$

- f is a *conformism* from **A** to **B** when it satisfies the following condition: for every $\varphi \in \Omega$ and $\underline{a} \in (\operatorname{Dom} f)^{w(\varphi)}$

$$f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}} \Longrightarrow \left\{\begin{array}{l} \underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}} \\ \varphi^{\mathbf{A}}(\underline{a}) \in (\operatorname{Dom} f)_{\sigma(\varphi)} \\ \varphi^{\mathbf{B}}(f(\underline{a})) = f_{\sigma(\varphi)}(\varphi^{\mathbf{A}}(\underline{a})) \end{array}\right\}$$

- f is a *c-quomorphism* from **A** to **B** when it satisfies the following condition: for every $\varphi \in \Omega$ and $\underline{a} \in (\operatorname{Dom} f)^{w(\varphi)}$

$$\left\{\begin{array}{l} \underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}} \\ \varphi^{\mathbf{A}}(\underline{a}) \in (\operatorname{Dom} f)_{\sigma(\varphi)} \end{array}\right\} \Rightarrow f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}} \Rightarrow \left\{\begin{array}{l} \underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}} \\ \varphi^{\mathbf{A}}(\underline{a}) \in (\operatorname{Dom} f)_{\sigma(\varphi)} \\ \varphi^{\mathbf{B}}(f(\underline{a})) = f_{\sigma(\varphi)}(\varphi^{\mathbf{A}}(\underline{a})) \end{array}\right\}$$

- f is a *cdc-quomorphism* from **A** to **B** when it satisfies the following condition: for every $\varphi \in \Omega$ and $\underline{a} \in (\operatorname{Dom} f)^{w(\varphi)}$

$$\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}} \Longrightarrow f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}} \Longrightarrow \left\{\begin{array}{l} \underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}} \\ \varphi^{\mathbf{A}}(\underline{a}) \in (\operatorname{Dom} f)_{\sigma(\varphi)} \\ \varphi^{\mathbf{B}}(f(\underline{a})) = f_{\sigma(\varphi)}(\varphi^{\mathbf{A}}(\underline{a})) \end{array}\right\}$$

Example 1. Let Γ be a signature with a unique sort (which we shall omit) and a unary operation φ. Let **A** be a partial Γ-algebra supported on $A = \{a_1, a_2, a_3\}$ with the operation $\varphi^{\mathbf{A}}$ given by $\varphi^{\mathbf{A}}(a_1) = a_2$, $\varphi^{\mathbf{A}}(a_2) = a_3$, and let **B** be a partial Γ-algebra supported on $B = \{b_1, b_2, b_3\}$ with the operation $\varphi^{\mathbf{B}}$ given by $\varphi^{\mathbf{B}}(b_1) = b_2$, $\varphi^{\mathbf{B}}(b_2) = b_3$. Then (see Fig. 1)

- the mapping $f : A \to B$ given by $f(a_2) = f(a_3) = b_3$ is a conformism, but not a quomorphism, from **A** to **B**;
- the mapping $f' : A \to B$ given by $f'(a_1) = b_1$, $f'(a_2) = b_2$ is a quomorphism, but not a conformism, from **A** to **B**;
- the mapping $f'' : A \to B$ given by $f''(a_1) = b_2$, $f''(a_2) = b_3$ is a c-quomorphism, but not a cdc-quomorphism, from **A** to **B**;
- and the mapping $f''' : A \to B$ given by $f'''(a_2) = b_2$, $f'''(a_3) = b_3$ is a cdc-quomorphism from **A** to **B**.

For a given signature Σ, we shall denote by Q-Alg$_\Sigma$, CF-Alg$_\Sigma$, CQ-Alg$_\Sigma$ and CDCQ-Alg$_\Sigma$ the categories whose objects are all partial Σ-algebras and whose morphisms are, respectively, the quomorphisms, conformisms, c-quomorphisms and

[5] Where logical formulae of the shape $A \Rightarrow B \Rightarrow C$ are intended to be parsed as $(A \Rightarrow B) \wedge (B \Rightarrow C)$.

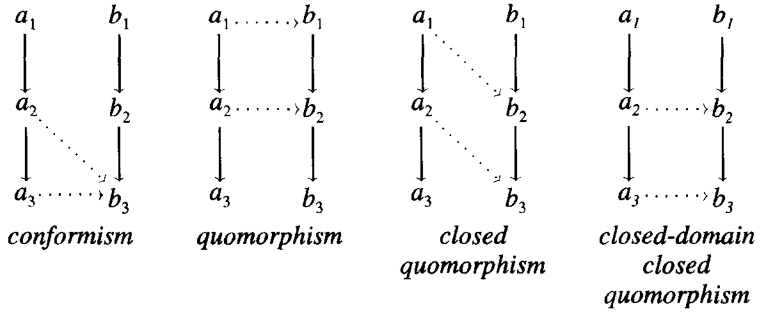


Fig. 1.

cdc-quomorphisms. Notice in particular that the following relations hold:

$$\text{C-Alg}_\Sigma \subseteq \text{CDCQ-Alg}_\Sigma \subseteq \text{CQ-Alg}_\Sigma \subseteq \text{CF-Alg}_\Sigma,$$

$$\text{Alg}_\Sigma \subseteq \text{Q-Alg}_\Sigma,$$

$$\text{Q-Alg}_\Sigma \cap \text{CF-Alg}_\Sigma = \text{CQ-Alg}_\Sigma.$$

Furthermore, total (that is, totally defined) quomorphisms are homomorphisms and total c-quomorphisms and total cdc-quomorphisms are closed homomorphisms. But a total conformism $f : \mathbf{A} \to \mathbf{B}$ need not be even a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Consider for instance a total mapping between a total algebra and a discrete algebra: it is always a conformism, but it is never a homomorphism, as long as the signature has operation symbols. Let TCF-Alg$_\Sigma$ denote the category of all partial Σ-algebras with total conformisms as morphisms.

The categories CF-Alg$_\Sigma$, Q-Alg$_\Sigma$, CQ-Alg$_\Sigma$, CDCQ-Alg$_\Sigma$ and TCF-Alg$_\Sigma$ are thoroughly studied in [2, 6, 8, 9, 16, 19]. In particular:

- Q-Alg$_\Sigma$ is never complete or cocomplete if $\Omega \neq \emptyset$ [9].
- For every one of the categories CF-Alg$_\Sigma$ [9], CQ-Alg$_\Sigma$ [9], CDCQ-Alg$_\Sigma$ [2] and TCF-Alg$_\Sigma$ [19], one has that it is complete iff it is cocomplete iff Σ is a graph structure.

We shall denote by TAlg$_\Sigma$ the usual category of total Σ-algebras, and by P-TAlg$_\Sigma$ the category of total Σ-algebras with partial homomorphisms in the sense of [15]: [6] homomorphisms from subalgebras of the source algebra. Since the subalgebras of a total algebra are exactly its closed subalgebras, and since for total algebras the concepts of homomorphism (in the total algebras sense) and closed homomorphism are exactly the same, it turns out that P-TAlg$_\Sigma$ is a full subcategory of CDCQ-Alg$_\Sigma$. It is proved in [15] that P-TAlg$_\Sigma$ is cocomplete iff Σ is a graph structure.

[6] It is denoted $\text{Alg}^{\text{P}}(\Sigma)$ in [15]

2.2. Partial algebras versus total algebras

Given a graph structure $\Gamma = (S, \Omega, \eta)$, with $\eta(\varphi) = (w(\varphi), \sigma(\varphi)) \in S \times S$ for every operation symbol $\varphi \in \Omega$, one can encode partial Γ-algebras as total algebras in two ways.

The first way roughly consists in splitting the universe of a partial Γ-algebra in layers in such a way that the operations in Γ become total operations between these layers; this is done for instance in [24] to model partial graphs as total algebras.

Let $\mathbf{T} = \mathbf{T}_\Gamma(X)$ be the total Γ-algebra of terms with variables in the set $X = (\{x_s\})_{s \in S}$ (with a single variable x_s for every sort $s \in S$), and let $T = (T_s)$ be its universe, that is, the S-set of Γ-terms defined over the set of variables X.

We shall say that a non-empty initial segment A of $\mathbf{T}$ over the set X of variables is *homogeneous* when all terms in it have the same variable. Let $\mathscr{G}$ denote the set of all (non-empty) homogeneous initial segments of $\mathbf{T}$.

Given a homogeneous initial segment $I \in \mathscr{G}$ and a term $\mathbf{t} \in T_{s_0}$, we define $I/\mathbf{t} = ((I/\mathbf{t})_s)_{s \in S}$ as follows:

$$(I/\mathbf{t})_s = \left\{\mathbf{t}'(x_{s_0}) \in T_s \mid \mathbf{t}'^{\mathbf{T}}(\mathbf{t}) \in I_s\right\}.$$

Notice that $I/\mathbf{t}$ is either empty (when $\mathbf{t} \notin I_{s_0}$) or a homogeneous initial segment (when $\mathbf{t} \in I_{s_0}$).

Consider then the graph structure $\tilde{\Gamma} = (\mathscr{G}, \tilde{\Omega}, \tilde{\eta})$ with set of sorts $\mathscr{G}$, set of operations

$$\tilde{\Omega} = \{\varphi_I \mid I \in \mathscr{G}, \varphi \in \Omega, \varphi(x_{w(\varphi)}) \in I_{\sigma(\varphi)}\}$$

and the arity mapping given by $\tilde{\eta}(\varphi_I) = (I, I/\varphi(x_{w(\varphi)}))$ for every $\varphi_I \in \tilde{\Omega}$.

Now, to every partial Γ-algebra $\mathbf{A}$ we can associate a total $\tilde{\Gamma}$-algebra

$$\tilde{\mathbf{A}} = (\tilde{A}, (\varphi_I^{\tilde{\mathbf{A}}})_{\varphi_I \in \tilde{\Omega}})$$

in the following way. Its carrier $\tilde{A} = (\tilde{A}_I)_{I \in \mathscr{G}}$ is given by

$$\tilde{A}_I = \left(\bigcap_{\mathbf{t} \in I} \operatorname{dom} \mathbf{t}^{\mathbf{A}}\right) - \left(\bigcup_{\mathbf{s} \in T - I} \operatorname{dom} \mathbf{s}^{\mathbf{A}}\right), \quad I \in \mathscr{G}$$

and for every $\varphi_I \in \tilde{\Omega}$, the operation $\varphi_I^{\tilde{\mathbf{A}}} : \tilde{A}_I \to \tilde{A}_{I/\varphi(x_{w(\varphi)})}$ is given by the restriction to $\tilde{A}_I \subseteq A_{w(\varphi)}$ of the corresponding operation $\varphi^{\mathbf{A}} : A_{w(\varphi)} \to A_{\sigma(\varphi)}$ on $\mathbf{A}$.

For instance, consider a monounary graph structure Γ. In this case $\mathscr{G} = \{I_n \mid n \in \mathbb{N} \cup \{\infty\}\}$, where $I_0 = \{x\}$, $I_n = \{x, \varphi(x), \ldots, \varphi^n(x)\}$ for any $n \in \mathbb{Z}^+$, and $I_\infty = T$, and then

$$I_n/\varphi(x) = \begin{cases} \emptyset & \text{if } n = 0, \\ I_{n-1} & \text{if } n \in \mathbb{Z}^+, \\ I_\infty & \text{if } n = \infty. \end{cases}$$

We identify then $\mathscr{G}$ with $\mathbb{N} \cup \{\infty\}$, and we have that, in this case,

$$\tilde{\Gamma} = (\mathbb{N} \cup \{\infty\}, (\varphi_n)_{n \in \mathbb{Z}^+ \cup \{\infty\}}, \tilde{\eta})$$

with $\tilde{\eta}(\varphi_\infty) = (\infty, \infty)$ and $\tilde{\eta}(\varphi_n) = (n, n-1)$ for every $n \in \mathbb{Z}^+$.

And then, given a partial Γ-algebra $\mathbf{A} = (A, \varphi^{\mathbf{A}})$, we obtain the total $\tilde{\Gamma}$-algebra $\tilde{\mathbf{A}} = ((A_n)_{n\in\mathbb{N}\cup\{\infty\}}, (\varphi_n^{\tilde{\mathbf{A}}})_{n\in\mathbb{Z}^+\cup\{\infty\}})$ by splitting its universe into the layers $\tilde{A}_0 = \{a \mid a \notin \operatorname{dom}\varphi^{\mathbf{A}}\}$, $\tilde{A}_n = \operatorname{dom}\varphi^{n^{\mathbf{A}}} - \operatorname{dom}\varphi^{n+1^{\mathbf{A}}}$ for every $n \in \mathbb{Z}^+$, and $\tilde{A}_\infty = \bigcap_{n\in\mathbb{N}} \operatorname{dom}\varphi^{n^{\mathbf{A}}}$, and then breaking the partial operation $\varphi^{\mathbf{A}}$ into total operations $\varphi_n^{\tilde{\mathbf{A}}} : A_n \to A_{n-1}$ $(n \in \mathbb{Z}^+)$ and $\varphi_\infty^{\tilde{\mathbf{A}}} : A_\infty \to A_\infty$.

Returning to the case of an arbitrary graph structure, it turns out that the association $\mathbf{A} \mapsto \tilde{\mathbf{A}}$ yields an equivalence between the categories C-Alg$_\Gamma$ and TAlg$_{\tilde{\Gamma}}$. But plain homomorphisms of partial Γ-algebras would yield mappings between $\tilde{\Gamma}$-algebras which do not even preserve the sorts. So, plain homomorphisms do not find a good translation into the $\tilde{\Gamma}$-algebras setting, and since conformisms have a part depending on plain homomorphisms (they are closed homomorphisms from *weak* subalgebras, whose embeddings are, in general, only plain homomorphisms), we should not expect a good translation for them. The same assertion can be made, for similar reasons, regarding c-quomorphisms.

But, on the other hand, since cdc-quomorphisms of partial Γ-algebras are closed homomorphisms from closed subalgebras, and the embeddings of the latter are then closed homomorphisms, one should expect that cdc-quomorphisms have a good translation to the total $\tilde{\Gamma}$ setting. And, indeed, it is not difficult to check that the association $\mathbf{A} \mapsto \tilde{\mathbf{A}}$ yields an equivalence between the categories CDCQ-Alg$_\Gamma$ and P-TAlg$_{\tilde{\Gamma}}$.

Before leaving this approach, we want to point out that passing from Γ to $\tilde{\Gamma}$ blows up the size of the signature: for instance, in the one-sorted case, if Γ has finitely many operations then $\tilde{\Gamma}$ has infinitely many sorts and operations, and if Γ has a countably infinite set of operations then $\tilde{\Gamma}$ has non-countably many sorts and operations.

A second way to encode partial Γ-algebras into total unary algebras consists in associating to a partial Γ-algebra its colored directed graph, replacing the operations by arcs with color the operation symbol.

Specifically, given the graph structure Γ from the beginning, consider the new graph structure $\hat{\Gamma} = (S \cup \Omega, \hat{\Omega}, \hat{\eta})$ with set of sorts $S \cup \Omega$, set of operations

$$\hat{\Omega} = \{s_\varphi, t_\varphi \mid \varphi \in \Omega\}$$

and arity function $\hat{\eta}$ given by $\hat{\eta}(s_\varphi) = (\varphi, w(\varphi))$ and $\hat{\eta}(t_\varphi) = (\varphi, \sigma(\varphi))$, for every $\varphi \in \Omega$.

Then, to every partial Γ-algebra $\mathbf{A}$ we can associate a total $\hat{\Gamma}$-algebra $\hat{\mathbf{A}} = (\hat{A}, (s_\varphi^{\hat{\mathbf{A}}}, t_\varphi^{\hat{\mathbf{A}}})_{\varphi\in\Omega})$ as follows. Its carrier $\hat{A}$ is given by $\hat{A}_s = A_s$ for every $s \in S$ and $\hat{A}_\varphi = \{(a, \varphi^{\mathbf{A}}(a)) \mid a \in \operatorname{dom}\varphi^{\mathbf{A}}\}$ for every $\varphi \in \Omega$. And for every $\varphi \in \Omega$ and $(a, b) \in \hat{A}_\varphi$, $s_\varphi^{\hat{\mathbf{A}}}(a, b) = a$ and $t_\varphi^{\hat{\mathbf{A}}}(a, b) = b$. Notice that such $\hat{\mathbf{A}}$ has all its operations s_φ injective.

We can understand then such a $\hat{\Gamma}$-algebra $\hat{\mathbf{A}}$ as a colored directed graph (with nodes in A and arcs the operations defined in $\mathbf{A}$, with color the corresponding operation symbol) such that no two same-colored arcs have the same source node.

For instance, if Γ is again a monounary graph structure, then the signature $\hat{\Gamma}$ has two sorts, say V and E (corresponding to the sort and the operation of Γ, respectively), and two operations, say s, t, with arity $\hat{\eta}(s) = \hat{\eta}(t) = (E, V)$: that is, it is (equivalent to) the graph structure that allows to understand directed graphs as total unary algebras; cf. [7, Example 4]. And then to every partial Γ-algebra $\mathbf{A}$ we associate its directed graph $\hat{\mathbf{A}}$, with carrier $\hat{A}_V = A$, $\hat{A}_E = \{(a, \varphi^{\mathbf{A}}(a)) \mid a \in \operatorname{dom} \varphi^{\mathbf{A}}\}$ (the operation $\varphi^{\mathbf{A}}$ as a set of ordered pairs), and "source" and "target" operations $s^{\hat{\mathbf{A}}}$ and $t^{\hat{\mathbf{A}}}$ given by the first and second projection from the graph of $\varphi^{\mathbf{A}}$ to A.

For a general graph structure Γ, this association $\mathbf{A} \to \hat{\mathbf{A}}$ yields an equivalence between Alg_Γ and the full subcategory of $\mathrm{TAlg}_{\hat{\Gamma}}$ supported on the class of all total $\hat{\Gamma}$-algebras with all operations s_φ injective. However, closed homomorphisms of Γ-algebras do not find a simple translation to $\hat{\Gamma}$-algebras: they correspond to the homomorphisms $f : \hat{\mathbf{A}} \to \hat{\mathbf{B}}$ such that if an arc $e \in \hat{B}_\varphi$ has source (respectively, target) node $b \in f_s(A_s) \subseteq \hat{B}_s$, then for every pre-image a of b there exists a pre-image of e (in $\hat{A}_\varphi$) with a as its source (respectively, target) node. This means that conformisms (or, for that matter, c-quomorphisms or cdc-quomorphisms) do not find a simple translation into this setting, either.

2.3. Pushouts of partial mappings

We recall here a description of the pushout of two partial mappings of S-sets that lies at the basis of all constructions of pushouts that appear in the next sections. We shall freely use in the rest of this paper the notations introduced in this subsection, usually without any further notice.

Given a non-empty set S (of sorts), let S-**Set** and S-**PSet** denote the categories of S-sets with total and partial mappings, respectively, as morphisms.

Let $f : K \to A$ and $g : K \to B$ be two partial mappings of S-sets. Let $\theta(K)'$ be the least equivalence relation on the disjoint union $A \sqcup B$ such that it contains [7]

$$\left(\{(f_s(x), g_s(x)) \in A_s \times B_s \subseteq (A_s \sqcup B_s)^2 \mid x \in \operatorname{Dom} f_s \cap \operatorname{Dom} g_s\}\right)_{s \in S}.$$

Definition 2. The gluing set of f and g is the greatest subset $K' = (K'_s)_{s \in S}$ of K compatible with $\theta(K)'$; that is,

$$K'_s = \{x \in \operatorname{Dom} f_s \cap \operatorname{Dom} g_s \mid f_s^{-1}([f_s(x)]_{\theta(K)'_s}) \cup g_s^{-1}([g_s(x)]_{\theta(K)'_s}) \subseteq \operatorname{Dom} f_s \cap \operatorname{Dom} g_s\}.$$

(Recall in this definition and in the sequel that f^{-1} and g^{-1} are, in general, relations and not mappings.)

Notice that this definition of K' entails that if $x \in K'_s$ then $f_s^{-1}([f_s(x)]_{\theta(K)'_s}) = g_s^{-1}([g_s(x)]_{\theta(K)'_s}) \subseteq K'_s$.

The gluing set of two partial mappings can also be characterized by means of the following property, whose easy proof we leave to the reader.

[7] As in [7], for the sake of simplicity we shall always identify an element of a given set with its image in the disjoint union of this set with other sets.

Lemma 3. *The gluing set K' of f and g is the greatest subset of K such that $f^{-1}(f(K')) = K'$ and $g^{-1}(g(K')) = K'$.*

Set now $A' = (A - f(K)) \cup f(K')$ and $B' = (B - g(K)) \cup g(K')$ and consider the restrictions

$$f|_{K'} : K' \to A', \quad g|_{K'} : K' \to B'$$

of f and g to K'. Set $H = (A' \sqcup B')/_{\theta(K')}$, where $\theta(K')$ stands for the least equivalence relation on $A' \sqcup B'$ containing

$$(\{(f_s(x), g_s(x)) \mid x \in K'_s\})_{s \in S},$$

and let $\tilde{f} : B' \to H$ and $\tilde{g} : A' \to H$ denote the restrictions to B' and A', respectively, of the quotient map $A' \sqcup B' \to (A' \sqcup B')/_{\theta(K')}$. Then the set H, together with the mappings $\tilde{f} : B' \to H$ and $\tilde{g} : A' \to H$, is a pushout of $f|_{K'} : K' \to A'$ and $g|_{K'} : K' \to B'$ in S-**Set**; we shall refer to this construction as the *usual pushout* of $f|_{K'}$ and $g|_{K'}$ in S-**Set**.

Let $\tilde{g} : A \to H$ and $\tilde{f} : B \to H$ also denote the partial mappings corresponding to the homonymous total mappings in the corresponding square, with domains $\operatorname{Dom} \tilde{g} = A'$ and $\operatorname{Dom} \tilde{f} = B'$ respectively.

Proposition 4. *The square*

$$\begin{array}{ccc} K & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ B & \xrightarrow[\tilde{f}]{} & H \end{array}$$

*is a pushout square in S-**PSet**.*

Proof. If Γ is a graph structure with set of sorts S and empty set of operations, then the pushout of two partial homomorphisms of total Γ-algebras (that is, two partial mappings of S-sets) described in [15, Construction 2.6] (cf. [23, Proposition 3]) amounts to the construction summarized in this proposition. □

From this description of the pushout of two partial mappings of S-sets the following properties, which shall be used later, are easily deduced; cf. [15, Corollary 2.8].

