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Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means

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ABSTRACT

In this paper, some inequalities of Hadamard's type for quasi-convex functions are given. Some error estimates for the Trapezoidal formula are obtained. Applications to some special means are considered.

1. Introduction

Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex mapping defined on the interval I of real numbers and $a,\ b \in I$, with $a < b$. The following inequality:

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This inequality is known as the Hermite–Hadamard inequality for convex mappings.

In recent years, many authors established several inequalities connected to Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hadamard's-type inequalities, see [1–15].

In [3], Dragomir and Agarwal obtained inequalities for differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove it.

Lemma 1.1. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,\mathrm{d}t. \tag{2}$$

The main inequality in [3], *pointed out as follows:*

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Theorem 1.1. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $\left|f'\right|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{3}$$

In [13] Pearce and Pečarić using the same Lemma 1.1 proved the following theorem.

Theorem 1.2. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is convex on $[a, b]$, for some $q \ge 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left[\frac{|f(a)|^q + |f(b)|^q}{2}\right]^{1/q}. \tag{4}$$

If $|f|^q$ is concave on $[a, b]$ for some $q \ge 1$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{5}$$

Now, we recall that the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f : [a, b] \to \mathbf{R}$ is said quasi-convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda)y) \le \sup\{f(x), f(y)\},$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex, (see [9]).

Recently, Ion [9] introduced two inequalities of the right hand side of Hadamard's type for quasi-convex functions, as follow:

Theorem 1.4. *Let $f : I^\circ \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\sup\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\}. \tag{6}$$

Theorem 1.5. *Let $f : I^\circ \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|^{p/(p-1)}$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)}{2(p+1)^{1/p}}\left(\sup\left\{\left|f'(a)\right|^{p/(p-1)}, \left|f'(b)\right|^{p/(p-1)}\right\}\right)^{(p-1)/p}. \tag{7}$$

The main purpose of this paper is to establish refinements inequalities of the right-hand side of Hadamard's type for quasi-convex functions. We will show that our results can be used in order to give best estimates for the approximation error of the integral $\int_a^b f(x)\,\mathrm{d}x$ in the trapezoid formula which is better than in [9].

2. Hadamard's type inequalities quasi-convex functions

Lemma 2.1. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{4}\left[\int_0^1 (-t)f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t + \int_0^1 tf'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\mathrm{d}t\right].$$

Proof. It suffices to note that

$$\begin{aligned} I_1 &= \int_0^1 -tf'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t \\ &= -\frac{2}{a-b}f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)t\Big|_0^1 + \frac{2}{a-b}\int_0^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t \\ &= -\frac{2}{a-b}f(a) + \frac{2}{a-b}\int_0^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t. \end{aligned}$$

Setting $x = \frac{1+t}{2}a + \frac{1-t}{2}b$, and $\mathrm{d}x = \frac{a-b}{2}\mathrm{d}t$, which gives

$$I_1 = \frac{2}{b-a} f(a) - \frac{4}{(a-b)^2} \int_a^{\frac{a+b}{2}} f(x)\,\mathrm{d}x.$$

Similarly, we can show that

$$\begin{aligned} I_2 &= \int_0^1 t f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\mathrm{d}t \\ &= \frac{2}{b-a} f(b) - \frac{4}{(b-a)^2} \int_{\frac{a+b}{2}}^{b} f(x)\,\mathrm{d}x. \end{aligned}$$

Thus,

$$\frac{b-a}{4}[I_1 + I_2] = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

which is required. □

In the following theorem, we shall propose some new upper bound for the right-hand side of Hadamard's inequality for quasi-convex mappings, which is better than the inequality had done in [9].

Theorem 2.2. *Let $f : I \subset [0,\infty) \to \mathbf{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is an quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{8}\left[\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(a)|\right\} + \sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(b)|\right\}\right]. \tag{8}$$

Proof. From Lemma 2.1, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| = \frac{b-a}{4}\left|\int_0^1 (-t) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t + \int_0^1 t f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\mathrm{d}t\right|.$$

Since $|f'|$ is quasi-convex on $[a,b]$, for any $t \in [0,1]$ we have

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{b-a}{4}\left[\int_0^1 |(-t)| \left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|\mathrm{d}t \right. \\ &\quad \left. + \int_0^1 |t| \left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|\mathrm{d}t\right] \\ &\le \frac{b-a}{4}\left[\int_0^1 t \sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(a)|\right\}\mathrm{d}t \right. \\ &\quad \left. + \int_0^1 t \sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(b)|\right\}\mathrm{d}t\right] \\ &= \frac{b-a}{8}\left[\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(a)|\right\} + \sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|, |f'(b)|\right\}\right] \end{aligned}$$

which completes the proof. □

Corollary 2.1. *Let f as in Theorem 2.2, if in addition*

(1) *$|f'|$ is increasing, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{8}\left[|f'(b)| + \left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{