Corollary 5. *Let H, together with $\tilde{g} : A \to H$ and $\tilde{f} : B \to H$, be a pushout in S-**PSet** of $f : K \to A$ and $g : K \to B$.*

(i) *The partial mappings $\tilde{g}$ and $\tilde{f}$ are jointly surjective ($\tilde{g}(A) \cup \tilde{f}(B) = H$).*
(ii) *The partial mapping $\tilde{g}$ is total iff $g^{-1}(g(\operatorname{Dom} f)) = \operatorname{Dom} f$.*
(iii) *If f and g are total then $\tilde{g}$ and $\tilde{f}$ are also total.*

Finally, notice that if $x \in K'_s$, then $[f_s(x)]_{\theta(K)'_s} = [f_s(x)]_{\theta(K')_s}$. We shall use it later, usually without any further mention.

2.4. Pushouts of quomorphisms

It is well known [9, 27] that for every signature Σ with non-empty set of operations (even a graph structure), there exist quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of partial Σ-algebras whose pushout in Q-Alg$_\Sigma$ does not exist. Wagner and Gogolla give in [27] a characterization of those pairs of quomorphisms that have a pushout. Although we shall not use it in this paper, we recall this characterization here because of its close relation with our constructions.

Proposition 6 (Wagner and Gogolla [27, Theorem 7]). *Let $\Sigma = (S, \Omega, \eta)$ be a signature with $\Omega \neq \emptyset$, and let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two quomorphisms of partial Σ-algebras. Let H, together with $\tilde{f} : B \to H$ and $\tilde{g} : A \to H$, be the pushout of f and g in S-***PSet*** *described in Proposition* 4, *and let* $\mathbf{A}'$ *and* $\mathbf{B}'$ *denote respectively the relative subalgebras of* $\mathbf{A}$ *and* $\mathbf{B}$ *supported on the domains A' and B' of $\tilde{g}$ and $\tilde{f}$.*

Then, the quomorphisms f and g have a pushout in Q-Alg$_\Sigma$ *iff for every operation $\varphi \in \Omega$, the following description of $\varphi^{\mathbf{H}}$ yields a well defined partial mapping $\varphi^{\mathbf{H}} : H^{w(\varphi)} \to H_{\sigma(\varphi)}$: the domain of $\varphi^{\mathbf{H}}$ is*

$$\operatorname{dom}\varphi^{\mathbf{H}} = \{\tilde{f}(\underline{b}) \mid \underline{b} \in \operatorname{dom}\varphi^{\mathbf{B}'}\} \cup \{\tilde{g}(\underline{a}) \mid \underline{a} \in \operatorname{dom}\varphi^{\mathbf{A}'}\}$$

and if $\underline{b} \in \operatorname{dom}\varphi^{\mathbf{B}'}$ (respectively, $\underline{a} \in \operatorname{dom}\varphi^{\mathbf{A}'}$) then

$$\varphi^{\mathbf{H}}(\tilde{f}(\underline{b})) = \tilde{f}_{\sigma(\varphi)}(\varphi^{\mathbf{B}'}(\underline{b})) \quad (\textit{respectively, } \varphi^{\mathbf{H}}(\tilde{g}(\underline{a})) = \tilde{g}_{\sigma(\varphi)}(\varphi^{\mathbf{A}'}(\underline{a}))).$$

And when it exists, the pushout of f and g in Q-Alg$_\Sigma$ *is given by $\mathbf{H} = (H, (\varphi^{\mathbf{H}})_{\varphi\in\Omega})$, together with the quomorphisms $\tilde{f} : \mathbf{B} \to \mathbf{H}$ and $\tilde{g} : \mathbf{A} \to \mathbf{H}$.*

This proposition can be rephrased as follows.

Proposition 7. *With the notations of the previous proposition, $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ have a pushout in* Q-Alg$_\Sigma$ *iff $\theta(K')$ (see Section 2.3) is a congruence on the coproduct $\mathbf{A}' + \mathbf{B}'$. And when f and g have a pushout, it is $\mathbf{H} = (\mathbf{A}' + \mathbf{B}')/\theta(K')$ together with the quomorphisms $\tilde{f} : \mathbf{B} \to \mathbf{H}$ and $\tilde{g} : \mathbf{A} \to \mathbf{H}$ given respectively by the compositions*

$$\mathbf{B}' \hookrightarrow \mathbf{A}' + \mathbf{B}' \overset{\mathrm{nat}_{\theta(K')}}{\to} (\mathbf{A}' + \mathbf{B}')/\theta(K'), \quad \mathbf{A}' \hookrightarrow \mathbf{A}' + \mathbf{B}' \overset{\mathrm{nat}_{\theta(K')}}{\to} (\mathbf{A}' + \mathbf{B}')/\theta(K')$$

3. Single-pushout transformation in CF-Alg$_\Gamma$

Let Γ denote, in this and the following sections, a *graph structure*: a signature (S, Ω, η) with $\eta(\Omega) \subseteq S \times S$.

Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two conformisms of partial Γ-algebras. In order to compute their pushout in CF-Alg$_\Gamma$, we shall define suitable weak subalgebras $\mathbf{K}'_w$, $\mathbf{A}'_w$ and $\mathbf{B}'_w$ of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported, respectively, on the sets K', A' and B' defined in Section 2.3, in such a way that $f|_{K'}$ and $g|_{K'}$ become closed homomorphisms between them and their pushout in C-Alg$_\Gamma$ turns out to yield the desired pushout of f and g in CF-Alg$_\Gamma$. Such $\mathbf{K}'_w$, $\mathbf{A}'_w$ and $\mathbf{B}'_w$ will actually be the greatest weak subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K', A' and B', respectively, such that $f|_{K'}$ and $g|_{K'}$ are closed homomorphisms between them.

Specifically, let $\mathbf{K}'_w$, $\mathbf{A}'_w$ and $\mathbf{B}'_w$ be the following weak subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$:

– $\mathbf{K}'_w = (K', (\varphi^{\mathbf{K}'_w})_{\varphi\in\Omega})$ is the weak subalgebra of $\mathbf{K}$ supported on K', whose operations $\varphi^{\mathbf{K}'_w}$, for every $\varphi \in \Omega$ with $\eta(\varphi) = (s, s')$, have their domain given by

$$\operatorname{dom}\varphi^{\mathbf{K}'_w} = \left\{x \in K'_s \mid [f_s(x)]_{\theta(K')_s} \subseteq \operatorname{dom}\varphi^{\mathbf{A}+\mathbf{B}} \text{ and } \varphi^{\mathbf{K}}(f_s^{-1}([f_s(x)]_{\theta(K')_s})) \subseteq K'_{s'}\right\}$$

where $\mathbf{A}+\mathbf{B}$ denotes the coproduct (disjoint union) of $\mathbf{A}$ and $\mathbf{B}$, and $\theta(K')$ denotes the equivalence on its universe defined in Section 2.3.

There are some comments we want to make on this definition:

- If $[f_s(x)]_{\theta(K')_s} \subseteq \operatorname{dom}\varphi^{\mathbf{A}+\mathbf{B}}$ then $f_s^{-1}([f_s(x)]_{\theta(K')_s}) \subseteq \operatorname{dom}\varphi^{\mathbf{K}}$ because f is a conformism.
- f_s could be replaced by g_s anywhere in this definition. Indeed, if $x \in K'_s$ then $[f_s(x)]_{\theta(K')_s} = [g_s(x)]_{\theta(K')_s}$ and $f_s^{-1}([f_s(x)]_{\theta(K')_s}) = g_s^{-1}([g_s(x)]_{\theta(K')_s}) \subseteq K'_s$.

– $\mathbf{A}'_w = (A', (\varphi^{\mathbf{A}'_w})_{\varphi\in\Omega})$ is the weak subalgebra of $\mathbf{A}$ supported on A', whose operations $\varphi^{\mathbf{A}'_w}$, for every $\varphi \in \Omega$ with $\eta(\varphi) = (s, s')$, have their domain given by

$$\begin{aligned}\operatorname{dom}\varphi^{\mathbf{A}'_w} = &\left\{a \in A_s - f_s(K_s) \mid a \in \operatorname{dom}\varphi^{\mathbf{A}} \text{ and } \varphi^{\mathbf{A}}(a) \in A'_{s'}\right\}\\ &\cup\{f_s(x) \mid x \in \operatorname{dom}\varphi^{\mathbf{K}'_w}\}\end{aligned}$$

– In a similar way, $\mathbf{B}'_w = (B', (\varphi^{\mathbf{B}'_w})_{\varphi\in\Omega})$ is the weak subalgebra of $\mathbf{B}$ supported on B', whose operations $\varphi^{\mathbf{B}'_w}$, for every $\varphi \in \Omega$ with $\eta(\varphi) = (s, s')$, have their domain given by

$$\begin{aligned}\operatorname{dom}\varphi^{\mathbf{B}'_w} = &\left\{b \in B_s - g_s(K_s) \mid b \in \operatorname{dom}\varphi^{\mathbf{B}} \text{ and } \varphi^{\mathbf{B}}(b) \in B'_{s'}\right\}\\ &\cup\{g_s(x) \mid x \in \operatorname{dom}\varphi^{\mathbf{K}'_w}\}.\end{aligned}$$

Lemma 8. *Both $f|_{K'}$ and $g|_{K'}$ are closed homomorphisms from $\mathbf{K}'_w$ to $\mathbf{A}'_w$ and $\mathbf{B}'_w$, respectively.*

Furthermore, if $\mathbf{K}''$, $\mathbf{A}''$ and $\mathbf{B}''$ are any weak subalgebras of $\mathbf{K}, \mathbf{A}$ and $\mathbf{B}$ supported on K', A' and B', respectively, such that $f|_{K'}$ and $g|_{K'}$ are closed homomorphisms between them , then they are also weak subalgebras of $\mathbf{K}'_w, \mathbf{A}'_w$ and $\mathbf{B}'_w$, respectively.

Proof. As far as the first assertion goes, it is clear from the definitions that $f|_{K'} : \mathbf{K}'_w \to \mathbf{A}'_w$ and $g|_{K'} : \mathbf{K}'_w \to \mathbf{B}'_w$ are homomorphisms. To show that $f|_{K'} : \mathbf{K}'_w \to \mathbf{A}'_w$ is closed, let $\varphi \in \Omega$ be an operation symbol with $\eta(\varphi) = (s, s')$ and let $x \in K'_s$ be an element such that $f_s(x) \in \operatorname{dom}\varphi^{\mathbf{A}'_w}$. Then there exists $x_0 \in \operatorname{dom}\varphi^{\mathbf{K}'_w}$ such that

$f_s(x) = f_s(x_0)$. But then the equality $[f_s(x)]_{\theta(K')_s} = [f_s(x_0)]_{\theta(K')_s}$ implies $x \in \operatorname{dom} \varphi^{\mathbf{K}'_w}$. Therefore, $f|_{K'} : \mathbf{K}'_w \to \mathbf{A}'_w$ is indeed closed. By symmetry, $f|_{K'} : \mathbf{K}'_w \to \mathbf{A}'_w$ is also closed.

As far as the second assertion goes, let $\mathbf{K}''$, $\mathbf{A}''$ and $\mathbf{B}''$ be any weak subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K', A' and B', respectively, such that $f|_{K'} : \mathbf{K}'' \to \mathbf{A}''$ and $g|_{K'} : \mathbf{K}'' \to \mathbf{B}''$ are closed homomorphisms.

Let $\varphi \in \Omega$ be an operation symbol with $\eta(\varphi) = (s, s')$. We have to show that $\operatorname{dom} \varphi^{\mathbf{K}''} \subseteq \operatorname{dom} \varphi^{\mathbf{K}'_w}$, $\operatorname{dom} \varphi^{\mathbf{A}''} \subseteq \operatorname{dom} \varphi^{\mathbf{A}'_w}$ and $\operatorname{dom} \varphi^{\mathbf{B}''} \subseteq \operatorname{dom} \varphi^{\mathbf{B}'_w}$.

If $x \in \operatorname{dom} \varphi^{\mathbf{K}''}$ then $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}''}$, and since $\theta(K')$ is a closed congruence on $\mathbf{A}'' + \mathbf{B}''$ by [9, Lemma 1], we have that $[f_s(x)]_{\theta(K')_s} \subseteq \operatorname{dom} \varphi^{\mathbf{A}''+\mathbf{B}''} \subseteq \operatorname{dom} \varphi^{\mathbf{A}+\mathbf{B}}$. And then, $f|_{K'} : \mathbf{K}'' \to \mathbf{A}''$ being a closed homomorphism, we have that $f_s^{-1}([f_s(x)]_{\theta(K')_s}) \subseteq \operatorname{dom} \varphi^{\mathbf{K}''}$, which implies that $x \in \operatorname{dom} \varphi^{\mathbf{K}'_w}$.

If $a \in \operatorname{dom} \varphi^{\mathbf{A}''}$ then we must distinguish two cases. If $a \in A_s - f_s(K_s)$ then $a \in \operatorname{dom} \varphi^{\mathbf{A}''} \subseteq \operatorname{dom} \varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(a) = \varphi^{\mathbf{A}''}(a) \in A'_{s'}$ imply $a \in \operatorname{dom} \varphi^{\mathbf{A}'_w}$. And if $a = f_s(z)$ with $z \in K'_s$ then $z \in \operatorname{dom} \varphi^{\mathbf{K}''} \subseteq \operatorname{dom} \varphi^{\mathbf{K}'_w}$ because $f|_{K'} : \mathbf{K}'' \to \mathbf{A}''$ is closed, and therefore $a \in \operatorname{dom} \varphi^{\mathbf{A}'_w}$. This proves that $\operatorname{dom} \varphi^{\mathbf{A}''} \subseteq \operatorname{dom} \varphi^{\mathbf{A}'_w}$; a similar argument proves that $\operatorname{dom} \varphi^{\mathbf{B}''} \subseteq \operatorname{dom} \varphi^{\mathbf{B}'_w}$. □

Let

$$\begin{array}{ccc} \mathbf{K}'_w & \xrightarrow{f|_{K'}} & \mathbf{A}'_w \\ {\scriptstyle g|_{K'}}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B}'_w & \xrightarrow[\tilde{f}]{} & \mathbf{H} \end{array}$$

be the pushout of $f|_{K'} : \mathbf{K}'_w \to \mathbf{A}'_w$ and $g|_{K'} : \mathbf{K}'_w \to \mathbf{B}'_w$ in C-Alg_Γ described in [7, Proposition 19 and Lemma 20]. That is, $\mathbf{H} = (\mathbf{A}'_w + \mathbf{B}'_w)/\theta(K')$ and the underlying commutative square of mappings of S-sets is the usual pushout of $f|_{K'}$ and $g|_{K'}$ in S-**Set**, described in Section 2.3. Recall moreover from [7] that the pushout of two closed homomorphisms in C-Alg_Γ is also their pushout in Alg_Γ.

Let $\tilde{g} : \mathbf{A} \to \mathbf{H}$ and $\tilde{f} : \mathbf{B} \to \mathbf{H}$ also denote the conformisms corresponding to the closed homomorphisms $\tilde{g} : \mathbf{A}'_w \to \mathbf{H}$ and $\tilde{f} : \mathbf{B}'_w \to \mathbf{H}$.

Now we have the following result.

Proposition 9. *The commutative square*

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B} & \xrightarrow[\tilde{f}]{} & \mathbf{H} \end{array}$$

is a pushout square in CF-Alg_Γ.

Proof. Let $\mathbf{C} = (C, (\varphi^{\mathbf{C}})_{\varphi \in \Omega})$ be a partial Γ-algebra, and let $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two conformisms of partial Γ-algebras such that $p \circ f = q \circ g$. We know from

Proposition 4 that there exists a unique partial mapping $h : H \to C$ such that $h \circ \tilde{g} = p$ and $h \circ \tilde{f} = q$. It is enough to check that h is a conformism.

Let $\varphi \in \Omega$ be any operation symbol with $\eta(\varphi) = (s, s')$, and let $x \in \operatorname{Dom} h_s$ such that $h_s(x) \in \operatorname{dom} \varphi^{\mathbf{C}}$. Since $\tilde{f}$ and $\tilde{g}$ are jointly surjective, we can assume without any loss of generality that $x = \tilde{g}_s(a)$ for some $a \in A'_s$. Then, $a \in \operatorname{Dom} h_s \circ \tilde{g}_s = \operatorname{Dom} p_s$ and $p_s(a) = h_s(x) \in \operatorname{dom} \varphi^{\mathbf{C}}$ and therefore, p being a conformism, $a \in \operatorname{dom} \varphi^{\mathbf{A}}$, $\varphi^{\mathbf{A}}(a) \in \operatorname{Dom} p_{s'}$ and $p_{s'}(\varphi^{\mathbf{A}}(a)) = \varphi^{\mathbf{C}}(p_s(a)) = \varphi^{\mathbf{C}}(h_s(x))$.

There are two possibilities for such $a \in A'_s$:

- If $a \in A_s - f_s(K_s)$ then $a \in \operatorname{dom} \varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(a) \in \operatorname{Dom} p_{s'} \subseteq \operatorname{Dom} \tilde{g}_{s'} = A'_{s'}$ imply that $a \in \operatorname{dom} \varphi^{\mathbf{A}'_w}$.
- If $a \in f_s(K'_s)$, say $a = f_s(z)$ with $z \in K'_s$, we shall prove that $z \in \operatorname{dom} \varphi^{\mathbf{K}'_w}$, which will imply, by the definition of $\mathbf{A}'_w$, that $a \in \operatorname{dom} \varphi^{\mathbf{A}'_w}$.

To do that, we have to check that $[f_s(z)]_{\theta(K')_s} \subseteq \operatorname{dom} \varphi^{\mathbf{A+B}}$ and

$$\varphi^{\mathbf{K}}(f_s^{-1}([f_s(z)]_{\theta(K')_s})) \subseteq K'_{s'}.$$

Let $f_s(z') \in [f_s(z)]_{\theta(K')_s}$. Then there exist $x_0, \ldots, x_{2n} \in K'_s$ such that $f_s(z) = f_s(x_0)$, $g_s(x_{2k-2}) = g_s(x_{2k-1})$ and $f_s(x_{2k-1}) = f_s(x_{2k})$ for every $k = 1, \ldots, n$, and finally $f_s(x_{2n}) = f_s(z')$. A simple argument by induction on k shows then that $p_s f_s(x_{2k}) = p_s f_s(z) = p_s(a)$ for every $k = 1, \ldots, n$, and finally that $p_s f_s(z') = p_s(a)$. Since $p_s(a) \in \operatorname{dom} \varphi^{\mathbf{C}}$, this implies that $f_s(z') \in \operatorname{dom} \varphi^{\mathbf{A}}$ (because p is a conformism) and

$$\varphi^{\mathbf{K}}(z') \in \operatorname{Dom} p_{s'} \circ f_{s'} = f_{s'}^{-1}(\operatorname{Dom} p_{s'}) \subseteq f_{s'}^{-1}(A'_{s'}) = K'_{s'}$$

(because $p \circ f$ is a conformism).

A similar argument shows that if $g_s(z') \in [f_s(z)]_{\theta(K')_s}$ then $g_s(z') \in \operatorname{dom} \varphi^{\mathbf{B}}$ and $\varphi^{\mathbf{K}}(z') \in K'_{s'}$. This finishes the proof of $z \in \operatorname{dom} \varphi^{\mathbf{K}'_w}$.

Therefore $a \in \operatorname{dom} \varphi^{\mathbf{A}'_w}$, and then $x = \tilde{g}_s(a) \in \operatorname{dom} \varphi^{\mathbf{H}}$, $\varphi^{\mathbf{H}}(x) = \tilde{g}_{s'}(\varphi^{\mathbf{A}}(a)) \in \operatorname{Dom} h_{s'}$ (because $\varphi^{\mathbf{A}}(a) \in \operatorname{Dom} p_{s'} = \operatorname{Dom} h_{s'} \circ \tilde{g}_{s'}$) and

$$h_{s'}(\varphi^{\mathbf{H}}(x)) = h_{s'}(\tilde{g}_{s'}(\varphi^{\mathbf{A}}(a))) = p_{s'}(\varphi^{\mathbf{A}}(a)) = \varphi^{\mathbf{C}}(h_s(x)).$$

The symbol φ standing for any operation symbol in the signature, this shows that h is a conformism. □

From this proposition we deduce that the derived partial Γ-algebra $\mathbf{H}$ of a partial Γ-algebra $\mathbf{G}$ by the application of a production rule $r : \mathbf{L} \to \mathbf{R}$ in CF-Alg$_\Gamma$ (that is, a conformism of partial Γ-algebras) through a total conformism $m : \mathbf{L} \to \mathbf{G}$ can be obtained (up to isomorphism) as follows:

(i) Compute the *gluing set* L' of $r : L \to R$ and $m : L \to G$ (Definition 2).

(ii) *Remove* from R and G the elements of $r(L) - r(L')$ and $m(L) - m(L')$, respectively. Define on the resulting subsets of R and G the weak subalgebras $\mathbf{R}'_w$ and $\mathbf{G}'_w$ of $\mathbf{R}$ and $\mathbf{G}$, respectively, described above.

(iii) *Add* $\mathbf{R}'_w$ to $\mathbf{G}'_w$, by first forming their coproduct (disjoint union) $\mathbf{G}'_w + \mathbf{R}'_w$ and then identifying in this coproduct all images of elements of L' by r and m (by means of the least equivalence relation containing all pairs $(r_s(z), m_s(z))$, for every $s \in S$ and $z \in L'_s$).

In the sequel we give some examples of application of single-pushout transformation using conformisms, to show some features of this approach as well as its differences with the classical approach to single-pushout transformation using partial homomorphisms of total algebras.

Example 10. Let Γ be a graph structure, let $\mathbf{L}$ be any partial Γ-algebra, let $\mathbf{R}$ be a weak subalgebra of $\mathbf{L}$, with universe R, and let $r : \mathbf{L} \to \mathbf{R}$ be the conformism given by the set-theoretical identity on R. This is, in general, a non-quomorphic conformism.

Let now $m : \mathbf{L} \to \mathbf{G}$ be any total conformism. To compute the derived partial algebra of $\mathbf{G}$ by the application of $r : \mathbf{L} \to \mathbf{R}$ through $m : \mathbf{L} \to \mathbf{G}$, we must find first the sets L', R' and G'. To do that, notice first that $\theta(L)_s$ only identifies on the one hand $x \in R_s$ with $m_s(x) \in G_s$, and on the other hand $x \in R_s$ with $x' \in R_s$ such that $m_s(x) = m_s(x')$. Therefore, for every $s \in S$:

- $L'_s = \{x \in L_s \mid m_s^{-1}(m_s(x)) \subseteq R_s\}$,
- $R'_s = L'_s$,
- $G'_s = \{y \in G_s \mid m_s^{-1}(y) \subseteq R_s\}$

(since $(G_s - m_s(L_s)) \cup m_s(L'_s) = \{y \in G_s \mid m_s^{-1}(y) = \emptyset\} \cup \{y \in G_s \mid m_s^{-1}(y) \neq \emptyset, m_s^{-1}(y) \subseteq R_s\}$).

Now the weak subalgebras $\mathbf{L}'_w$, $\mathbf{R}'_w$ and $\mathbf{G}'_w$ of $\mathbf{L}$, $\mathbf{R}$ and $\mathbf{G}$, respectively, have as universes these sets L', R' and G', and the domains of their operations are given as follows. For every $\varphi \in \Omega$, say with $\eta(\varphi) = (s, s')$:

- $x \in \operatorname{dom} \varphi^{\mathbf{L}'_w}$ iff $m_s(x) \in \operatorname{dom} \varphi^{\mathbf{G}}$, $m_s^{-1}(m_s(x)) \subseteq \operatorname{dom} \varphi^{\mathbf{R}}$ and $m_{s'}^{-1}(\varphi^{\mathbf{G}}(m_s(x))) \subseteq R_{s'}$.
- $x \in \operatorname{dom} \varphi^{\mathbf{R}'_w}$ iff $x \in \operatorname{dom} \varphi^{\mathbf{L}'_w}$ (that is, $\mathbf{R}'_w = \mathbf{L}'_w$, and r becomes the identity between these two algebras).
- $y \in \operatorname{dom} \varphi^{\mathbf{G}'_w}$ iff either $y \notin m_s(L'_s)$, $y \in \operatorname{dom} \varphi^{\mathbf{G}}$ and $\varphi^{\mathbf{G}}(y) \in G'$, or $y = m_s(x)$ with $x \in \operatorname{dom} \varphi^{\mathbf{L}'_w}$; that is, iff $y \in \operatorname{dom} \varphi^{\mathbf{G}}$, $m_s^{-1}(y) \subseteq \operatorname{dom} \varphi^{\mathbf{R}}$, and $m_{s'}^{-1}(\varphi^{\mathbf{G}}(y)) \subseteq R_{s'}$.

And then the pushout object of r and m is obtained as the pushout in C-Alg$_\Gamma$ of the identity $\mathbf{L}'_w \to \mathbf{R}'_w = \mathbf{L}'_w$ and the restriction $m|_{L'} : \mathbf{L}'_w \to \mathbf{G}'_w$. It is $\mathbf{G}'_w$ itself.

So, the derived partial Γ-algebra of $\mathbf{G}$ by the application of r through $m : \mathbf{L} \to \mathbf{G}$ in CF-Alg$_\Gamma$ is obtained from $\mathbf{G}$ as follows:

- We first remove from G all points whose preimage is not contained in R;
- Then we remove all operations (on the remaining points) which are not defined in $\mathbf{R}$ on all preimages of their argument.

In particular, if $R = L$ (that is, if $\mathbf{R}$ is a weak subalgebra of $\mathbf{L}$ with its same universe) then this derived partial Γ-algebra is obtained by removing from $\mathbf{G}$ those operations that are not defined in $\mathbf{R}$ on all preimages of their argument.

Example 11. Let Γ be a graph structure with two operations, φ and ϕ. Suppose we want to perform on partial Γ-algebras such a simple operation as removing all end φ-loops (that is, all φ-loops on points where ϕ is not defined) in them without touching anything else, and in particular without deleting the point where the loop is defined.

In CF-Alg$_\Gamma$ it can be done by simply applying the single-pushout rule $r : \mathbf{L} \to \mathbf{R}$ where $\mathbf{L}$ is a Γ-algebra with a single point, say $L = \{a\}$, and a loop $\varphi^{\mathbf{L}}(a) = a$, $\mathbf{R}$ is a discrete Γ-algebra supported on $\{a\}$ and r is given by $r(a) = a$.

Indeed, if there exists some total conformism $m : \mathbf{L} \to \mathbf{G}$ to some Γ-algebra $\mathbf{G}$, then either $m(a) \notin \operatorname{dom} \varphi^{\mathbf{G}}$ or $m(a) \in \operatorname{dom} \varphi^{\mathbf{G}}$ and $\varphi^{\mathbf{G}}(m(a)) = m(a)$. In the first case, the application of r to $\mathbf{G}$ through m produces $\mathbf{G}$ again, without any change,[8] while in the second case the application of r to $\mathbf{G}$ through m produces the weak subalgebra $\mathbf{H}$ of $\mathbf{G}$ with its same universe and with $\operatorname{dom} \varphi^{\mathbf{H}} = \operatorname{dom} \varphi^{\mathbf{G}} - \{m(a)\}$.

Then the rule $r : \mathbf{L} \to \mathbf{R}$, applied to a Γ-algebra $\mathbf{A}$, removes a φ-loop in a point where $\phi^{\mathbf{A}}$ is not defined, without touching anything else. So, successive applications of r remove from a Γ-algebra its end φ-loops, and only these ones.

One could also try to do this by first encoding partial Γ-algebras by means of total algebras, in any of the ways recalled in Section 2.2, and then using single-pushout transformation of total algebras in the sense of [15].

A first attempt could be to encode partial Γ-algebras $\mathbf{A}$ by means of the corresponding total $\tilde{\Gamma}$-algebras $\tilde{\mathbf{A}}$, as explained in Section 2.2. That is, one could try to remove all end φ-loops in a partial Γ-algebra $\mathbf{A}$ by using single-pushout transformation of total unary algebras on $\tilde{\mathbf{A}}$. However, it is quite evident that, since a single-pushout transformation rule must change the sort of an element, if one wants that the rest of the structure remains unchanged, then one needs one rule for every possible *context*: one needs an infinite set of rules that essentially replace a whole connected component of an algebra by the desired result.

One can also encode partial Γ-algebras as total $\hat{\Gamma}$-algebras, as also explained in Section 2.2. But then the application of the corresponding rule $r : \hat{\mathbf{L}} \to \hat{\mathbf{R}}$ of total $\hat{\Gamma}$-algebras to the $\hat{\Gamma}$-algebra $\hat{\mathbf{A}}$ corresponding to a Γ-algebra $\mathbf{A}$ can remove *any* φ-loop, and not only an end one. And, indeed, it is impossible to remove only the end φ-loops without application conditions (in this case, that the image of the point be not a source node of a ϕ-arc).

Let us consider now a more involved example.

Example 12. In this example we show how to use single-pushout transformation based on conformisms to produce the hypergraph underlying a hierarchic higher-order hypergraph [1].

Let $\Sigma_H = (\{V, E\}, \Omega_H, \eta)$ be a graph structure with

$$\Omega_H = \{s_i \mid i \in \mathbb{Z}^+\} \cup \{t_i \mid i \in \mathbb{Z}^+\} \cup \{a_i \mid i \in \mathbb{Z}^+\}$$

[8] If one wants to get rid of this possibility, it is enough to impose derivations to be made through closed homomorphisms, instead of through general total conformisms.

and the arity function η defined by $\eta(s_i) = \eta(t_i) = (E, V)$ and $\eta(a_i) = (E,E)$, $i \in \mathbb{Z}^+$. Recall from [7, Example 3] that a *higher-order hypergraph* can be understood then as a finite partial Σ_H-algebra $\mathbf{G}$ satisfying the following conditions:

- $\operatorname{dom} s_{i+1}^{\mathbf{G}} \subseteq \operatorname{dom} s_i^{\mathbf{G}}$, $\operatorname{dom} t_{i+1}^{\mathbf{G}} \subseteq \operatorname{dom} t_i^{\mathbf{G}}$ and $\operatorname{dom} a_{i+1}^{\mathbf{G}} \subseteq \operatorname{dom} a_i^{\mathbf{G}}$, for all $i \in \mathbb{Z}^+$,
- there exists an $n(\mathbf{G}) \in \mathbb{Z}^+$ such that $\operatorname{dom} s_{n(\mathbf{G})}^{\mathbf{G}} = \operatorname{dom} t_{n(\mathbf{G})}^{\mathbf{G}} = \operatorname{dom} a_{n(\mathbf{G})}^{\mathbf{G}} = \emptyset$.

The elements of G_V and G_E are called, respectively, the *nodes* and the *arcs* of $\mathbf{G}$, and the operations $s_i^{\mathbf{G}}$, $t_i^{\mathbf{G}}$ and $a_i^{\mathbf{G}}$ are called, respectively, the ith *source*, *target* and *abstraction* operations on $\mathbf{G}$.

For every $e \in G_E$, we shall denote by $n_s(e)$ (respectively, $n_t(e)$, $n_a(e)$) the greatest $i \geqslant 1$ such that $e \in \operatorname{dom} s_i^{\mathbf{G}}$ (respectively, $e \in \operatorname{dom} t_i^{\mathbf{G}}$, $e \in \operatorname{dom} a_i^{\mathbf{G}}$) or 0 if $e \notin \operatorname{dom} s_1^{\mathbf{G}}$ (respectively, $e \notin \operatorname{dom} t_1^{\mathbf{G}}$, $e \notin \operatorname{dom} a_1^{\mathbf{G}}$).

We shall say that such a higher-order hypergraph $\mathbf{G}$ is *hierarchic* when there is no sequence of abstraction operations $a_{i_1}^{\mathbf{G}}, \ldots, a_{i_n}^{\mathbf{G}}$, $n \geqslant 1$, such that

$$a_{i_1}^{\mathbf{G}}(\ldots(a_{i_n}^{\mathbf{G}}(e))\ldots) = e$$

for some arc $e \in G_E$ (that is, such that $a_{i_1}^{\mathbf{G}}(\ldots(a_{i_n}^{\mathbf{G}}(e))\ldots)$ is defined and yields the same e). For instance, the higher-order hypergraphs involved in the methods of knowledge base verification using higher-order hypergraph transformation developed in [25, 26] are always hierarchic.

We show in the sequel how to compute the hypergraph *underlying* a hierarchic higher-order hypergraph (that is, its weak subalgebra with the same sets of nodes and arcs and the same source and target operations, but no abstraction operation defined in it) by means of the application of suitable production rules in CF-Alg_{Σ_H}.

For every $n \geqslant 1$ let $\mathbf{G}(n)$ be the partial Σ_H-algebra with $G(n)_E = \{e, e_1, \ldots, e_n\}$ and

$$G(n)_V = \{v_j \mid 1 \leqslant j \leqslant 2n\} \cup \{w_{i,j} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant 2n\}$$

and with operations defined as follows:

$$a_j^{\mathbf{G}(n)}(e) = e_j, \quad s_j^{\mathbf{G}(n)}(e) = v_j, \quad t_j^{\mathbf{G}(n)}(e) = v_{n+j}, \qquad j = 1, \ldots, n,$$
$$s_j^{\mathbf{G}(n)}(e_i) = w_{i,j}, \quad t_j^{\mathbf{G}(n)}(e_i) = w_{i,n+j}, \qquad i,j = 1, \ldots, n$$

and let $\tilde{\mathbf{G}}(n)$ be the weak subalgebra of $\mathbf{G}(n)$ with the same carrier and the same source and target operations, but all abstraction operations discrete. Let $r_n : \mathbf{G}(n) \to \tilde{\mathbf{G}}(n)$ be the conformism given by the identity mapping.

Let now $\mathbf{G}$ be a higher-order hypergraph and let $m : \mathbf{G}(n) \to \mathbf{G}$ be a total conformism. Let $m_E(e) = e'$, $m_E(e_i) = e_i'$, $m_V(v_j) = v_j'$ and $m_V(w_{i,j}) = w_{i,j}'$ ($i = 1, \ldots, n$, $j = 1, \ldots, 2n$). Since m is a conformism, the following conditions are satisfied:

- If $e' \in \operatorname{dom} a_i^{\mathbf{G}}$ then $i \leqslant n$ and $a_i^{\mathbf{G}}(e') = e_i'$.
- $e_i' \notin \operatorname{dom} a_1^{\mathbf{G}}$, $i = 1, \ldots, n$.
- If $e' \in \operatorname{dom} s_j^{\mathbf{G}}$ then $j \leqslant n$ and $s_j^{\mathbf{G}}(e') = v_j'$. Let $n_s = n_s(e') \leqslant n$.
- If $e_i' \in \operatorname{dom} s_j^{\mathbf{G}}$ then $j \leqslant n$ and $s_j^{\mathbf{G}}(e_i') = w_{i,j}'$. Let $n_{s,i} = n_s(e_i') \leqslant n$.
- If $e' \in \operatorname{dom} t_j^{\mathbf{G}}$ then $j \leqslant n$ and $t_j^{\mathbf{G}}(e') = v_{n+j}'$. Let $n_t = n_t(e') \leqslant n$.
- If $e_i' \in \operatorname{dom} t_j^{\mathbf{G}}$ then $j \leqslant n$ and $t_j^{\mathbf{G}}(e_i') = w_{i,n+j}'$. Let $n_{t,i} = n_t(e_i') \leqslant n$.

The pushout object of $r_n : \mathbf{G}(n) \to \tilde{\mathbf{G}}(n)$ and $m : \mathbf{G}(n) \to \mathbf{G}$ in CF-Alg$_{\Sigma_H}$ is then given (by Example 10) by the weak subalgebra $\mathbf{G}_w$ of $\mathbf{G}$ obtained by deleting from $\mathbf{G}$ all abstraction operations defined on $m_E(e)$ and not touching anything else.

Let now $\mathbf{G}$ be a hierarchic higher-order hypergraph with $G_V, G_E \neq \emptyset$, and let n_0 be an upper bound for $\{n_s(e), n_t(e), n_a(e) \mid e \in G_E\}$. The argument developed above shows that successive applications of the production rule $r_{n_0} : \mathbf{G}(n_0) \to \tilde{\mathbf{G}}(n_0)$ through total conformisms end up by deleting all abstraction operations in $\mathbf{G}$ without modifying either its carrier or its source and target operations. In other words, they end up by computing the hypergraph underlying $\mathbf{G}$.

We need the condition $G_V, G_E \neq \emptyset$ to guarantee the existence of some total conformism $\mathbf{G}(n_0) \to \mathbf{G}$. If $G_E = \emptyset$ then it is clear that nothing must be done: $\mathbf{G}$ itself is the hypergraph we are looking for. If $G_E \neq \emptyset$ but $G_V = \emptyset$ then, we need a different set of rules.

Let $\mathbf{G}^0(n)$ be the relative subalgebra of $\mathbf{G}(n)$ with its same set of arcs, but no node (that is, with $G^0(n)_V = \emptyset$ and $G^0(n)_E = \{e, e_1, \ldots, e_n\}$), let $\tilde{\mathbf{G}}^0(n)$ its discrete weak subalgebra, and let $r_n^0 : \mathbf{G}^0(n) \to \tilde{\mathbf{G}}^0(n)$ be the conformism given by the identity mapping. The pushout object of $r_n^0 : \mathbf{G}^0(n) \to \tilde{\mathbf{G}}^0(n)$ and a total conformism $m : \mathbf{G}^0(n) \to \mathbf{G}$ in CF-Alg$_{\Sigma_H}$ is again given by the weak subalgebra $\mathbf{G}_w$ of $\mathbf{G}$ obtained by deleting from $\mathbf{G}$ all abstraction operations defined on $m_E(e)$ and not touching anything else (notice in this case that the existence of the total conformism m implies that no source or target operation is defined on any image $m_E(e)$ or $m_E(e_i)$, $i = 1, \ldots, n$).

Therefore, if $\mathbf{G}$ is a non-empty hierarchic higher-order hypergraph with $G_V = \emptyset$, and if n_0 is an upper bound for $\{n_a(e) \mid e \in G_E\}$, then successive applications of the production rule $r_{n_0}^0 : \mathbf{G}^0(n_0) \to \tilde{\mathbf{G}}^0(n_0)$ through total conformisms end up by deleting all abstraction operations in $\mathbf{G}$ without modifying its carrier.

If $\mathbf{G}$ is not hierarchic then such a procedure does not delete all abstraction operations, neither when $G_V = \emptyset$ nor when $G_V \neq \emptyset$. Actually, it may happen that there does not exist any total conformism $m : \mathbf{G}(n) \to \mathbf{G}$ or $m : \mathbf{G}^0(n) \to \mathbf{G}$ and therefore that we could never apply any rule r_n or r_n^0 to $\mathbf{G}$.

It is worth mentioning that one could also try to delete all abstraction operations in higher-order hypergraphs by understanding them as (special) total unary algebras over the corresponding signature $\tilde{\Sigma}_H$ (see Section 2.2; notice that in this case the corresponding set of sorts turns out to be non-countable), and then using single-pushout transformation of total unary algebras. But an argument similar to the one used in Example 11 when discussing the encoding of partial Γ-algebras by means of total $\tilde{\Gamma}$-algebras, shows that we would need here a non-countably infinite number of rules to take care of all possibilities, as well as application conditions.

The description of the pushout of two conformisms as the pushout of two closed homomorphisms given in Proposition 9, similar in spirit to the description of the pushout of two partial homomorphisms in P-TAlg$_\Gamma$ given in [15], motivates to investigate whether there is some relation between single-pushout transformation in CF-Alg$_\Gamma$ and double-pushout transformation in Alg$_\Gamma$, as it is the case for single-pushout transformation in P-TAlg$_\Gamma$ [15, Section 3.3] and double-pushout transformation of total unary algebras. This relation should come through the following notion of a (double-pushout) production rule in Alg$_\Gamma$ associated to a (single-pushout) production rule in CF-Alg$_\Gamma$ (cf. [15, Section 3.3]).

Definition 13. Given a conformism of partial Γ-algebras $r : \mathbf{L} \to \mathbf{R}$, let $\mathbf{L}_r$ be the weak subalgebra of $\mathbf{L}$ supported on $\mathrm{Dom}\, r$ such that r is a closed homomorphism from it, let $i : \mathbf{L}_r \to \mathbf{L}$ be the corresponding embedding and let $r : \mathbf{L}_r \to \mathbf{R}$ denote also the corresponding closed homomorphism. Then the production rule $P(r)$ associated to r is the production rule in Alg$_\Gamma$

$$P(r) = (\mathbf{L} \xleftarrow{i} \mathbf{L}_r \xrightarrow{r} \mathbf{R}).$$

The question we ask is, then, whether or not the derived partial algebra of $\mathbf{G}$ by the production rule r in CF-Alg$_\Gamma$ and by the production rule $P(r)$ in Alg$_\Gamma$, through a total conformism that is also a homomorphism, that is, a closed homomorphism, $m : \mathbf{L} \to \mathbf{G}$ satisfying the gluing condition [7, Definition 8] w.r.t. i, are the same (up to isomorphism). The answer is, in general, no, as the following example shows.

Example 14. Let Γ be a monounary graph structure. Let $\mathbf{L}$ be a Γ-algebra with $L = \{a,b\}$ and $\varphi^{\mathbf{L}}(a) = \varphi^{\mathbf{L}}(b) = b$, and let $\mathbf{R}$ be the discrete relative subalgebra of $\mathbf{L}$ supported on $R = \{a\}$. Let $r : \mathbf{L} \to \mathbf{R}$ be the conformism (actually, a c-quomorphism) given by the identity on R.

Let now $\mathbf{G}$ be equal to $\mathbf{L}$, and let $m : \mathbf{L} \to \mathbf{G}$ be the identity mapping, which is a closed homomorphism. A simple category-theoretical argument (or also Example 10) shows that the derived partial Γ-algebra of $\mathbf{G}$ by the application of r through m is isomorphic to $\mathbf{R}$.

On the other hand, in this case the production rule $P(r)$ would be $(\mathbf{L} \xleftarrow{i} \mathbf{L}_r \xrightarrow{r} \mathbf{R})$, with $\mathbf{L}_r$ the relative (discrete) subalgebra of $\mathbf{L}$ supported on $\{a\}$. It is clear that m satisfies the identification, dangling and relative closedness conditions in [7, Definition 8] w.r.t. i (for instance, because $\mathbf{L} = C_{\mathbf{L}}(\{a\})$). And a simple computation shows that the derived partial Γ-algebra of $\mathbf{G}$ through g by the application of rule $P(r)$ in Alg$_\Gamma$ [7, Defintion 15] is isomorphic to $\mathbf{G}$, and therefore clearly non-isomorphic to $\mathbf{R}$.

Another example is given by the c-quomorphism $r : \mathbf{L} \to \mathbf{R}$ and the closed homomorphism $g : \mathbf{L} \to \mathbf{G}'$, with $\mathbf{G}'$ a total Γ-algebra with carrier $G' = \{a'\}$ and $\varphi^{\mathbf{G}'}(a') = a'$, given by $g(a) = g(b) = a'$; again, g satisfies the gluing condition w.r.t. i. In this case, it is easy to check (using Example 10) that the derived partial Γ-algebra of $\mathbf{G}'$ by the application of r through g is empty, while there is no double-pushout diagram

$$\begin{array}{ccccc}
\mathbf{L} & \xleftarrow{i} & \mathbf{L}_r & \xrightarrow{r} & \mathbf{R} \\
\downarrow g & & \downarrow g' & & \downarrow g'' \\
\mathbf{G}' & \xleftarrow[i']{} & \mathbf{D} & \xrightarrow[r']{} & \emptyset
\end{array}$$

yielding an empty algebra as an algebra derived from $\mathbf{G}'$ by the application of $P(r)$ through g.

This example also shows that, in general, the derivation of a partial Γ-algebra $\mathbf{G}$ by the application of a rule $r : \mathbf{L} \to \mathbf{R}$ in CF-Alg$_\Gamma$ through a closed homomorphism $m : \mathbf{L} \to \mathbf{G}$ fits better into our intuitive expectations than the corresponding derivation by the application of $P(r)$ in Alg$_\Gamma$. For instance, it is clear from the intuitive notion of transformation that the application of a rule through the identity on its left-hand side object should give its right-hand side object. And it happens in this way with the application of a single-pushout production rule $r : \mathbf{L} \to \mathbf{R}$ in CF-Alg$_\Gamma$, because (by category theoretical reasons, but also by Proposition 9) the pushout algebra of a conformism $r : \mathbf{L} \to \mathbf{R}$ and the identity $\mathrm{Id}_{\mathbf{L}} : \mathbf{L} \to \mathbf{L}$ is $\mathbf{R}$. But it does not happen in general in this way with the application of the double-pushout production rule $P(r) = (\mathbf{L} \xleftarrow{i} \mathbf{L}_r \xrightarrow{r} \mathbf{R})$ in Alg$_\Gamma$, because in this case the algebra derived from $\mathbf{G}$ by the application of $P(r)$ through $\mathrm{Id}_{\mathbf{L}} : \mathbf{L} \to \mathbf{L}$ (which always satisfies the gluing condition w.r.t. $i : \mathbf{L}_r \to \mathbf{L}$) is, according to [7, Definition 15], the pushout algebra of $r : \mathbf{L}_r \to \mathbf{R}$ and $i : \mathbf{L}_r \to \mathbf{C}_{\mathbf{L}}(L_r)$ (the inclusion of $\mathbf{L}_r$ in the closed subalgebra generated by it), and it need not be isomorphic to $\mathbf{R}$ (as it shows the first case in the last example).

To close this section, we show that the pushout of two total conformisms in CF-Alg$_\Gamma$ is also their pushout in TCF-Alg$_\Gamma$. We shall use it in Section 6.

Proposition 15. *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two total conformisms of partial Γ-algebras , and let*

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B} & \xrightarrow[\tilde{f}]{} & \mathbf{H} \end{array}$$

be their pushout in CF-Alg$_\Gamma$ *described in Proposition 9. Then it is also their pushout in* TCF-Alg$_\Gamma$.

Proof. If f and g are total conformisms then $\tilde{f}$ and $\tilde{g}$ are also total conformisms by Corollary 5(iii). And if $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ are two total conformisms such that $p \circ f = q \circ g$ then the unique conformism $h : \mathbf{H} \to \mathbf{C}$ such that $h \circ \tilde{f} = p$ and $h \circ \tilde{g} = q$ is total because $\tilde{f}$ and $\tilde{g}$ are jointly surjective by Corollary 5(i). □

4. Single-pushout transformation in CQ-Alg$_\Gamma$

Let Γ be a graph structure and let now $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two c-quomorphisms of partial Γ-algebras. In this section we prove that their pushout in CF-Alg$_\Gamma$ is also their pushout in CQ-Alg$_\Gamma$. We shall freely use the notations introduced in the previous sections.

Let $\mathbf{K}'$, $\mathbf{A}'$ and $\mathbf{B}'$ be the relative subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported respectively on the sets K', A' and B' defined in Section 2.3.

Lemma 16. $\mathbf{K}' = \mathbf{K}'_w$, $\mathbf{A}' = \mathbf{A}'_w$ *and* $\mathbf{B}' = \mathbf{B}'_w$.

Proof. By Lemma 8, and since $\mathbf{K}'_w$, $\mathbf{A}'_w$ and $\mathbf{B}'_w$ are weak subalgebras of $\mathbf{K}'$, $\mathbf{A}'$ and $\mathbf{B}'$, respectively, it is enough to check that $f|_{K'} : \mathbf{K}' \to \mathbf{A}'$ and $g|_{K'} : \mathbf{K}' \to \mathbf{B}'$ are closed homomorphisms.

Let $\varphi \in \Omega$ be an operation symbol with $\eta(\varphi) = (s, s')$. If $x \in \operatorname{dom} \varphi^{\mathbf{K}'}$ then $x \in (\operatorname{Dom} f_s) \cap \operatorname{dom} \varphi^{\mathbf{K}}$ and $\varphi^{\mathbf{K}}(x) \in K'_{s'} \subseteq \operatorname{Dom} f_{s'}$. f being a quomorphism, this implies that $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}} \cap A'_s$ and $\varphi^{\mathbf{A}}(f_s(x)) = f_{s'}(\varphi^{\mathbf{K}}(x)) \in f_{s'}(K'_{s'}) \subseteq A'_{s'}$, that is, $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}'}$ and $\varphi^{\mathbf{A}'}(f_s(x)) = f_{s'}(\varphi^{\mathbf{K}}(x))$.

Now let $x \in K'_s$ such that $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}'}$. Then $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}}$ and $\varphi^{\mathbf{A}}(f_s(x)) \in A'_{s'}$. Since $f : \mathbf{K} \to \mathbf{A}$ is a c-quomorphism, $f_s(x) \in \operatorname{dom} \varphi^{\mathbf{A}}$ implies that $x \in \operatorname{dom} \varphi^{\mathbf{K}}$, $\varphi^{\mathbf{K}}(x) \in \operatorname{Dom} f_{s'}$ and $f_{s'}(\varphi^{\mathbf{K}}(x)) = \varphi^{\mathbf{A}}(f_s(x))$. But since the latter belongs to $A'_{s'} \cap f_{s'}(K_{s'}) = f_{s'}(K'_{s'})$, we deduce from Lemma 3 that $\varphi^{\mathbf{K}}(x) \in K'_{s'}$ and therefore that $x \in \operatorname{dom} \varphi^{\mathbf{K}'}$.

Since φ stands for any operation symbol in the signature, this shows that $f|_{K'} : \mathbf{K}' \to \mathbf{A}'$ is a closed homomorphism. By symmetry, $g|_{K'} : \mathbf{K}' \to \mathbf{B}'$ is also a closed homomorphism. $\square$

Let now

$$\begin{array}{ccc} \mathbf{K}' & \xrightarrow{f|_{K'}} & \mathbf{A}' \\ {\scriptstyle g|_{K'}}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B}' & \xrightarrow[\tilde{f}]{} & \mathbf{H} \end{array}$$

be the pushout of $f|_{K'} : \mathbf{K}' \to \mathbf{A}'$ and $g|_{K'} : \mathbf{K}' \to \mathbf{B}'$ in C-Alg$_\Gamma$. As in the case of conformisms, let $\tilde{g} : \mathbf{A} \to \mathbf{H}$ and $\tilde{f} : \mathbf{B} \to \mathbf{H}$ also denote the c-quomorphisms corresponding to (the closed homomorphisms) $\tilde{g}$ and $\tilde{f}$, respectively.

In this way, $\mathbf{H}$ together with $\tilde{g} : \mathbf{A} \to \mathbf{H}$ and $\tilde{f} : \mathbf{B} \to \mathbf{H}$, is the pushout of f and g in CF-Alg$_\Gamma$ by Lemma 16 and Proposition 9. And it turns out that it is also their pushout in CQ-Alg$_\Gamma$.

Proposition 17. *Let* $f : \mathbf{K} \to \mathbf{A}$ *and* $g : \mathbf{K} \to \mathbf{B}$ *be two c-quomorphisms of partial* Γ*-algebras, and let*

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B} & \xrightarrow[\tilde{f}]{} & \mathbf{H} \end{array}$$

be their pushout in CF-Alg$_\Gamma$. *Then it is also their pushout in* CQ-Alg$_\Gamma$.

Proof. Let $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two c-quomorphisms of partial Γ-algebras such that $p \circ f = q \circ g$. We know from Proposition 9 that there exists a unique conformism $h : \mathbf{H} \to \mathbf{C}$ such that $h \circ \tilde{g} = p$ and $h \circ \tilde{f} = q$. We must check that h is also a quomorphism.

So, let $\varphi \in \Omega$, say with $\eta(\varphi) = (s, s')$, and let $x \in \operatorname{Dom} h_s \cap \operatorname{dom} \varphi^{\mathbf{H}}$ such that $\varphi^{\mathbf{H}}(x) \in \operatorname{Dom} h_{s'}$. Since $\tilde{f}$ and $\tilde{g}$ are jointly surjective, we can assume without any loss of generality that $x = \tilde{g}(a)$ for some $a \in A'_s$, and since $\tilde{g} : \mathbf{A}' \to \mathbf{H}$ is a closed homomorphism, we have that $a \in \operatorname{dom} \varphi^{\mathbf{A}'}$ and $\tilde{g}_{s'}(\varphi^{\mathbf{A}'}(a)) = \varphi^{\mathbf{H}}(x)$. Therefore $a \in \operatorname{dom} \varphi^{\mathbf{A}} \cap \operatorname{Dom} p_s$ and $\varphi^{\mathbf{A}}(a) \in \operatorname{Dom} p_{s'}$, and then ($p$ being a quomorphism) $h_s(x) = p_s(a) \in \operatorname{dom} \varphi^{\mathbf{C}}$ and

$$h_{s'}(\varphi^{\mathbf{H}}(x)) = p_{s'}(\varphi^{\mathbf{A}}(a)) = \varphi^{\mathbf{C}}(p_s(a)) = \varphi^{\mathbf{C}}(h_s(x)).$$

The symbol φ standing for any operation symbol in the signature, this shows that h is a quomorphism, and therefore also a c-quomorphism. □

So, single-pushout transformation using c-quomorphisms is a special case of single-pushout transformation using conformisms (in the sense that the rules – c-quomorphisms – and the occurrences – closed homomorphisms – to be used in CQ-Alg$_\Gamma$ are special cases of those used in CF-Alg$_\Gamma$), and therefore it actually does not yield a new approach to transformation of partial unary algebras. Moreover, Example 14 entails that, in general, there is no relation between single-pushout transformation in CQ-Alg$_\Gamma$ and double-pushout transformation in Alg$_\Gamma$, in the sense explained therein (Definition 13).

Example 18. Let Γ be a graph structure, let $\mathbf{L}$ be any partial Γ-algebra, let $\mathbf{R}$ be a relative subalgebra of $\mathbf{L}$, with universe R, and let $r : \mathbf{L} \to \mathbf{R}$ be the c-quomorphism given by the set-theoretical identity on R; this is then a special case of Example 10.

Let now $m : \mathbf{L} \to \mathbf{G}$ be any closed homomorphism. Since, as we have just seen, the derived partial algebra $\mathbf{H}$ of $\mathbf{G}$ by the application of $r : \mathbf{L} \to \mathbf{R}$ through $m : \mathbf{L} \to \mathbf{G}$ in CQ-Alg$_\Gamma$ is the same as in CF-Alg$_\Gamma$, Example 10 applies, and we obtain that $\mathbf{H}$ is the relative subalgebra of $\mathbf{G}$ supported on $H = (H_s)_{s \in S}$ where $H_s = \{y \in G_s \mid m_s^{-1}(y) \subseteq R_s\}$. That is, $\mathbf{H}$ is simply obtained by removing from $\mathbf{G}$ all points whose preimage is not contained in R.

To close this section, we give two results connecting pushouts in CQ-Alg$_\Gamma$ with pushouts in other categories. First, notice that arguing as in Proposition 15 we obtain the following result, which we shall use later.

Proposition 19. *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two closed homomorphisms of partial Γ-algebras, and let*

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B} & \xrightarrow[\tilde{f}]{} & \mathbf{H} \end{array}$$

be their pushout in CQ-Alg$_\Gamma$. *Then it is also their pushout in* C-Alg$_\Gamma$.

And second, since the pushout of two c-quomorphisms $f : \mathbf{K} \rightharpoonup \mathbf{A}$ and $g : \mathbf{K} \rightharpoonup \mathbf{B}$ is given by the quotient of $\mathbf{A}' + \mathbf{B}'$ by $\theta(K')$, which in this case is a congruence on this disjoint sum, Proposition 7 entails the following result.

Proposition 20. *Let $f : \mathbf{K} \rightharpoonup \mathbf{A}$ and $g : \mathbf{K} \rightharpoonup \mathbf{B}$ be two c-quomorphisms of partial Γ-algebras. Then they have a pushout in* Q-Alg$_\Gamma$, *and it is given by their pushout in* CQ-Alg$_\Gamma$.

5. Single-pushout transformation in CDCQ-Alg$_\Gamma$

In the previous section we have seen that the pushout of two c-quomorphisms $f : \mathbf{K} \rightharpoonup \mathbf{A}$ and $g : \mathbf{K} \rightharpoonup \mathbf{B}$ in CQ-Alg$_\Gamma$ is the same as their pushout in CF-Alg$_\Gamma$, that is, their pushout as conformisms. This is no longer true for pushouts of cdc-quomorphisms in CDCQ-Alg$_\Gamma$ and in CF-Alg$_\Gamma$, as the following example shows.

Example 21. Let Γ be a monounary graph structure, and let $\mathbf{K}$ be a Γ-algebra with carrier $K = \{a, b\}$ and with $\varphi^{\mathbf{K}}$ total and given by $\varphi^{\mathbf{K}}(a) = \varphi^{\mathbf{K}}(b) = b$. Let $\mathbf{A}$ and $\mathbf{B}$ be both equal to $\mathbf{K}$.

Let $f : \mathbf{K} \rightharpoonup \mathbf{A}$ be the cdc-quomorphism with domain $\operatorname{Dom} f = \{b\}$ and $f(b) = b$. Let $g : \mathbf{K} \rightharpoonup \mathbf{B}$ be the closed homomorphism with $g(a) = g(b) = b$.

It turns out that $K' = \emptyset$ and then $A' = B' = \{a\}$, so that the pushout $\mathbf{H}$ of f and g in CF-Alg$_\Gamma$ is a discrete Γ-algebra with two elements. But then the c-quomorphisms $\tilde{f} : \mathbf{B} \rightharpoonup \mathbf{H}$ and $\tilde{g} : \mathbf{A} \rightharpoonup \mathbf{H}$ are not cdc-quomorphisms, because A' and B' are not closed subsets of $\mathbf{A}$ and $\mathbf{B}$. Therefore, the pushout of f and g in CQ-Alg$_\Gamma$ is not their pushout in CDCQ-Alg$_\Gamma$. And, in fact, a simple argument shows that the pushout algebra of f and g in CDCQ-Alg$_\Gamma$ is the empty algebra (see also Proposition 25 below).

The drawback found in this example is general. Given two cdc-quomorphisms $f : \mathbf{K} \rightharpoonup \mathbf{A}$ and $g : \mathbf{K} \rightharpoonup \mathbf{B}$ of partial Γ-algebras, the sets A' and B' defined in Section 2.3 need not be closed subsets of their respective home algebras. In this section we shall show that the naïve modification to the construction given in Section 3, consisting in replacing A' and B' by the greatest closed subsets of $\mathbf{A}$ and $\mathbf{B}$ contained in them (which exist, because all operations in the signature are unary), works, so that one gets in this way a pushout for f and g in CDCQ-Alg$_\Gamma$.

So, let $f : \mathbf{K} \rightharpoonup \mathbf{A}$ and $g : \mathbf{K} \rightharpoonup \mathbf{B}$ be two cdc-quomorphisms of partial Γ-algebras, and let

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B} & \xrightarrow[\tilde{f}]{} & \mathbf{H} \end{array}$$

be their pushout in CF-Alg$_\Gamma$ as described in Section 3. Let A^{c} and B^{c} be the greatest closed subsets of $\mathbf{A}$ and $\mathbf{B}$ contained in A' and B', respectively. To compute the subsets

A^c and B^c, we can use the following well known description of the greatest closed subset of a partial Γ-algebra contained in a given subset.

Lemma 22. *Let* $\mathbf{A}$ *be a partial* Γ*-algebra, and let* $X \subseteq A$. *The greatest closed subset* $X^c = (X_s^c)_{s\in S}$ *of* $\mathbf{A}$ *contained in* X *is given by*

$$X_s^c = \{x \in X_s \mid \textit{for every } \Gamma\textit{-term } \mathbf{t} \textit{ of sort } s' \in S, \textit{ if } x \in \operatorname{dom} \mathbf{t}^{\mathbf{A}} \textit{ then } \mathbf{t}^{\mathbf{A}}(x) \in X_{s'}\}.$$

Alternatively, it is also given by

$$X^c = \bigcup \{C_{\mathbf{A}}(\{x\}) \mid x \in \textstyle\bigcup_{s\in S} X_s \text{ and } C_{\mathbf{A}}(\{x\}) \subseteq X\}$$

where $C_{\mathbf{A}}(\{x\})$ *denotes the closed subset of* $\mathbf{A}$ *generated by the S-set with all its carriers of all sorts empty, except the one of the sort corresponding to x, which is* $\{x\}$.

Let now $\mathbf{H}^c$ be the relative subalgebra of $\mathbf{H}$ supported on $H^c = \tilde{g}(A^c) \cup \tilde{f}(B^c)$. Since A^c and B^c are closed subsets of A' and B', respectively, and $\tilde{g} : \mathbf{A}' \to \mathbf{H}$ and $\tilde{f} : \mathbf{B}' \to \mathbf{H}$ are closed homomorphisms, we have that $\mathbf{H}^c$ is a closed subalgebra of $\mathbf{H}$.

Lemma 23. *Let* $\hat{f} : B \to H^c$ *and* $\hat{g} : A \to H^c$ *be the restrictions of* $\tilde{f}$ *and* $\tilde{g}$ *to* B^c *and* A^c *(considered with target set* H^c*), respectively. Then* $\hat{g}$ *and* $\hat{f}$ *are cdc-quomorphisms* $\hat{g} : \mathbf{A} \to \mathbf{H}^c$ *and* $\hat{f} : \mathbf{B} \to \mathbf{H}^c$.

Proof. A^c is a closed subset of $\mathbf{A}$ and $\tilde{g}|_{A^c} : \mathbf{A}^c \to \mathbf{H}$ is the restriction of the closed homomorphism $\tilde{g} : \mathbf{A}' \to \mathbf{H}$ to a closed subalgebra of its domain, and therefore a closed homomorphism [5, Proposition 3.1.11(ii)]. $\mathbf{H}^c$ is a relative subalgebra of $\mathbf{H}$ containing the closed subset $\tilde{g}(A^c)$ of $\mathbf{H}$. Then $\hat{g} = \tilde{g}|_{A^c} : \mathbf{A}^c \to \mathbf{H}^c$ is still a closed homomorphism. By a similar reason, $\hat{f} = \tilde{f}|_{B^c} : \mathbf{B}^c \to \mathbf{H}^c$ is also a closed homomorphism. □

It turns out that $\mathbf{H}^c$, together with these cdc-quomorphisms $\hat{g} : \mathbf{A} \to \mathbf{H}^c$ and $\hat{f} : \mathbf{B} \to \mathbf{H}^c$, is indeed the pushout of f and g in CDCQ-Alg$_\Gamma$. The following lemma will be used to prove it.

Lemma 24. *Let* K^c *be the greatest closed subset of* $\mathbf{K}$ *contained in* K'. *Then* $\operatorname{Dom}(\hat{g} \circ f) = f^{-1}(A^c)$ *and* $\operatorname{Dom}(\hat{f} \circ g) = g^{-1}(B^c)$ *are equal to* K^c.

Proof. On the one hand, $\operatorname{Dom}(\hat{g} \circ f)$ is closed, because it is the domain of a cdc-quomorphism, and it is contained in K', because $A^c \subseteq A'$ and therefore $\operatorname{Dom}(\hat{g} \circ f) = f^{-1}(A^c) \subseteq f^{-1}(A') = K'$. Thus, $\operatorname{Dom}(\hat{g} \circ f) \subseteq K^c$. By symmetry, $\operatorname{Dom}(\hat{f} \circ g) \subseteq K^c$.

On the other hand, since $f : \mathbf{Dom}\, f \to \mathbf{A}$ is a closed homomorphism and $K^c \subseteq K' \subseteq \operatorname{Dom} f$, we have that $f(K^c)$ is a closed subset of $\mathbf{A}$ (by [5, Proposition 3.1.9]) contained in $f(K') \subseteq A'$ and therefore $f(K^c) \subseteq A^c$, which implies $K^c \subseteq f^{-1}(A^c)$; a similar argument shows that $K^c \subseteq g^{-1}(B^c)$. □

Proposition 25. *The square*

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ \mathbf{B} & \xrightarrow[\tilde{f}]{} & \mathbf{H}^{\mathrm{c}} \end{array}$$

is a pushout square in CDCQ-Alg$_\Gamma$.

Proof. This square commutes because $\mathrm{Dom}(\hat{g} \circ f) = \mathrm{Dom}(\hat{f} \circ g)$ by the previous lemma, and $\tilde{g} \circ f = \tilde{f} \circ g$.

Let now $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two cdc-quomorphisms of partial Γ-algebras such that $p \circ f = q \circ g$. We must prove that there exists a unique cdc-quomorphism $h : \mathbf{H}^{\mathrm{c}} \to \mathbf{C}$ such that $h \circ \hat{g} = p$ and $h \circ \hat{f} = q$.

We know from Proposition 17 that there exists a unique c-quomorphism $h : \mathbf{H} \to \mathbf{C}$ such that $h \circ \tilde{g} = p$ and $h \circ \tilde{f} = q$. By construction, this c-quomorphism is a closed homomorphism from the relative subalgebra $\tilde{\mathbf{H}}$ of $\mathbf{H}$ supported on $\tilde{H} = \tilde{f}(\mathrm{Dom}\, q) \cup \tilde{g}(\mathrm{Dom}\, p)$. If $\tilde{\mathbf{H}}$ is a closed subalgebra of $\mathbf{H}^{\mathrm{c}}$ then this c-quomorphism $h : \mathbf{H} \to \mathbf{C}$ will yield a cdc-quomorphism $h : \mathbf{H}^{\mathrm{c}} \to \mathbf{C}$ such that $h \circ \hat{g} = p$ and $h \circ \hat{f} = q$. And it will be unique with this property, because of the uniqueness of the c-quomorphism $h : \mathbf{H} \to \mathbf{C}$.

So, we must prove that $\tilde{H}$ is a closed subset of $\mathbf{H}^{\mathrm{c}}$, and, to do that, it is enough to prove that it is a subset of H^{c} that is closed in $\mathbf{H}$.

Since $\mathrm{Dom}\, p$ and $\mathrm{Dom}\, q$ are closed subsets of $\mathbf{A}$ and $\mathbf{B}$ contained in A' and B', respectively, we have that $\mathrm{Dom}\, p \subseteq A^{\mathrm{c}}$ and $\mathrm{Dom}\, q \subseteq B^{\mathrm{c}}$ and then

$$\tilde{H} = \tilde{f}(\mathrm{Dom}\, q) \cup \tilde{g}(\mathrm{Dom}\, p) \subseteq \tilde{f}(B^{\mathrm{c}}) \cup \tilde{g}(A^{\mathrm{c}}) = H^{\mathrm{c}}.$$

As to the closedness of $\tilde{H}$ in $\mathbf{H}$, let $\varphi \in \Omega$ be an operation symbol with $\eta(\varphi) = (s, s')$ and let $x \in \tilde{H}_s \cap \mathrm{dom}\, \varphi^{\mathbf{H}}$. Since $\tilde{H} = \tilde{f}(\mathrm{Dom}\, q) \cup \tilde{g}(\mathrm{Dom}\, p)$, we can assume without any loss of generality that $x = \tilde{g}_s(a)$ for some $a \in \mathrm{Dom}\, p_s$. Then, $\tilde{g}$ being a c-quomorphism, $x \in \mathrm{dom}\, \varphi^{\mathbf{H}}$ implies that $a \in \mathrm{dom}\, \varphi^{\mathbf{A}}$, $\varphi^{\mathbf{A}}(a) \in \mathrm{Dom}\, \tilde{g}_{s'}$ and $\tilde{g}_{s'}(\varphi^{\mathbf{A}}(a)) = \varphi^{\mathbf{H}}(x)$. And $\mathrm{Dom}\, p$ being a closed subset of $\mathbf{A}$, we have that $\varphi^{\mathbf{A}}(a) \in \mathrm{Dom}\, p_{s'}$, so that $\varphi^{\mathbf{H}}(x) \in \tilde{g}_{s'}(\mathrm{Dom}\, p_{s'}) \subseteq \tilde{H}_{s'}$. Since φ stands for any operation symbol in Ω, this proves that $\tilde{H}$ is a closed subset of $\mathbf{H}$. □

So, the pushout algebra $\mathbf{H}^{\mathrm{c}}$ in CDCQ-Alg$_\Gamma$ of two cdc-quomorphisms of partial Γ-algebras is a (proper, in general) closed subalgebra of their pushout algebra $\mathbf{H}$ in CF-Alg$_\Gamma$. As a matter of fact, $\mathbf{H}^{\mathrm{c}}$ is the greatest closed subalgebra of $\mathbf{H}$ such that its preimages by $\tilde{g}$ and $\tilde{f}$ are closed subsets of $\mathbf{A}$ and $\mathbf{B}$, respectively.

The pushout of two cdc-quomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of partial Γ-algebras is also described by the following proposition.

Proposition 26. *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two cdc-quomorphisms of partial Γ-algebras. As before, let K^{c} be the greatest closed subset of* $\mathbf{K}$ *contained in the* *gluing set K' of f and g, let $\mathbf{K}^c$ be the closed subalgebra of $\mathbf{K}$ supported on K^c, let A^c and B^c be the greatest closed subsets of $\mathbf{A}$ and $\mathbf{B}$ contained respectively in A' and B', and let $\mathbf{A}^c$ and $\mathbf{B}^c$ be the corresponding closed subalgebras of $\mathbf{A}$ and $\mathbf{B}$ supported on A^c and B^c, respectively.*

(a) *$f|_{K^c} : \mathbf{K}^c \to \mathbf{A}^c$ and $g|_{K^c} : \mathbf{K}^c \to \mathbf{B}^c$ are closed homomorphisms.*
Let now

$$\begin{array}{ccc} \mathbf{K}^c & \xrightarrow{f|_{K^c}} & \mathbf{A}^c \\ {\scriptstyle g|_{K^c}}\downarrow & & \downarrow{\scriptstyle \bar{g}} \\ \mathbf{B}^c & \xrightarrow[\bar{f}]{} & \mathbf{H}_0 \end{array}$$

be the pushout of $f|_{K^c}$ and $g|_{K^c}$ in C-Alg$_\Gamma$, *and let also $\bar{g} : \mathbf{A} \to \mathbf{H}_0$ and $\bar{f} : \mathbf{B} \to \mathbf{H}_0$ denote the cdc-quomorphisms corresponding to the homonymous closed homomorphisms.*

(b) *The square*

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar{g}} \\ \mathbf{B} & \xrightarrow[\bar{f}]{} & \mathbf{H}_0 \end{array}$$

is a pushout square in CDCQ-Alg$_\Gamma$.

Proof. (a) To prove that $f|_{K^c} : \mathbf{K}^c \to \mathbf{A}^c$ is closed, notice that $f|_{K^c} : \mathbf{K}^c \to \mathbf{A}$ is the restriction of the closed homomorphism $f : \mathbf{Dom}\, f \to \mathbf{A}$ to the closed subalgebra $\mathbf{K}^c$ and therefore it is a closed homomorphism, and that $f(K^c) \subseteq A^c$, as we have seen in the proof of Lemma 24. A similar proof applies to $g|_{K^c} : \mathbf{K}^c \to \mathbf{B}^c$.

(b) Let $\mathbf{H}^c$, together with cdc-quomorphisms $\hat{g} : \mathbf{A} \to \mathbf{H}^c$ and $\hat{f} : \mathbf{B} \to \mathbf{H}^c$, be the pushout of f and g described in Proposition 25. In particular, $\hat{g}$ and $\hat{f}$ are the cdc-quomorphisms obtained by restricting the c-quomorphisms $\tilde{g} : \mathbf{A}' \to \mathbf{H}'$ and $\tilde{f} : \mathbf{B}' \to \mathbf{H}'$ (appearing in the pushout of f and g in CQ-Alg$_\Gamma$ described in Proposition 17) to $\mathbf{A}^c$ and $\mathbf{B}^c$, respectively. From the first part of the proof of Proposition 25 (and Lemmas 23 and 24) we know that $\hat{g} : \mathbf{A}^c \to \mathbf{H}^c$ and $\hat{f} : \mathbf{B}^c \to \mathbf{H}^c$ are closed homomorphisms such that $\hat{g} \circ f|_{K^c} = \hat{f} \circ g|_{K^c}$. Therefore, by the universal property of pushouts, there exists a unique closed homomorphism $h : \mathbf{H}_0 \to \mathbf{H}^c$ such that $h \circ \bar{f} = \hat{f}$ and $h \circ \bar{g} = \hat{g}$. Such h is clearly surjective, because $\hat{f}$ and $\hat{g}$ are jointly surjective by the construction of $\mathbf{H}^c$. It is enough to prove that h is also injective and total (and therefore an isomorphism).

That h is total follows from the facts that $\mathrm{Dom}\, \bar{f} = \mathrm{Dom}\, \hat{f} = \mathbf{B}^c$ and $\mathrm{Dom}\, \bar{g} = \mathrm{Dom}\, \hat{g} = \mathbf{A}^c$, and that $\bar{f}$ and $\bar{g}$ are jointly surjective.

Notice now that this closed homomorphism is nothing but the mapping

$$h : H_0 = (A^c \sqcup B^c)/\theta(K^c) \longrightarrow H = (A' \sqcup B')/\theta(K')$$

induced by the inclusion $A^c \sqcup B^c \hookrightarrow A' \sqcup B'$, and considered as a mapping with target $H^c = \mathrm{Im}\, h$. To show that this mapping is injective, we shall prove that

$$\theta(K') \cap ((A^c \sqcup B^c) \times (A'_s \sqcup B'_s)) = \theta(K^c)$$

(this equality implies that h is not only injective, but actually the inclusion of a subset).

So, let $s \in S$ and let $(z,z') \in \theta(K')_s$ with $z \in A_s^c \sqcup B_s^c$ and $z' \in A_s' \sqcup B_s'$. Since the four cases to be considered ($z \in A_s^c$ and $z' \in A_s'$, $z \in A_s^c$ and $z' \in B_s'$, $z \in B_s^c$ and $z' \in A_s'$, $z \in B_s^c$ and $z' \in B_s'$) are similar, we shall only prove it in a sample case, namely the first one, leaving the others to the reader. So, let $(z,z') \in \theta(K')_s \cap (A_s^c \times A_s')$. Then there exist $x_0,\dots,x_{2n} \in K_s'$ such that

$$f_s(x_0) = z, \; f_s(x_{2n}) = z',$$

$$g_s(x_{2k-2}) = g_s(x_{2k-1}), \; f_s(x_{2k-1}) = f_s(x_{2k}), \quad k = 1,\dots,n.$$

Using that $K_s^c = f_s^{-1}(A_s^c) = g_s^{-1}(B_s^c)$ (see Lemma 24) and starting with $f_s(x_0) = z \in A_s^c$, a simple argument by induction on k shows that $x_{2k}, x_{2k+1} \in K_s^c$ for every k. In this way we finally obtain on the one hand that $x_{2n} \in K_s^c$, and therefore $z' = f_s(x_{2n}) \in A_s^c$, and on the other hand that $(z,z') \in \theta(K^c)$, as desired. □

Corollary 27. *Given two partial homomorphisms $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ of total Γ-algebras, their pushout in* CDCQ-Alg$_\Gamma$ *is also their pushout in* P-TAlg$_\Gamma$.

Proof. Comparing the construction of the pushout of two cdc-quomorphisms in CDCQ-Alg$_\Gamma$ given in Proposition 26 with the construction of the pushout of two partial homomorphisms in P-TAlg$_\Gamma$ given in [15, Construction 2.6], one easily sees that both constructions, when applied to two partial homomorphisms of total algebras, are exactly the same, provided that the following two assertions hold.

- K^c is the greatest closed subset of K such that $f^{-1}(f(K^c)) = K^c$ and $g^{-1}(g(K^c)) = K^c$.
- The greatest closed subset A^c of $\mathbf{A}$ contained in A' is also the greatest closed subset of $\mathbf{A}$ contained in $(A - f(K)) \cup f(K^c)$.

The first assertion turns out to be true. Indeed, on the one hand by Lemma 3 we have that $f^{-1}(f(K^c)) \subseteq f^{-1}(f(K')) = K'$, and on the other hand, f being a closed homomorphism from its domain **Dom** f, which is closed in $\mathbf{K}$, and K^c being a closed subset of this domain, by [5, Proposition 3.1.9] we have that $f^{-1}(f(K^c))$ is a closed subset of $\mathbf{K}$. Then, K^c being the greatest closed subset of $\mathbf{K}$ contained in K', we deduce that $f^{-1}(f(K^c)) \subseteq K^c$, and this inclusion becomes an equality because $K^c \subseteq \mathrm{Dom}\, f$. A similar argument shows that $g^{-1}(g(K^c)) = K^c$. And if K^* is some closed subset of K such that $f^{-1}(f(K^*)) = K^*$ and $g^{-1}(g(K^*)) = K^*$ then, by Lemma 3 again, $K^* \subseteq K'$ and therefore $K^* \subseteq K^c$.

And the second assertion is also true, because by Lemma 24 we have that $f^{-1}(A^c) = K^c$ and therefore $A^c \subseteq (A - f(K)) \cup f(K^c)$. □

This entails that the current approach to single-pushout transformation of total unary algebras [15] is a special case of the single-pushout transformation in CDCQ-Alg$_\Gamma$ (but not of the single-pushout transformation in CQ-Alg$_\Gamma$ or CF-Alg$_\Gamma$). On the other hand, the equivalence between CDCQ-Alg$_\Gamma$ and P-TAlg$_{\hat{\Gamma}}$ mentioned in Section 2.2

entails on its turn that single-pushout transformation in CDCQ-Alg$_\Gamma$ (but not the other approaches introduced in this paper) can be considered as a special case of single-pushout transformation of total unary algebras: namely, of single-pushout transformation in P-TAlg$_{\hat{\Gamma}}$.

Notice that Proposition 26 and the proof of Corollary 27 yield a simple way of obtaining the derived partial Γ-algebra of $\mathbf{G}$ by the application of a production rule $r : \mathbf{L} \to \mathbf{R}$ in CDCQ-Alg$_\Gamma$ through a closed homomorphism $m : \mathbf{L} \to \mathbf{G}$. This method is formally similar to the aforementioned Construction 2.6 in [15].

(i) Compute the gluing set L' of r and m.

(ii) *Remove* from L' those elements for which there exists some term that applied to them gives an element not in L'. Let L^c be the subset of L' obtained in this way.

(iii) Also, *remove* from R and G those elements for which there exists some term that applied to them gives an element in $r(L) - r(L^c)$ and $m(L) - m(L^c)$, respectively. Let $\mathbf{R}^c$ and $\mathbf{G}^c$ be the corresponding closed subalgebras of $\mathbf{R}$ and $\mathbf{G}$ supported on the sets obtained in this way.

(iv) *Add* $\mathbf{R}^c$ to $\mathbf{G}^c$, by first forming the coproduct $\mathbf{G}^c + \mathbf{R}^c$ and then identifying in this coproduct all images of elements of L^c by r and m.

Another nice feature of single-pushout transformation of cdc-quomorphisms is that there is a relation between single-pushout transformation in CDCQ-Alg$_\Gamma$ and double-pushout transformation in C-Alg$_\Gamma$, similar to the relation between single-pushout and double-pushout transformation of total unary algebras shown in [15, Section 3.3].

Definition 28. Given a cdc-quomorphism of partial Γ-algebras $r : \mathbf{L} \to \mathbf{R}$, let $\mathbf{L}_r$ be the closed subalgebra of $\mathbf{L}$ supported on $\operatorname{Dom} r$. Let $i : \mathbf{L}_r \to \mathbf{L}$ be the corresponding (closed) embedding and let $r : \mathbf{L}_r \to \mathbf{R}$ denote also the corresponding closed homomorphism. We shall say that the production rule in C-Alg$_\Gamma$

$$P(r) = (\mathbf{L} \xleftarrow{i} \mathbf{L}_r \xrightarrow{r} \mathbf{R})$$

is the *production rule associated to r*.

Let us recall from [7, Definition 21] that, given two closed homomorphisms of partial Γ-algebras $r : \mathbf{K} \to \mathbf{L}$ and $m : \mathbf{L} \to \mathbf{G}$, we say that m satisfies the gluing condition w.r.t. r when $m(r(K)) \cap m(L - r(K)) = \emptyset$ and the restriction of m to $L - r(K)$ is injective (*identification condition*), and $m(L - r(K))$ is an initial segment of $\mathbf{G}$ (*dangling condition*).

Proposition 29. *Let $r : \mathbf{L} \to \mathbf{R}$ be a cdc-quomorphism of partial Γ-algebras, and let $m : \mathbf{L} \to \mathbf{G}$ be a closed homomorphism satisfying the gluing condition w.r.t. $i : \mathbf{L}_r \to \mathbf{L}$.*

Then the derived partial Γ-algebras of $\mathbf{G}$ through m by the production rule $P(r)$ in C-Alg$_\Gamma$ [7, *Definition* 20] *and by the production rule r in* CDCQ-Alg$_\Gamma$ *are the same (that is, isomorphic).*

Proof. Let us follow the construction given in Proposition 26 of the derived partial Γ-algebra of $\mathbf{G}$ through m by means of the production rule r.

We must compute first the greatest closed subset L^c of $\mathbf{L}$ contained in the gluing set L' of r and m. To do this, notice that $r^{-1}(r(L_r)) = L_r$ (because $L_r = \operatorname{Dom} r$) and $m^{-1}(m(L_r)) = L_r$ (because m is total and it satisfies the gluing condition w.r.t. i). Therefore, by Lemma 3, $L' = L_r$. And since L_r is a closed subset of $\mathbf{L}$, because r is a cdc-quomorphism, we also have that $L^c = L_r$.

Now, R^c is the greatest closed subset of $(R - r(L)) \cup r(L_r) = R$, that is, R itself, and then $\mathbf{R}^c = \mathbf{R}$. And G^c is the greatest closed subset of $\mathbf{G}$ contained in $G' = (G - m(L)) \cup m(L_r)$, but by the gluing condition this set G' is closed (cf. the proof of Proposition 10 of [7]), and therefore $G^c = G'$. This implies that $\mathbf{G}^c = \mathbf{G}'$, which is the context algebra of the application of rule $P(r)$ to $\mathbf{G}$ through m [7, Definition 15].

Then, the pushout algebra of $r : \mathbf{L} \to \mathbf{R}$ and $m : \mathbf{L} \to \mathbf{G}$ in CDCQ-Alg$_\Gamma$ is obtained by Proposition 26 as the pushout of the closed homomorphisms $r : \mathbf{L}_r \to \mathbf{R}$ and $m|_{L_r} : \mathbf{L}_r \to \mathbf{G}^c$, and the derived partial Γ-algebra of $\mathbf{G}$ through m by the application of rule $P(r)$ in C-Alg$_\Gamma$ is obtained by [7, Def. 15] exactly in the same way. □

The converse assertion also holds.

Proposition 30. *Let* $P = (\mathbf{L} \xleftarrow{l} \mathbf{K} \xrightarrow{r} \mathbf{R})$ *be a production rule in* C-Alg$_\Gamma$ *with* l *injective, and let* $m : \mathbf{L} \to \mathbf{G}$ *be a closed homomorphism satisfying the gluing condition w.r.t.* l. *Let* $\tilde{r} : \mathbf{L} \to \mathbf{R}$ *be the cdc-quomorphism corresponding to* r *considered as a closed homomorphism from the closed subalgebra of* $\mathbf{L}$ *supported on* $l(K)$.

Then the derived partial Γ*-algebras of* $\mathbf{G}$ *through* m *by the production rule* P *in* C-Alg$_\Gamma$ *and by the production rule* $\tilde{r}$ *in* CDCQ-Alg$_\Gamma$ *are the same.*

Proof. Notice that P and $P(\tilde{r})$ are the same rule, up to an isomorphism $\mathbf{K} \cong \mathbf{L}_{\tilde{r}}$ compatible with the left- and right-hand-side morphisms of both rules, so that the previous proposition can be applied. We leave the details to the reader. □

Therefore, and in a similar way to what happens with double-pushout transformation of total unary algebras, double-pushout transformation of unary partial algebras using rules in C-Alg$_\Gamma$ with left-hand side homomorphism injective can be seen as a particular case of single-pushout transformation in CDCQ-Alg$_\Gamma$.

Remark. Actually, these two last propositions could have also been proved using the corresponding result for total unary algebras given in [15, Section 3.3], through the equivalences between C-Alg$_\Gamma$ and TAlg$_{\hat{\Gamma}}$, on the one hand, and between CDCQ-Alg$_\Gamma$ and P-TAlg$_{\hat{\Gamma}}$, on the other hand, stated in Section 2.2.

Example 31. Let Γ be a graph structure, let $\mathbf{L}$ be any partial Γ-algebra, let $\mathbf{R}$ be a closed subalgebra of $\mathbf{L}$, with universe R, and let $r : \mathbf{L} \to \mathbf{R}$ be the cdc-quomorphism given by the set-theoretical identity on R. This is a special case of the rules considered in Examples 10 and 18.

Let now $m : \mathbf{L} \to \mathbf{G}$ be any closed homomorphism; we want to compute the derived partial algebra $\mathbf{H}_0$ of $\mathbf{G}$ by the application of $r : \mathbf{L} \to \mathbf{R}$ through $m : \mathbf{L} \to \mathbf{G}$ in CDCQ-Alg$_\Gamma$. Since we have already computed the corresponding derived algebra $\mathbf{H}$ in CQ-Alg$_\Gamma$ in Example 18, we can use Proposition 25 to compute this derived algebra in CDCQ-Alg$_\Gamma$, and it turns out to be the greatest closed subalgebra of $\mathbf{G}$ contained in $\mathbf{H}$.

So, $\mathbf{H}_0$ is obtained from $\mathbf{G}$ by simply removing all points y for which there exists some term $\mathbf{t}$ such that when applied to them yields a point $\mathbf{t}^{\mathbf{G}}(y)$ whose preimage under m is not fully contained in R. In general, such $\mathbf{H}_0$ is properly contained in $\mathbf{H}$.

Example 32. The cdc-quomorphisms between (higher-order) hypergraphs (considered as special partial Σ_H-algebras) correspond to the usual partial morphisms between them; see [1]. And, as it was already mentioned in [7, Example 3], (higher-order) hypergraphs considered as partial algebras are closed under pushouts w.r.t. closed homomorphisms, which are the usual morphisms of (higher-order) hypergraphs. Therefore, usual single-pushout transformations of (higher-order) hypergraphs are examples of transformations in CDCQ-Alg$_{\Sigma_H}$.

As an example, we shall show how to delete from (finite) hierarchic higher-order hypergraphs all arcs where some abstraction operation is defined, such an operation being of special interest for the structural verification of knowledge bases through transformations of partial algebras; see [26]. The argument will be similar to the one in Example 12, but it is worth pointing out that using closed homomorphisms as redices instead of total conformisms (as we could do therein) makes us to need a specific rule to delete each arc (a rule that will depend on "how many operations" are defined on it and its images), instead of a single rule for all arcs.

For every $\underline{n} = (n_0, n_1, n_2, (m_{i,j})_{i=1,\dots,n_0, j=1,2})$, $n_0 \geqslant 1$, $n_1, n_2 \geqslant 0$ and $m_{i,j} \geqslant 0, 1 \leqslant i, j \leqslant n_0$, let $\mathbf{G}(\underline{n})$ be the partial Σ_H-algebra with $G(\underline{n})_E = \{e, e_1, \dots, e_{n_0}\}$ and

$$G(\underline{n})_V = \{v_j \mid 1 \leqslant j \leqslant n_1 + n_2\} \cup \{w_{i,j} \mid 1 \leqslant i \leqslant n_0, 1 \leqslant j \leqslant m_{i,1} + m_{i,2}\}$$

and with operations defined as follows:

$$\begin{aligned}
a_j^{\mathbf{G}(\underline{n})}(e) &= e_j, & & j = 1, \dots, n_0,\\
s_j^{\mathbf{G}(\underline{n})}(e) &= v_j, & & j = 1, \dots, n_1,\\
t_j^{\mathbf{G}(\underline{n})}(e) &= v_{n_1+j}, & & j = 1, \dots, n_2,\\
s_j^{\mathbf{G}(\underline{n})}(e_i) &= w_{i,j}, & & i = 1, \dots, n_0, \quad j = 1, \dots, m_{i,1},\\
t_j^{\mathbf{G}(\underline{n})}(e_i) &= w_{i,m_{i,1}+j}, & & i = 1, \dots, n_0, \quad j = 1, \dots, m_{i,2}
\end{aligned}$$

and let $\hat{\mathbf{G}}(\underline{n})$ be the relative subalgebra of $\mathbf{G}(\underline{n})$ obtained by removing e from $G(\underline{n})_E$. Let $r_{\underline{n}} : \mathbf{G}(\underline{n}) \to \hat{\mathbf{G}}(\underline{n})$ be the cdc-quomorphism given by the identity on the carrier of $\hat{\mathbf{G}}(\underline{n})$.

Let now $\mathbf{G}$ be a higher-order hypergraph and let $m : \mathbf{G}(\underline{n}) \to \mathbf{G}$ be a closed homomorphism. Let $m_E(e) = e'$, $m_E(e_i) = e'_i$, $m_V(v_j) = v'_j$ and $m_V(w_{i,j}) = w'_{i,j}$. Since m is a closed homomorphism, the following conditions hold:

- $n_a(e') = n_0$, and $a_i^{\mathbf{G}}(e') = e'_i$ for every $i \leqslant n_0$.
- $n_a(e'_i) = 0$.
- $n_s(e') = n_1$, and $s_j^{\mathbf{G}}(e') = v'_j$ for every $j \leqslant n_1$.
- $n_t(e') = n_2$, and $t_j^{\mathbf{G}}(e') = v'_j$ for every $n_1 < j \leqslant n_1 + n_2$.
- $n_s(e'_i) = m_{i,1}$, and $s_j^{\mathbf{G}}(e'_i) = w'_{i,j}$, for every $i \leqslant n_0$ and $j \leqslant m_{i,1}$.
- $n_t(e'_i) = m_{i,2}$, and $t_j^{\mathbf{G}}(e'_i) = w'_{i,m_{i,1}+j}$ for every $i \leqslant n_0$ and $m_{i,1} < j \leqslant m_{i,1} + m_{i,2}$.

Arguing as in Example 12, and using Proposition 25, it is easy to see that the pushout algebra of $r_{\underline{n}} : \mathbf{G}(\underline{n}) \to \hat{\mathbf{G}}(\underline{n})$ and $m : \mathbf{G}(\underline{n}) \to \mathbf{G}$ in CDCQ-Alg$_{\Sigma_H}$ is given by the greatest closed subalgebra of $\mathbf{G}$ contained in $(G_E - \{e'\}, G_V)$. Since closed subalgebras of higher-order hypergraphs are again higher-order hypergraphs [1], this pushout algebra is again a higher-order hypergraph.

Finally, let $\mathbf{G}$ be a hierarchic (see Example 12) higher-order hypergraph. Let $e' \in G_E$ be an arc such that $n_a(e') \geqslant 1$, but $n_a(a_i^{\mathbf{G}}(e')) = 0$ for every $i = 1, \ldots, n_a(e')$. Then there is one and only one higher-order hypergraph $\mathbf{G}(\underline{n})$ with a closed homomorphism $m : \mathbf{G}(\underline{n}) \to \mathbf{G}$ such that $m_E(e) = e'$. And the application of rule $r_{\underline{n}}$ to $\mathbf{G}$ through such m removes e' (and all operations defined on it, of course) and, in cascade, all other arcs from which some sequence of abstraction operations yields e'. This shows that applying rules $r_{\underline{n}}$ in a recurrent way, one finally removes all arcs where some abstraction operation is defined.

As in Example 12, if $\mathbf{G}$ is not hierarchic then it may happen that there does not exist any closed homomorphism $m : \mathbf{G}(\underline{n}) \to \mathbf{G}$ for any $\underline{n}$.

To close this section, notice that Propositions 19 and 25 imply the following result, which shall be used in the next section.

Proposition 33. *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two closed homomorphisms of partial Γ-algebras, and let*

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar{g}} \\ \mathbf{B} & \xrightarrow[\bar{f}]{} & \mathbf{H} \end{array}$$

be their pushout in CDCQ-Alg$_\Gamma$. *Then it is also their pushout in* C-Alg$_\Gamma$.

6. HLR conditions

Contrary to the case of double-pushout algebraic transformation, where many HLR conditions for many results have been introduced (see [21] for a survey), the field of HLR conditions in single-pushout algebraic transformation is at its beginnings. In particular, and to our knowledge, only four HLR conditions, that entail the Local Church-Rosser Theorem, the Parallelism Closure and the Parallelism Theorem, respectively (cf. [12]), have been introduced so far in the literature; see Section 6.1 below.

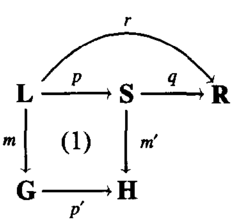
In this section, and in order to motivate our claim that single-pushout transformation in CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ and in CDCQ-Alg$_\Gamma$ has the same good properties as single-pushout transformation in P-TAlg$_\Gamma$ [15], on the one hand we prove that the aforementioned HLR conditions introduced in [12] are satisfied in all three cases taking total morphisms as occurrences. And, on the other hand, we also introduce several HLR conditions that are proved to entail some basic amalgamation properties, and we show that they are also satisfied in CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ and CDCQ-Alg$_\Gamma$, again taking total morphisms as occurrences. We plan to study elsewhere other features of the single-pushout transformation approaches introduced in this paper.

This section is organized as follows. In Section 6.1, we recall (and somehow simplify) the HLR conditions for parallelism introduced in [12]; in Section 6.2, we study the amalgamation properties of single-pushout HLR systems, and in particular we give HLR conditions entailing them; and, finally, in Section 6.3 we show that single-pushout transformation in CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ and CDCQ-Alg$_\Gamma$ satisfies all these HLR conditions.

Throughout this section we deal with a general category $\mathscr{C}$, whose morphisms shall be called *rules*, and a subcategory $\mathcal{O}$ of $\mathscr{C}$, whose morphisms shall be called *occurrences*, such that $\mathscr{C}$ and $\mathcal{O}$ have the same class of objects. In Section 6.3, $\mathscr{C}$ will be CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ or CDCQ-Alg$_\Gamma$ (and in some remarks P-TAlg$_\Gamma$), and the occurrences will be the corresponding total morphisms.

Before proceeding with the different subsections, let us recall the important concept of clean occurrences.

Definition 34. An occurrence $m : \mathbf{L} \to \mathbf{G}$ is said to be *clean* for a rule $r : \mathbf{L} \to \mathbf{R}$ if whenever r is factorized into two rules, $r = q \circ p$ with $\mathbf{L} \xrightarrow{p} \mathbf{S} \xrightarrow{q} \mathbf{R}$, and if the square (1) in



is a pushout square in $\mathscr{C}$ then the morphism m' is an occurrence.

6.1. HLR conditions for parallelism

Let us recall the HLR conditions introduced in [12] (where they are called "SPO-conditions for parallelism of HLR-systems of type SPO").

Definition 35 (*Ehrig and Löwe* [12, *Definition* 3.13]). We say that the pair $(\mathscr{C}, \mathcal{O})$ *satisfies the HLR conditions for parallelism* when it satisfies the following four conditions:

(HLRP1) There exists a pushout in $\mathscr{C}$ of every rule $r : \mathbf{L} \to \mathbf{R}$ and every occurrence $m : \mathbf{L} \to \mathbf{G}$.

(HLRP2) $\mathcal{O}$ has all coproducts, and they are preserved by the inclusion functor $\mathcal{O} \hookrightarrow \mathscr{C}$. (We shall denote the coproduct of two objects $\mathbf{A}$ and $\mathbf{B}$ by $\mathbf{A} + \mathbf{B}$.)

(HLRP3) If $f, g \in \mathscr{C}$ and $f \circ g \in \mathcal{O}$ then $g \in \mathcal{O}$.

(HLRP4) Let $f : \mathbf{A}_1 \to \mathbf{A}_2$ and $g : \mathbf{B}_1 \to \mathbf{B}_2$ be two rules in $\mathscr{C}$, let $f+g : \mathbf{A}_1+\mathbf{B}_1 \to \mathbf{A}_2 + \mathbf{B}_2$ be their coproduct morphism (their *parallel rule*), let $i : \mathbf{A}_1 \to \mathbf{A}_1 + \mathbf{B}_1$ be the corresponding coproduct embedding, and let $p : \mathbf{A}_1 + \mathbf{B}_1 \to \mathbf{C}$ be any rule in $\mathscr{C}$. If there exists some $q : \mathbf{C} \to \mathbf{A}_2 + \mathbf{B}_2$ such that $q \circ p = f + g$ then there exists some $q' : \mathbf{C} \to \mathbf{A}_2$ such that $q' \circ (p \circ i) = f$.

In [12] it is proved that single-pushout algebraic transformation in a category $\mathscr{C}$ through occurrences in $\mathcal{O}$ satisfies the *local Church-Rosser theorem*, the *parallelism closure* and the *parallelism theorem* (respectively, point (a), point (c) and points (b)–(d) in Theorem 37 below) whenever $(\mathscr{C}, \mathcal{O})$ satisfies these HLR conditions for parallelism.

Definition 36. Two direct derivations $\mathbf{G} \overset{p}{\Longrightarrow} \mathbf{H}$ and $\mathbf{G} \overset{q}{\Longrightarrow} \mathbf{H}'$, given by the pushout squares

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{p} & \mathbf{R} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ \mathbf{G} & \xrightarrow[p']{} & \mathbf{H} \end{array} \qquad \begin{array}{ccc} \mathbf{L}' & \xrightarrow{q} & \mathbf{R}' \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle n'} \\ \mathbf{G} & \xrightarrow[q']{} & \mathbf{H}' \end{array}$$

are said to be *parallel independent* when the occurrences of both productions are not destroyed by the other derivation: that is, when $q' \circ m, p' \circ n \in \mathcal{O}$.

Theorem 37 (Ehrig and Löwe [12, Theorems 3.5 and 3.12, Corollary 3.11]). *Let $\mathscr{C}$ and $\mathcal{O}$ be as in Definition* 35, *and assume that $(\mathscr{C}, \mathcal{O})$ satisfies the HLR conditions for parallelism.*

(a) *For every pair of parallel independent direct derivations $\mathbf{G} \overset{p}{\Longrightarrow} \mathbf{H}$ and $\mathbf{G} \overset{q}{\Longrightarrow} \mathbf{H}'$ there exists an object $\mathbf{X}$ and direct derivations $\mathbf{H} \overset{q}{\Longrightarrow} \mathbf{X}$ and $\mathbf{H}' \overset{p}{\Longrightarrow} \mathbf{X}$.*

(b) *If $\mathbf{G} \overset{p}{\Longrightarrow} \mathbf{H}$ and $\mathbf{G} \overset{q}{\Longrightarrow} \mathbf{H}'$ are two parallel independent direct derivations, such that we have $\mathbf{H} \overset{q}{\Longrightarrow} \mathbf{X}$ and $\mathbf{H}' \overset{p}{\Longrightarrow} \mathbf{X}$ by* (a), *then there exists a parallel derivation $\mathbf{G} \overset{p+q}{\Longrightarrow} \mathbf{X}$.*

(c) *Assume that all derivations are made through clean occurrences. For every pair of rules p, q and for every parallel direct derivation $\mathbf{G} \overset{p+q}{\Longrightarrow} \mathbf{X}$ there exist derivation* *sequences* $\mathbf{G} \overset{p}{\Longrightarrow} \mathbf{H} \overset{q}{\Longrightarrow} \mathbf{X}$ *and* $\mathbf{G} \overset{q}{\Longrightarrow} \mathbf{H}' \overset{p}{\Longrightarrow} \mathbf{X}$ *such that* $\mathbf{G} \overset{p}{\Longrightarrow} \mathbf{H}$ *and* $\mathbf{G} \overset{q}{\Longrightarrow} \mathbf{H}'$ *are parallel independent.*

(d) *Assume again that all derivations are made through clean occurrences. Then the constructions given by points* (b) *and* (c) *are in bijective correspondence, up to isomorphism.*

More specifically, condition (HLRP1) *entails* (a); *conditions* (HLRP1) *and* (HLRP2) *entail* (b) *and* (c); *and* (HLRP1) *to* (HLRP4) *entail* (d).

To close this subsection, we want to point out that (HLRP4) is equivalent to a much simpler property.

Proposition 38. *Let* $(\mathscr{C}, \mathscr{O})$ *be a pair satisfying condition* (HLRP2). *Then it satisfies condition* (HLRP4) *iff it satisfies the following additional property*:

(HLRP4′) *For every pair of objects* $\mathbf{A}, \mathbf{B}$ *in* $\mathscr{C}$, *there exists a rule* $r : \mathbf{A} \to \mathbf{B}$.

Proof. As far as the "only if" implication goes, apply (HLRP4) taking $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{B}$, $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{A}$, $f = \mathrm{Id}_{\mathbf{B}}$, $g = \mathrm{Id}_{\mathbf{A}}$ and $p = q = \mathrm{Id}_{\mathbf{B}+\mathbf{A}} : \mathbf{B} + \mathbf{A} \to \mathbf{B} + \mathbf{A}$. This yields a rule $q' : \mathbf{B} + \mathbf{A} \to \mathbf{B}$ such that $q' \circ i = \mathrm{Id}_{\mathbf{B}}$ (where $i : \mathbf{B} \to \mathbf{B} + \mathbf{A}$ stands for the corresponding coproduct embedding). Composing q' with the other pushout embedding $\mathbf{A} \to \mathbf{B} + \mathbf{A}$, we obtain a rule $\mathbf{A} \to \mathbf{B}$.

As to the "if" implication, let $f : \mathbf{A}_1 \to \mathbf{A}_2$, $g : \mathbf{B}_1 \to \mathbf{B}_2$, $i : \mathbf{A}_1 \to \mathbf{A}_1 + \mathbf{B}_1$, $p : \mathbf{A}_1 + \mathbf{B}_1 \to \mathbf{C}$ and $q : \mathbf{C} \to \mathbf{A}_2 + \mathbf{B}_2$ be as in the statement of condition (HLRP4), and let $i' : \mathbf{A}_2 \to \mathbf{A}_2 + \mathbf{B}_2$ be the corresponding coproduct embedding. The existence of a rule $r : \mathbf{B}_2 \to \mathbf{A}_2$ entails the existence of a rule $\pi : \mathbf{A}_2 + \mathbf{B}_2 \to \mathbf{A}_2$ such that $\pi \circ i' = \mathrm{Id}_{\mathbf{A}_2}$. Take $q' = \pi \circ q : \mathbf{C} \to \mathbf{A}_2$. Then we have

$$q' \circ p \circ i = \pi \circ q \circ p \circ i = \pi \circ (f + g) \circ i = \pi \circ i' \circ f = f$$

as we wanted to prove. □

Notice that condition (HLRP4′) is automatically satisfied by any category having a zero object $\mathbf{0}$, that is, an object that is both initial and terminal, since then there exists a rule $\mathbf{A} \to \mathbf{0} \to \mathbf{B}$ for every pair of objects $\mathbf{A}, \mathbf{B}$ in $\mathscr{C}$.

6.2. HLR conditions for amalgamation

Non-parallel-independent derivations can still be "merged" or amalgamated, by means of a construction known as *amalgamated sum*.

Amalgamation is a generalization of parallelism, motivated by the idea of rule gluing [11], and it has been widely studied for double-pushout derivations [4] and for single-pushout derivations [15]. However, no HLR conditions for amalgamation are known yet, neither for double-pushout nor for single-pushout derivations. In this section we give HLR conditions that entail the amalgamation properties of single-pushout HLR systems.

Recall that $\mathscr{C}$ is a category (whose morphisms are called rules) and $\mathscr{O}$ is a subcategory of $\mathscr{C}$ (whose morphisms are called occurrences) such that $\mathscr{C}$ and $\mathscr{O}$ have the same class of objects. Let us assume for the moment that the pair $(\mathscr{C}, \mathscr{O})$ satisfies the following two conditions (that we shall later impose as HLR conditions).

(HLRA1) There exists a pushout in $\mathscr{C}$ of every rule $r : \mathbf{L} \to \mathbf{R}$ and occurrence $m : \mathbf{L} \to \mathbf{G}$.[9]

(HLRA2) $\mathscr{O}$ has all pushouts, and they are preserved by the inclusion functor $\mathscr{O} \hookrightarrow \mathscr{C}$.

Then clean occurrences satisfy, under these assumptions, the following properties that shall be used in the sequel.

Lemma 39. *Let* $(\mathscr{C}, \mathscr{O})$ *satisfy condition* (HLRA1). *Then the following properties hold.*

(i) *For every pair of composable rules* $r_1 : \mathbf{L} \to \mathbf{R}$ *and* $r_2 : \mathbf{R} \to \mathbf{S}$, *every clean occurrence for* $r_2 \circ r_1$ *is also a clean occurrence for* r_1.

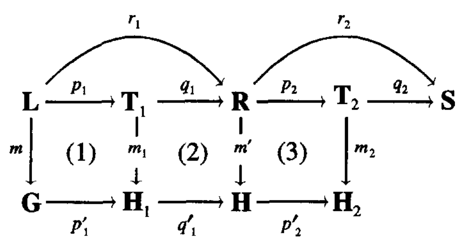
(ii) *For every pair of composable rules* $r_1 : \mathbf{L} \to \mathbf{R}$ *and* $r_2 : \mathbf{R} \to \mathbf{S}$, *if* $m : \mathbf{L} \to \mathbf{G}$ *is a clean occurrence for* $r_2 \circ r_1$ *and if* (1) *in the diagram*

$$\begin{array}{ccccc} \mathbf{L} & \xrightarrow{r_1} & \mathbf{R} & \xrightarrow{r_2} & \mathbf{S} \\ {\scriptstyle m}\downarrow & (1) & \downarrow{\scriptstyle m'} & & \\ \mathbf{G} & \xrightarrow[r_1']{} & \mathbf{H} & & \end{array}$$

is a pushout of r_1 *and* m *in* $\mathscr{C}$, *then* m' *is a clean occurrence for* r_2.

(iii) *Let* $r : \mathbf{L} \to \mathbf{R}$ *and* $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ *be two rules for which there exist a clean occurrence* $i : \mathbf{L} \to \mathbf{L}_1$ *for* r *and an occurrence* $j : \mathbf{R} \to \mathbf{R}_1$ *such that* $r_1 \circ i = j \circ r$. *If* m *is a clean occurrence for* r_1 *then* $m \circ i$ *is a clean occurrence for* r.

Proof. As far as points (i) and (ii) go, let $m : \mathbf{L} \to \mathbf{G}$ be a clean occurrence for $r_2 \circ r_1$, let $r_1 = q_1 \circ p_1$ be any factorization of r_1, with $\mathbf{L} \xrightarrow{p_1} \mathbf{T}_1 \xrightarrow{q_1} \mathbf{R}$, and let $r_2 = q_2 \circ p_2$ be any factorization of r_2, with $\mathbf{R} \xrightarrow{p_2} \mathbf{T}_2 \xrightarrow{q_2} \mathbf{S}$. Consider the diagram



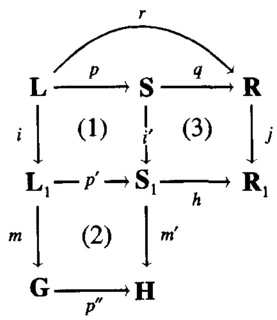
where squares (1)–(3) are pushout squares in $\mathscr{C}$.

Since m is clean for $r_2 \circ r_1 = (q_2 \circ p_2 \circ q_1) \circ p_1$, we have that m_1 is an occurrence. This proves that m is also clean for r_1, as it is stated in point (i).

On the other hand, since $r_2 \circ r_1 = q_2 \circ (p_2 \circ q_1 \circ p_1)$ and (1)+(2)+(3) is a pushout square, we have that m_2 is an occurrence. This proves that m' is clean for r_2, as it is stated in point (ii).

[9] Notice that condition (HLRA1) coincides with condition (HLRP1).

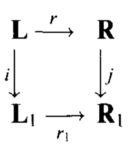
As far as point (iii) goes, let $r = q \circ p$ be any factorization of r, with $\mathbf{L} \xrightarrow{p} \mathbf{S} \xrightarrow{q} \mathbf{R}$, and consider the commutative diagram



where (1) and (2) are pushout squares, so that (1)+(2) yields a pushout square for p and $m \circ i$, and h is the unique morphism $h : \mathbf{S}_1 \to \mathbf{R}_1$ (induced by the fact that (1) is a pushout square) such that $h \circ i' = j \circ q$ and $h \circ p' = r_1$. Since m is a clean occurrence for r_1, we have that $m' \in \mathcal{O}$, and since i is a clean occurrence for r, we have that $i' \in \mathcal{O}$. Therefore $m' \circ i' \in \mathcal{O}$, which shows that $m \circ i$ is a clean occurrence for r. □

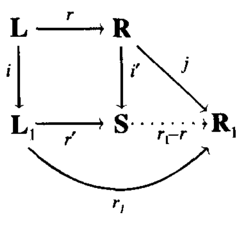
Under the hypothesis of point (iii) in this lemma, it is said that r is a *subrule* of r_1. Subrules and remainders, together with amalgamated sums to be defined later, are the key concepts in amalgamation.

Definition 40. A rule $r : \mathbf{L} \to \mathbf{R}$ is a *subrule* of a rule $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ when there exist a clean occurrence $i : \mathbf{L} \to \mathbf{L}_1$ for r and an occurrence $j : \mathbf{R} \to \mathbf{R}_1$ such that $j \circ r = r_1 \circ i$



We shall denote it by $r \leqslant_{(i,j)} r_1$, or simply by $r \leqslant r_1$, when the pair (i,j) is irrelevant or understood.

If $r \leqslant_{(i,j)} r_1$ then let $\mathbf{S}$ together with $i' : \mathbf{R} \to \mathbf{S}$ and $r' : \mathbf{L}_1 \to \mathbf{S}$ be a pushout of r and i in $\mathscr{C}$. The *(i,j)-remainder* of r_1 w.r.t. r, which shall be denoted by $r_1 -_{(i,j)} r : \mathbf{S} \to \mathbf{R}_1$ or simply by $r_1 - r : \mathbf{S} \to \mathbf{R}_1$ when the pair (i,j) is irrelevant or understood, is the unique rule $\mathbf{S} \to \mathbf{R}_1$ such that $(r_1 - r) \circ i' = j$ and $(r_1 - r) \circ r' = r_1$.



Subrules allow to decompose derivations.

Proposition 41. *Assume that all derivations are made through clean occurrences and that* $(\mathscr{C}, \mathscr{O})$ *satisfies condition* (HLRA1) *above, and let* $r : \mathbf{L} \to \mathbf{R}$ *and* $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ *be two rules in* $\mathscr{C}$ *such that* $r \leqslant r_1$. *If* $\mathbf{G} \overset{r_1}{\Longrightarrow} \mathbf{H}$ *then there exists an object* $\mathbf{H}'$ *such that* $\mathbf{G} \overset{r}{\Longrightarrow} \mathbf{H}' \overset{r_1 - r}{\Longrightarrow} \mathbf{H}$.

Proof. Let $i : \mathbf{L} \to \mathbf{L}_1$ and $j : \mathbf{R} \to \mathbf{R}_1$ be occurrences (the first one, clean for r) making r to be a subrule of r_1. Assume that $\mathbf{G} \overset{r_1}{\Longrightarrow} \mathbf{H}$ and let $m : \mathbf{L}_1 \to \mathbf{G}$ be a clean occurrence for r_1 used to derive $\mathbf{H}$ from $\mathbf{G}$ by the application of rule r_1.

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbf{L} & \xrightarrow{r} & \mathbf{R} & & \\
{\scriptstyle i}\downarrow & (1) & \downarrow{\scriptstyle i'} & & \\
\mathbf{L}_1 & \xrightarrow[r']{} & \mathbf{S} & \xrightarrow{r_1 - r} & \mathbf{R}_1 \\
{\scriptstyle m}\downarrow & (2) & \downarrow{\scriptstyle m'} & (3) & \downarrow{\scriptstyle m''} \\
\mathbf{G} & \xrightarrow[r'']{} & \mathbf{H}' & \xrightarrow[p]{} & \mathbf{H}''
\end{array}
$$

where all three squares (1)–(3) are pushout squares in $\mathscr{C}$.

Then on the one hand (1)+(2) is a pushout square for r and $m \circ i$, and since $m \circ i$ is a clean occurrence for r by Lemma 39(iii), we have that $\mathbf{G} \overset{r}{\Longrightarrow} \mathbf{H}'$. And on the other hand, since (2)+(3) is a pushout square for $r_1 = (r_1 - r) \circ r'$ and m, and since m is a clean occurrence for r_1, we have that $\mathbf{H}''$ is the derived object of $\mathbf{G}$ by the application of rule r_1 through occurrence m. Therefore $\mathbf{H}'' = \mathbf{H}$ (up to isomorphism). And since m' is a clean occurrence for $r_1 - r$ by Lemma 39(ii), we have that $\mathbf{H}' \overset{r_1 - r}{\Longrightarrow} \mathbf{H}$. □

Definition 42. Let $r : \mathbf{L} \to \mathbf{R}$, $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ and $r_2 : \mathbf{L}_2 \to \mathbf{R}_2$ be three rules in $\mathscr{C}$ such that $r \leqslant_{(i_1, j_1)} r_1$ and $r \leqslant_{(i_2, j_2)} r_2$. Let

$$
\begin{array}{ccc}
\mathbf{L} & \xrightarrow{i_1} & \mathbf{L}_1 \\
{\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle i_2'} \\
\mathbf{L}_2 & \xrightarrow[i_1']{} & \mathbf{L}'
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{R} & \xrightarrow{j_1} & \mathbf{R}_1 \\
{\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle j_2'} \\
\mathbf{R}_2 & \xrightarrow[j_1']{} & \mathbf{R}'
\end{array}
$$

be the pushout squares in $\mathscr{O}$ (and also in $\mathscr{C}$) of i_1 and i_2, and of j_1 and j_2 respectively. Then the *amalgamated sum rule* $r_1 +_r r_2 : \mathbf{L}' \to \mathbf{R}'$ is defined as the unique morphism in C (which exists by the universal property of pushouts) making the cube given in Fig. 2 commutative.

This is time now to introduce the HLR conditions for amalgamation, which, as we have said, will include the conditions introduced at the beginning of this subsection.

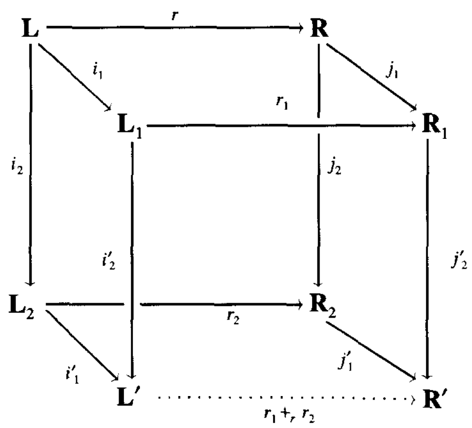


Fig. 2.

Definition 43. We say that the pair $(\mathscr{C}, \mathcal{O})$ satisfies the *HLR conditions for amalgamation* when it satisfies the following properties:[10]

(HLRA1) There exists a pushout in $\mathscr{C}$ of every rule $r : \mathbf{L} \to \mathbf{R}$ and occurrence $m : \mathbf{L} \to \mathbf{G}$.

(HLRA2) $\mathcal{O}$ has all pushouts, and they are preserved by the inclusion functor $\mathcal{O} \hookrightarrow \mathscr{C}$.

(HLRA3) If $f, g \in \mathscr{C}$ and $f \circ g \in \mathcal{O}$ then $g \in \mathcal{O}$.

(HLRA4) If m is a clean occurrence for $r : \mathbf{L} \to \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{S}$ is any occurrence, then m is also clean for $f \circ r$.

Lemma 44. *Assume that $(\mathscr{C}, \mathcal{O})$ satisfies the HLR conditions for amalgamation. If r is a subrule of r_1 and r_2 then r_1 and r_2 are subrules of $r_1 +_r r_2$.*

Proof. Assume $r \leqslant_{(i_1, j_1)} r_1$ and $r \leqslant_{(i_2, j_2)} r_2$, and consider the following commutative diagram given in Fig. 3 where $r_1 = q \circ p$, $r_1 +_r r_2 = q' \circ p'$, the left-hand side rectangle in the front face (the one involving $\mathbf{L}_1$, $\mathbf{K}_1$, $\mathbf{L}'$ and $\mathbf{K}'$) is a pushout square in $\mathscr{C}$, and both the left-hand side and the right-hand side faces of the cube are pushout squares in $\mathcal{O}$ (and therefore also in $\mathscr{C}$). Since i_2 is a clean occurrence for r, it is also clean for $j_1 \circ r$ by condition (HLRA4). Therefore, from $j_1 \circ r = r_1 \circ i_1 = q \circ (p \circ i_1)$ and the composition of pushout squares

$$\begin{array}{ccccc} \mathbf{L} & \xrightarrow{i_1} & \mathbf{L}_1 & \xrightarrow{p} & \mathbf{K}_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle i'_2} & & \downarrow{\scriptstyle \bar{i}_2} \\ \mathbf{L}_2 & \xrightarrow[i'_1]{} & \mathbf{L}' & \xrightarrow[p']{} & K' \end{array}$$

[10] Notice that conditions (HLRA1) and (HLRA3) coincide with conditions (HLRP1) and (HLRP3), respectively.

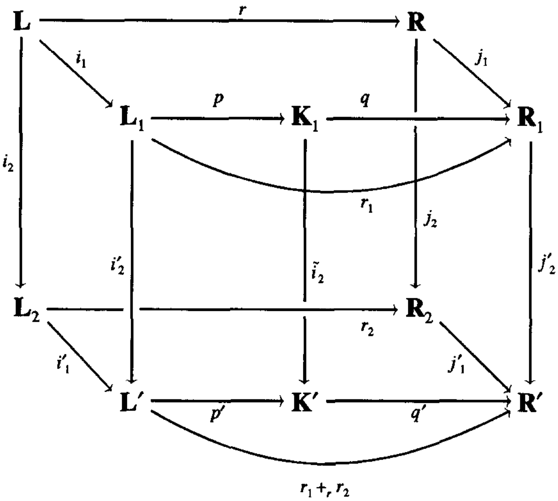


Fig. 3.

we obtain that $\tilde{\imath}_2$ is an occurrence. This shows that $i_2' : \mathbf{L}_1 \to \mathbf{L}'$ is a clean occurrence for r_1, and therefore, since $j_2' : \mathbf{R}_1 \to \mathbf{R}'$ is an occurrence by construction, that $r_1 \leqslant_{(i_2', j_2')} r_1 +_r r_2$. By symmetry, $r_2 \leqslant_{(i_1', j_1')} r_1 +_r r_2$. □

Common subrules also induce a synchronization mechanism for derivations. When two rules share a common subrule, the effect of the shared subrule is produced only once and the whole derivation is obtained by derivations through rule remainders.

Proposition 45. *Assume that all derivations are made through clean occurrences and that $(\mathscr{C}, \mathscr{O})$ satisfies the HLR conditions for amalgamation, and let $r : \mathbf{L} \to \mathbf{R}$, $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ and $r_2 : \mathbf{L}_2 \to \mathbf{R}_2$ be three rules in $\mathscr{C}$ such that $r \leqslant r_1$ and $r \leqslant r_2$. If $\mathbf{G} \overset{r_1 +_r r_2}{\Longrightarrow} \mathbf{H}$ then there exist $\mathbf{H}'$ and $\mathbf{H}''$ such that*

$$\mathbf{G} \overset{r}{\Longrightarrow} \mathbf{H}' \overset{r_1 - r}{\Longrightarrow} \mathbf{H}'' \overset{r_2 - r}{\Longrightarrow} \mathbf{H}.$$

Proof. Since by the previous lemma we have that $r_1 \leqslant r_1 +_r r_2$, by Proposition 41 there exists an object $\mathbf{H}''$ such that $\mathbf{G} \overset{r_1}{\Longrightarrow} \mathbf{H}'' \overset{(r_1 +_r r_2) - r_1}{\Longrightarrow} \mathbf{H}$. And since $r \leqslant r_1$, again by Proposition 41 we have that there exists an object $\mathbf{H}'$ such that $\mathbf{G} \overset{r}{\Longrightarrow} \mathbf{H}' \overset{r_1 - r}{\Longrightarrow} \mathbf{H}''$.

It is enough to show that $\mathbf{H}'' \overset{r_2 - r}{\Longrightarrow} \mathbf{H}$, and to do that it is enough to find a pushout square in $\mathscr{C}$

$$\begin{array}{ccc} \mathbf{K}_2 & \xrightarrow{r_2 - r} & \mathbf{R}_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \mathbf{N}' & \xrightarrow[(r_1 +_r r_2) - r_1]{} & \mathbf{R}' \end{array}$$

with f a clean occurrence for $r_2 - r$.

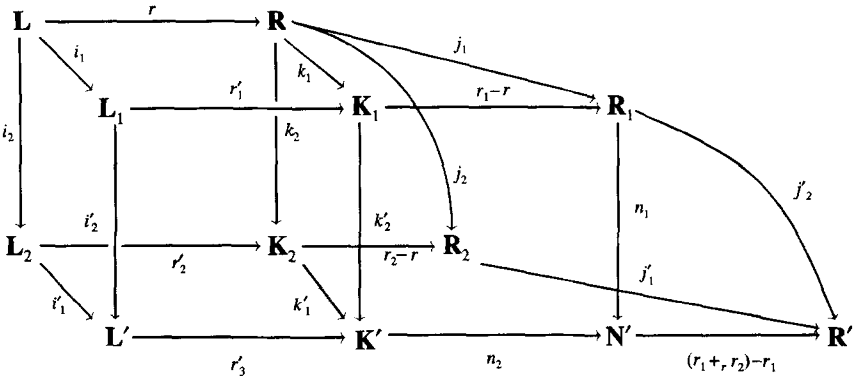


Fig. 4.

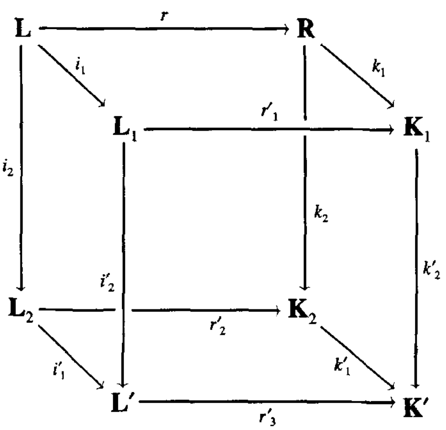


Fig. 5.

Consider Fig. 4 (referred to henceforth as *the main diagram*) which is obtained as follows:

- The squares $\mathbf{L}, \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}'$ and $\mathbf{R}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}'$ are pushout squares in $\mathcal{O}$, and therefore also in $\mathscr{C}$.
- The squares $\mathbf{L}, \mathbf{R}, \mathbf{L}_1, \mathbf{K}_1$, $\mathbf{L}, \mathbf{R}, \mathbf{L}_2, \mathbf{K}_2$ and $\mathbf{L}_2, \mathbf{K}_2, \mathbf{L}', \mathbf{K}'$ are pushout squares in $\mathscr{C}$. Let $r_1 - r : \mathbf{K}_1 \to \mathbf{R}_1$ and $r_2 - r : \mathbf{K}_2 \to \mathbf{R}_2$ be respectively the (i_1, j_1)-remainder of r_1 w.r.t. r and the (i_2, j_2)-remainder of r_2 w.r.t. r (Definition 40).
- $k'_2 : \mathbf{K}_1 \to \mathbf{K}'$ is the only rule such that $k'_2 \circ k_1 = k'_1 \circ k_2$ and $k'_2 \circ r'_1 = r'_3 \circ i'_2$. Then, we have the commutative cube of Fig. 5 (extracted from the left-hand side of the main diagram) whose top, bottom, back and left-hand side faces are pushout squares. Standard properties of pushouts entail that the remaining two faces, corresponding to squares $\mathbf{R}, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}'$ and $\mathbf{L}_1, \mathbf{K}_1, \mathbf{L}', \mathbf{K}'$, are also pushout squares.

- The square $\mathbf{K}_1, \mathbf{R}_1, \mathbf{K}', \mathbf{N}'$ is a pushout square. Then, the composition of pushout squares

$$\begin{array}{ccccc} \mathbf{L}_1 & \xrightarrow{r_1'} & \mathbf{K}_1 & \xrightarrow{r_1-r} & \mathbf{R}_1 \\ {\scriptstyle i_2'}\downarrow & & \downarrow{\scriptstyle k_2'} & & \downarrow{\scriptstyle n_1} \\ \mathbf{L}' & \xrightarrow[r_3']{} & \mathbf{K}' & \xrightarrow[n_2]{} & \mathbf{N}' \end{array}$$

yields a pushout

$$\begin{array}{ccc} \mathbf{L}_1 & \xrightarrow{r_1} & \mathbf{R}_1 \\ {\scriptstyle i_2'}\downarrow & & \downarrow{\scriptstyle n_1} \\ \mathbf{L}' & \xrightarrow[n_2 \circ r_3']{} & \mathbf{N}' \end{array}$$

Since i_2' is a clean occurrence for r_1 by the proof of Lemma 44, we have that k_2' and n_1 are occurrences.
- Let $(r_1 +_r r_2) - r_1 : \mathbf{N}' \to \mathbf{R}'$ be the (i_2', j_2')-remainder of $r_1 +_r r_2$ w.r.t. r_1; in particular it satisfies that $((r_1 +_r r_2) - r_1) \circ n_1 = j_2'$ and $((r_1 +_r r_2) - r_1) \circ (n_2 \circ r_3') = r_1 +_r r_2$. We shall show that the square

$$\begin{array}{ccc} \mathbf{K}_2 & \xrightarrow{r_2 - r} & \mathbf{R}_2 \\ {\scriptstyle f := n_2 \circ k_1'}\downarrow & & \downarrow{\scriptstyle j_1'} \\ \mathbf{N}' & \xrightarrow[(r_1 +_r r_2) - r_1]{} & \mathbf{R}' \end{array}$$

is a pushout square in $\mathscr{C}$ and that $n_2 \circ k_1'$ is a clean occurrence for $r_2 - r$. As we have mentioned at the beginning of this proof, this will finish the proof.

To do that, consider the commutative prism, given in Fig. 6 which is extracted from the right-hand side of the main diagram.

In this commutative diagram, the right-hand side face $\mathbf{R}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}'$ is a pushout square by construction. And the left-hand side face $\mathbf{R}, \mathbf{R}_1, \mathbf{K}_2, \mathbf{N}'$ is a pushout square, because it is the composition of pushout squares

$$\begin{array}{ccccc} \mathbf{R} & \xrightarrow{k_1} & \mathbf{K}_1 & \xrightarrow{r_1-r} & \mathbf{R}_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle k_2'} & & \downarrow{\scriptstyle n_1} \\ \mathbf{K}_2 & \xrightarrow[k_1']{} & \mathbf{K}' & \xrightarrow[n_2]{} & \mathbf{N}' \end{array} \quad = \quad \begin{array}{ccc} \mathbf{R} & \xrightarrow{j_1} & \mathbf{R}_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle n_1} \\ \mathbf{L}_2 & \xrightarrow[f]{} & \mathbf{N}' \end{array}$$

This has two consequences. On the one hand, standard properties of pushouts entail that the bottom square, which is the one we are interested in, is also a pushout square. And on the other hand, since j_1 and k_2 are occurrences (the latter, because i_2 is a clean occurrence for r), it turns out that $f : \mathbf{K}_2 \to \mathbf{N}'$ and $n_1 : \mathbf{R}_1 \to \mathbf{N}'$ are occurrences.

Finally, in order to show that f is clean for $r_2 - r$, let $r_2 - r = q \circ p$ be any decomposition of $r_2 - r$, with $\mathbf{K}_2 \xrightarrow{p} \mathbf{S}_2 \xrightarrow{q} \mathbf{R}_2$, and let

$$\begin{array}{ccc} \mathbf{K}_2 & \xrightarrow{p} & \mathbf{S}_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathbf{N}' & \xrightarrow[p']{} & \mathbf{S}_3 \end{array}$$

be a pushout square in $\mathscr{C}$. We have to prove that f' is an occurrence.

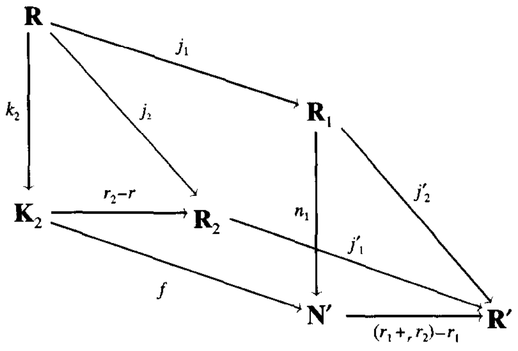


Fig. 6.

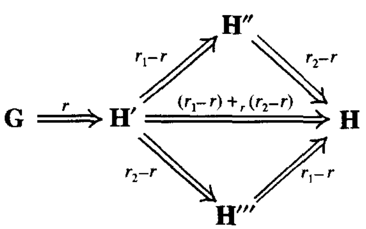
Notice now that we have the composition of pushouts

$$\begin{array}{ccccc} \mathbf{R} & \xrightarrow{k_2} & \mathbf{K}_2 & \xrightarrow{p} & \mathbf{S}_2 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathbf{R}_1 & \xrightarrow[n_1]{} & \mathbf{N}' & \xrightarrow[p']{} & \mathbf{S}_3 \end{array} \quad = \quad \begin{array}{ccc} \mathbf{R} & \xrightarrow{p\circ k_2} & \mathbf{S}_2 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathbf{R}_1 & \xrightarrow[p'\circ n_1]{} & \mathbf{S}_3 \end{array}$$

where j_1 is an occurrence by definition and $p \circ k_2$ is an occurrence because $q \circ (p \circ k_2) = j_2$ is an occurrence (and then condition (HLRA3) applies). Therefore f' is an occurrence, because of (HLRA2). □

Proposition 46. *Assume that all derivations are made through clean occurrences and that $(\mathscr{C},\mathscr{O})$ satisfies the HLR conditions for amalgamation, as well as condition (HLRP2) in Definition 35. Let $r : \mathbf{L} \to \mathbf{R}$, $r_1 : \mathbf{L}_1 \to \mathbf{R}_1$ and $r_2 : \mathbf{L}_2 \to \mathbf{R}_2$ be three rules in $\mathscr{C}$ such that $r \leqslant r_1$ and $r \leqslant r_2$. If $\mathbf{G} \stackrel{r_1+_r r_2}{\Longrightarrow} \mathbf{H}$ then there exists $\mathbf{H}'$ and two parallel independent derivations $\mathbf{H}' \stackrel{r_1-r}{\Longrightarrow} \mathbf{H}''$ and $\mathbf{H}' \stackrel{r_2-r}{\Longrightarrow} \mathbf{H}'''$ such that $\mathbf{G} \stackrel{r}{\Longrightarrow} \mathbf{H}' \stackrel{(r_1-r)+_r(r_2-r)}{\Longrightarrow} \mathbf{H}$.*

The situation dealt with in Proposition 46 is illustrated by the derivation diagram



where the upper path corresponds to Proposition 45.

Proof. If $(\mathscr{C},\mathscr{O})$ satisfies the HLR conditions for amalgamation and (HLRP2) then points (a) and (b) in Theorem 37 hold. Therefore, this proposition will be implied by

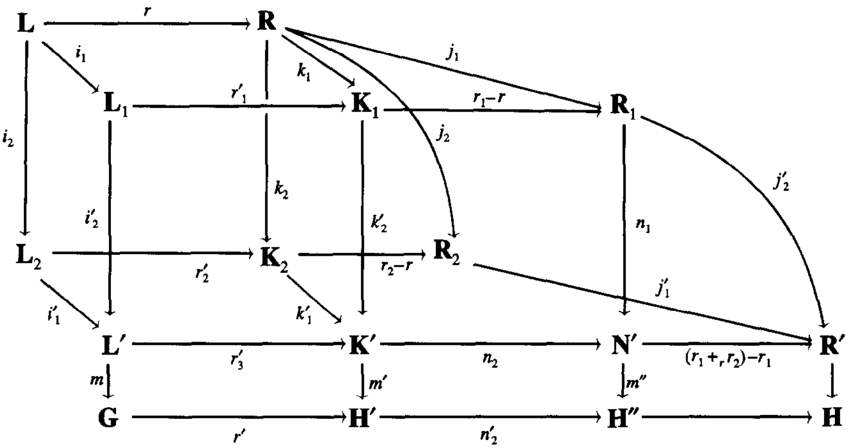


Fig. 7.

the previous one if, with the notations therein, we can find the two parallel independent derivations $\mathbf{H}' \overset{r_1-r}{\Longrightarrow} \mathbf{H}''$ and $\mathbf{H}' \overset{r_2-r}{\Longrightarrow} \mathbf{H}'''$.

Now consider the main diagram in the proof of the previous proposition, enlarged with the pushout square corresponding to the direct derivation $\mathbf{G} \overset{r_1+_r r_2}{\Longrightarrow} \mathbf{H}$ through a clean occurrence $m : \mathbf{L}' \longrightarrow \mathbf{G}$, decomposed into several pushout squares that correspond to the direct derivations $\mathbf{G} \overset{r}{\Longrightarrow} \mathbf{H}'$, $\mathbf{H}' \overset{r_1-r}{\Longrightarrow} \mathbf{H}''$ and $\mathbf{H}'' \overset{r_2-r}{\Longrightarrow} \mathbf{H}$ (Fig. 7). In this diagram, we have that $n_2' \circ (m' \circ k_1')$ is an occurrence, because $n_2' \circ (m' \circ k_1') = m'' \circ n_2 \circ k_1'$ where $n_2 \circ k_1'$ is an occurrence, as it has been seen in the proof of the previous proposition, and m'' is an occurrence, because m is a clean occurrence for $r_1 +_r r_2$ by assumption.

By symmetry, if

$$\begin{array}{ccc} \mathbf{K}_2 & \xrightarrow{r_2-r} & \mathbf{R}_2 \\ {\scriptstyle k_1'}\downarrow & & \downarrow{\scriptstyle m_1} \\ \mathbf{K}' & \xrightarrow[m_2]{} & \mathbf{M}' \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m'''} \\ \mathbf{H}' & \xrightarrow[m_2']{} & \mathbf{H}''' \end{array}$$

are two pushout squares in $\mathscr{C}$ then $m_2' \circ (m' \circ k_2')$ is an occurrence. Then the pushout squares

$$\begin{array}{ccc} \mathbf{K}_1 & \xrightarrow{r_1-r} & \mathbf{R}_1 \\ {\scriptstyle m' \circ k_2'}\downarrow & & \downarrow{\scriptstyle m'' \circ n_1} \\ \mathbf{H}' & \xrightarrow[n_2']{} & \mathbf{H}'' \end{array} \qquad \begin{array}{ccc} \mathbf{K}_2 & \xrightarrow{r_2-r} & \mathbf{R}_2 \\ {\scriptstyle m' \circ k_1'}\downarrow & & \downarrow{\scriptstyle m''' \circ m_1} \\ \mathbf{H}' & \xrightarrow[m_2']{} & \mathbf{H}''' \end{array}$$

correspond to parallel independent derivations $\mathbf{H}' \overset{r_1-r}{\Longrightarrow} \mathbf{H}''$ and $\mathbf{H}' \overset{r_2-r}{\Longrightarrow} \mathbf{H}'''$. As we mentioned earlier, this finishes the proof. □

6.3. The case of conformisms, c-quomorphisms and cdc-quomorphisms

It turns out that the HLR conditions for parallelism and amalgamation listed in Definitions 35 and 43, respectively, are satisfied by every pair of the form $(\mathscr{C}, \mathscr{O})$ when $\mathscr{C}$ is any one of the categories CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ or CDCQ-Alg$_\Gamma$ and $\mathscr{O}$ is the corresponding subcategory given by the total morphisms. Therefore, single-pushout algebraic transformation of partial unary algebras using conformisms, c-quomorphisms or cdc-quomorphisms (taking in all three cases totally defined morphisms as occurrences) satisfy points (a)–(d) in Theorem 37. We start by proving this for the case of parallelism.

Proposition 47. *The three pairs* (CF-Alg$_\Gamma$,TCF-Alg$_\Gamma$), (CQ-Alg$_\Gamma$,C-Alg$_\Gamma$) *and* (CDCQ-Alg$_\Gamma$,C-Alg$_\Gamma$) *satisfy the HLR conditions for parallelism, for every graph structure Γ.*

Proof. The categories CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ and CDCQ-Alg$_\Gamma$ have all pushouts. Moreover, the coproduct of two partial Γ-algebras $\mathbf{A}$ and $\mathbf{B}$ is given by the usual disjoint union $\mathbf{A}+\mathbf{B}$ in C-Alg$_\Gamma$, CQ-Alg$_\Gamma$, CF-Alg$_\Gamma$, CDCQ-Alg$_\Gamma$ and TCF-Alg$_\Gamma$ (it can be seen, for instance, noticing that the coproduct of $\mathbf{A}$ and $\mathbf{B}$ in any one of these categories is the pushout of the empty morphisms $\emptyset \to \mathbf{A}$ and $\emptyset \to \mathbf{B}$, and using also Corollaries 15, 19 and 33; see also [2, 9, 19]).

Condition (HLRP3) for all three pairs is trivial for set-theoretical reasons.

Finally, as far as condition (HLRP4) goes, by Proposition 38 it is enough to check condition (HLRP4′), which is easily seen to be satisfied in this case: given any two partial Γ-algebras $\mathbf{A}$ and $\mathbf{B}$, the empty mapping is always a cdc-quomorphism from $\mathbf{A}$ to $\mathbf{B}$. □

In order to give full content to the Parallelism Closure and the Parallelism Theorem for single-pushout transformation in CF-Alg$_\Gamma$, C-Alg$_\Gamma$ and CDCQ-Alg$_\Gamma$, we have to identify in each category which occurrences are clean for a given rule.

Definition 48. Let $\mathbf{L}$, $\mathbf{G}$ and $\mathbf{R}$ be partial Γ-algebras, with carriers L, G and R, respectively, and let $m : L \to G$ and $r : L \to R$ be respectively a total and a partial mapping of S-sets.

(i) m is *d-injective* w.r.t. r (see for instance [15, Definition 3.10(2)]) when for every $s \in S$ and $x, y \in L_s$, if $m_s(x) = m_s(y)$ then $x, y \in \operatorname{Dom} r_s$ or $x = y$.

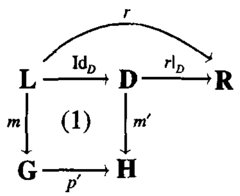
(ii) m is *closed-injective* w.r.t. r when for every $s \in S$ and $x, y \in L_s$, if $m_s(x) = m_s(y)$ then $x, y \in \operatorname{Dom} r_s$ or $C_{\mathbf{L}}(\{x\}) = C_{\mathbf{L}}(\{y\})$.[11]

D-injectivity and d-closed-injectivity are different notions: see, for instance, the Remark after Proposition 52.

[11] As in Lemma 22, by $C_{\mathbf{L}}(\{z\})$ we denote the closed subset of $\mathbf{L}$ generated by the S-set with all carriers of all sorts empty, except the carrier of sort s, which is $\{z\}$.

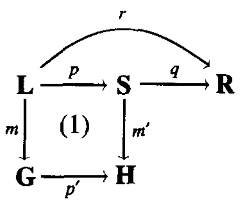
Proposition 49. *Given a rule* $r : \mathbf{L} \to \mathbf{R}$ *and an occurrence* $m : \mathbf{L} \to \mathbf{G}$ *in* CF-Alg$_\Gamma$, *m is clean for r iff it is d-injective w.r.t. r.*

Proof. Assume that m is a clean occurrence for r and let $x, y \in L_s$ such that $m_s(x) = m_s(y)$ and $x \neq y$. Consider the subset D of L with $D_s = \operatorname{Dom} r_s \cup \{y\}$ and $D_t = \operatorname{Dom} r_t$ for every $t \neq s$. Let $\mathbf{D}$ be the relative subalgebra of $\mathbf{L}$ supported on D and consider the factorization $r|_D \circ \mathrm{Id}_D$ of r with $\mathrm{Id}_D : \mathbf{L} \to \mathbf{D}$ the c-quomorphism given by the identity on D and $r|_D : \mathbf{D} \to \mathbf{R}$ the conformism $r : \mathbf{L} \to \mathbf{R}$ taken with source algebra $\mathbf{D}$. Consider the diagram



where (1) is a pushout square in CF-Alg$_\Gamma$. Since m is clean, m' has to be total and therefore $m^{-1}(m(D)) = D$ by Corollary 5(ii). And since $x \neq y$ and $y \in D_s$, $m_s^{-1}(m_s(x)) \subseteq D_s$ and then $x \in \operatorname{Dom} r_s$. Therefore, m is d-injective w.r.t. r.

Conversely, let $r : \mathbf{L} \to \mathbf{R}$ be a conformism of partial Γ-algebras and let $m : \mathbf{L} \to \mathbf{G}$ be a total conformism, d-injective w.r.t. r. Let $r = q \circ p$ be any factorization of r, with $\mathbf{L} \xrightarrow{p} \mathbf{S} \xrightarrow{q} \mathbf{R}$, and consider the diagram



where (1) is a pushout square in CF-Alg$_\Gamma$. Let $x \in m_s^{-1}(m_s(\operatorname{Dom} p_s))$, say $m_s(x) = m_s(y)$ with $y \in \operatorname{Dom} p_s$. Then d-injectivity implies that either $x = y \in \operatorname{Dom} p_s$ or $x, y \in \operatorname{Dom} r_s \subseteq \operatorname{Dom} p_s$. Since m is total and $s \in S$ any sort, this implies that $m^{-1}(m(\operatorname{Dom} p)) = \operatorname{Dom} p$ and therefore that m' is total, by Corollary 5(ii). Therefore, m is clean for r. $\square$

The proof of this proposition applies *mutatis mutandi* to c-quomorphisms, yielding:

Proposition 50. *Given a rule* $r : \mathbf{L} \to \mathbf{R}$ *and an occurrence* $m : \mathbf{L} \to \mathbf{G}$ *in* CQ-Alg$_\Gamma$, *m is clean for r iff it is d-injective w.r.t. r.*

As far as cdc-quomorphisms, we need the following lemma.

Lemma 51. *Let*

$$\begin{array}{ccc} \mathbf{K} & \xrightarrow{f} & \mathbf{A} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \hat{g}} \\ \mathbf{B} & \xrightarrow[\hat{f}]{} & \mathbf{H}^{c} \end{array}$$

be a pushout square in CDCQ-Alg$_\Gamma$. *Then* $\hat{g}$ *is total iff* $g^{-1}(g(\mathrm{Dom}\, f)) = \mathrm{Dom}\, f$.

Proof. Since $\mathrm{Dom}\,\hat{g}$ is the greatest closed subset of $\mathbf{A}$ contained in A', we have that $\hat{g}$ is total iff $A = A'$. And by Corollary 5(ii) the latter is equivalent to $g^{-1}(g(\mathrm{Dom}\, f)) = \mathrm{Dom}\, f$. □

Proposition 52. *Given a rule* $r : \mathbf{L} \to \mathbf{R}$ *and an occurrence* $m : \mathbf{L} \to \mathbf{G}$ *in* CDCQ-Alg$_\Gamma$, *m is clean for r iff it is d-closed-injective w.r.t. r.*

Proof. Assume that m is clean for r, and let $x, y \in L_s$ such that $m_s(x) = m_s(y)$. If $y \in \mathrm{Dom}\, r_s$ but $x \notin \mathrm{Dom}\, r_s$ then the cdc-quomorphism m' in the pushout square in CDCQ-Alg$_\Gamma$

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{r} & \mathbf{R} \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle m'} \\ \mathbf{G} & \xrightarrow[r']{} & \mathbf{H} \end{array}$$

is not total, because $m^{-1}(m(\mathrm{Dom}\, r)) \neq \mathrm{Dom}\, r$. Therefore, if m is clean and $x \in \mathrm{Dom}\, r_s$ or $y \in \mathrm{Dom}\, r_s$ then $x, y \in \mathrm{Dom}\, r_s$.

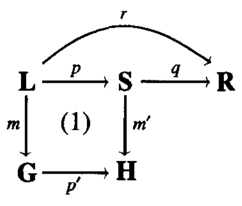
Assume now that $x, y \notin \mathrm{Dom}\, r_s$, and set $D = \mathrm{Dom}\, r \cup C_{\mathbf{L}}(\{y\})$, which is a closed subset of $\mathbf{L}$. Let $\mathbf{D}$ be the closed subalgebra of $\mathbf{L}$ supported on D and consider the factorization $r|_D \circ \mathrm{Id}_D$ of r with $\mathrm{Id}_D : \mathbf{L} \to \mathbf{D}$ the cdc-quomorphism given by the identity on D and $r|_D : \mathbf{D} \to \mathbf{R}$ the cdc-quomorphism $r : \mathbf{L} \to \mathbf{R}$ taken with source algebra $\mathbf{D}$. Consider the diagram

$$\begin{array}{ccccc} \mathbf{L} & \xrightarrow{\mathrm{Id}_D} & \mathbf{D} & \xrightarrow{r|_D} & \mathbf{R} \\ {\scriptstyle m}\big\downarrow & (1) & \big\downarrow{\scriptstyle m'} & & \\ \mathbf{G} & \xrightarrow[r']{} & \mathbf{H} & & \end{array} \qquad (r = r|_D \circ \mathrm{Id}_D : \mathbf{L} \to \mathbf{R})$$

where (1) is a pushout square in CDCQ-Alg$_\Gamma$. Since m is clean, m' is total. Therefore $m^{-1}(m(D)) = D$, and since $x \in m_s^{-1}(m_s(D_s))$ but $x \notin \mathrm{Dom}\, r_s$, we conclude that $x \in C_{\mathbf{L}}(\{y\})_s$ and therefore $C_{\mathbf{L}}(\{x\}) \subseteq C_{\mathbf{L}}(\{y\})$. By symmetry, we also have that $C_{\mathbf{L}}(\{y\}) \subseteq C_{\mathbf{L}}(\{x\})$.

Therefore, if m is clean for r and if $m_s(x) = m_s(y)$ then $x, y \in \mathrm{Dom}\, r_s$ or $C_{\mathbf{L}}(\{x\}) = C_{\mathbf{L}}(\{y\})$.

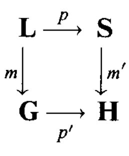
Conversely, let $r : \mathbf{L} \to \mathbf{R}$ be a cdc-quomorphism of partial Γ-algebras and let $m : \mathbf{L} \to \mathbf{G}$ a closed homomorphism d-closed-injective w.r.t. r. Let $r = q \circ p$ be any factorization of r, with $\mathbf{L} \xrightarrow{p} \mathbf{S} \xrightarrow{q} \mathbf{R}$, and consider the diagram



where (1) is a pushout square in CDCQ-Alg_Γ. Let $x \in m_s^{-1}(m_s(\mathrm{Dom}\, p_s))$, say $m_s(x) = m_s(y)$ with $y \in \mathrm{Dom}\, p_s$. Then d-closed-injectivity implies that either $x, y \in \mathrm{Dom}\, r_s \subseteq \mathrm{Dom}\, p_s$ or $C_{\mathbf{L}}(\{x\})_s = C_{\mathbf{L}}(\{y\})_s \subseteq \mathrm{Dom}\, p_s$ (because $\mathrm{Dom}\, p$ is closed), so that in both cases $x \in \mathrm{Dom}\, p_s$. Since m is total and $s \in S$ any sort, this implies that $m^{-1}(m(\mathrm{Dom}\, p)) = \mathrm{Dom}\, p$ and thus, by the previous lemma, that m' is total. Therefore, m is clean for r. □

Remark. It is stated in [12, p. 44] that the clean occurrences in P-TAlg_Γ are exactly the d-injective (total) homomorphisms. It is not true in general, as the following example shows. Let Γ be a monounary graph structure, let $\mathbf{L}$ be a Γ-algebra with carrier $L = \{a, b, b'\}$ and with operation $\varphi^{\mathbf{L}}$ given by $\varphi^{\mathbf{L}}(a) = a$, $\varphi^{\mathbf{L}}(b) = b'$ and $\varphi^{\mathbf{L}}(b') = b$, let $\mathbf{R}$ be its (closed) subalgebra supported on $R = \{a\}$, let $\mathbf{G}$ be a Γ-algebra with carrier $G = \{c, d\}$ and with operation $\varphi^{\mathbf{G}}$ given by $\varphi^{\mathbf{G}}(c) = c$ and $\varphi^{\mathbf{G}}(d) = d$. Let $r : \mathbf{L} \to \mathbf{R}$ be the partial homomorphism given by the identity on $\mathbf{R}$, and let $m : \mathbf{L} \to \mathbf{G}$ be the (closed) homomorphism given by $m(a) = c$ and $m(b) = m(b') = d$ (see Fig. 8).

It is clear that m is not d-injective w.r.t. r, and yet it is clean for it. Indeed, if $r = q \circ p$ then either $\mathrm{Dom}\, p = \{a\}$ or $\mathrm{Dom}\, p = L$ (because $\mathrm{Dom}\, p$ has to be a closed subset of $\mathbf{L}$ containing $\mathrm{Dom}\, r = \{a\}$), and in both cases $m^{-1}(m(\mathrm{Dom}\, p)) = \mathrm{Dom}\, p$. This entails that, if



is a pushout in P-TAlg_Γ (which is also a pushout in CDCQ-Alg_Γ) then m' is total by Lemma 51.

Actually, the proof of Proposition 52 also applies to partial homomorphisms of total Γ-algebras, showing that the clean occurrences in P-TAlg_Γ are exactly the d-closed-injective homomorphisms. Notice that in the case of an acyclic graph structure (that is, where there does not exist any sequence of operations $\varphi_1, \ldots, \varphi_n$, $n \geqslant 1$, such that $\eta(\varphi_i) = (s_i, s_{i+1})$ for every $i = 1, \ldots, n$ and $s_{n+1} = s_1$), as for instance the signature corresponding to hypergraphs [7, Example 4], "d-closed-injective" is equivalent to "d-injective." But this equivalence does not hold for instance for higher-order hypergraphs. □

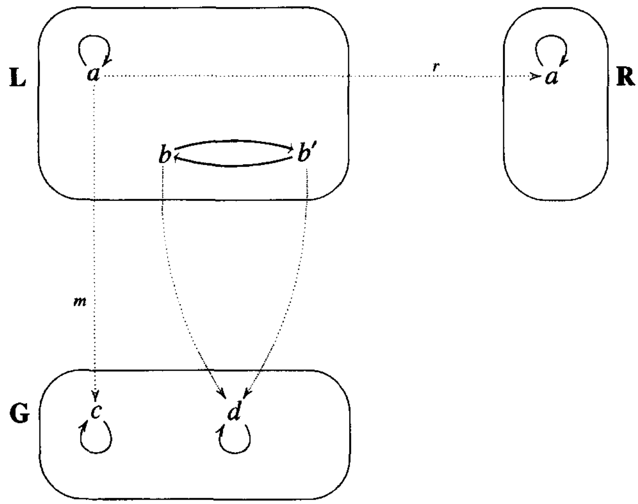


Fig. 8.

Now, we show that the pairs (CF-Alg$_\Gamma$,TCF-Alg$_\Gamma$), (CQ-Alg$_\Gamma$,C-Alg$_\Gamma$) and (CDCQ-Alg$_\Gamma$,C-Alg$_\Gamma$) satisfy the HLR conditions for amalgamation introduced in Definition 43, when Γ is a graph structure.

Proposition 53. *If Γ is a graph structure then the pairs* (CF-Alg$_\Gamma$,TCF-Alg$_\Gamma$), (CQ-Alg$_\Gamma$,C-Alg$_\Gamma$) *and* (CDCQ-Alg$_\Gamma$,C-Alg$_\Gamma$) *satisfy the HLR conditions for amalgamation.*

Proof. Conditions (HLRA1) and (HLRA3) have already been considered in Proposition 47. Condition (HLRA2) is given by Propositions 15, 19 and 33. And condition (HLRA4) is straightforward from the fact that if $r : L \to R$ is a partial mapping and $f : R \to S$ is a total mapping then $\mathrm{Dom}(f \circ r) = \mathrm{Dom}\, r$. □

Therefore, single-pushout transformation in CF-Alg$_\Gamma$, CQ-Alg$_\Gamma$ and CDCQ-Alg$_\Gamma$ through d-injective total conformisms, d-injective closed homomorphisms and d-closed-injective closed homomorphisms, respectively, satisfies the basic properties of amalgamation given in Propositions 41, 45 and 46.

Remark. The pair (P-TAlg$_\Gamma$,TAlg$_\Gamma$) also satisfies the HLR conditions for amalgamation (the proof is similar to the one for (CDCQ-Alg$_\Gamma$,C-Alg$_\Gamma$)). Therefore, Propositions 41, 45 and 46 are satisfied by single-pushout transformation in P-TAlg$_\Gamma$ through d-closed-injective homomorphisms. This generalizes Theorem 6.3 and the "only if" implication in Theorem 6.7 of [15], where these properties are proved for single-pushout transformation through d-injective homomorphisms. This contributes to the solution of the problem proposed in Footnote 35, p. 210, of the aforementioned paper [15], where it is asked to investigate conditions more general than d-injectivity under which the results on amalgamation stated therein still hold. Since d-closed-injective homomorphisms (and not d-injective homomorphisms) are the clean occurrences in P-TAlg$_\Gamma$, we believe that this is the right setting where to prove those results. And, indeed, it is not difficult to prove directly that the remaining "if" implication in Theorem 6.7 of [15] also holds if we replace in it d-injectivity by d-closed-injectivity.

7. Conclusion

The single-pushout approach to graph transformation is extended in this paper to the algebraic transformation of partial many-sorted unary algebras; that is, of partial Γ-algebras for Γ any graph structure. The algebraic characterization is developed in the category CF-Alg$_\Gamma$ of all conformisms of partial Γ-algebras, in the category CQ-Alg$_\Gamma$ of all closed quomorphisms of partial Γ-algebras, and in the category CDCQ-Alg$_\Gamma$ of all closed-domain closed quomorphisms of partial Γ-algebras. Such an algebraic characterization is accompanied by an operational characterization, which may serve as a basis for implementation.

These new single-pushout approaches to algebraic transformation relate to each other in a meaningful way. Single-pushout transformation in CQ-Alg$_\Gamma$ turns out to be a particular case of transformation in CF-Alg$_\Gamma$, and single-pushout transformation in CQ-Alg$_\Gamma$ (or in CF-Alg$_\Gamma$) and in CDCQ-Alg$_\Gamma$ are independent, yielding two completely different approaches to transformation of unary partial algebras: in general, applying a rule in CDCQ-Alg$_\Gamma$ to an algebra through a closed homomorphism yields a "smaller" result than applying it in CQ-Alg$_\Gamma$ (actually, a closed subalgebra of it).

The new single-pushout approaches to algebraic transformation also relate to previous approaches in a meaningful way. On the one hand, the single-pushout transformation of unary total algebras introduced by Löwe in [15] turns out to be a special case of single-pushout transformation in CDCQ-Alg$_\Gamma$, and in some sense the latter can also be understood as a particular case of the first. On the other hand, single-pushout transformation in CF-Alg$_\Gamma$ or CQ-Alg$_\Gamma$ has no relation to any double-pushout approach to transformation introduced in [7], while single-pushout transformation in CDCQ-Alg$_\Gamma$ generalizes double-pushout transformation in C-Alg$_\Gamma$, exactly in the same way as single-pushout transformation of unary total algebras (or hypergraphs) generalizes double-pushout transformation.

The categories of partial many-sorted unary algebras considered in this paper are also shown to satisfy all of the HLR conditions for parallelism introduced in [12]. Similar to the case of the double-pushout approach [7], the HLR conditions satisfied by a pair $(\mathscr{C}, \mathscr{O})$ entail the satisfaction of different rewriting theorems by single-pushout transformation systems in $(\mathscr{C}, \mathscr{O})$ formed by production rules in $\mathscr{C}$ and occurrences in $\mathscr{O}$. In particular, the HLR conditions for parallelism entail the Local Church–Rosser Theorem, the Parallelism Closure and the Parallelism Theorem, which are therefore satisfied by the three single-pushout transformation systems of unary partial algebras introduced here.

Table 1

		HLRP				HLRA			
$\mathscr{C}$	$\mathscr{E}$	1	2	3	4	1	2	3	4
CF-Alg_Γ	TCF-Alg_Γ	+	+	+	+	+	+	+	+
CQ-Alg_Γ	C-Alg_Γ	+	+	+	+	+	+	+	+
CDCQ-Alg_Γ	C-Alg_Γ	+	+	+	+	+	+	+	+

HLR conditions for amalgamation are also defined which allow to decompose non-parallel-independent derivations sharing a common subrule into a common derivation followed by parallel-independent derivations. The categories CF-Alg_Γ, CQ-Alg_Γ and CDCQ-Alg_Γ are also shown to satisfy all of the HLR conditions for amalgamation, taking the corresponding total morphisms as occurrences. This is summarized in Table 1, in parallel with the table given in the Conclusion of [7].

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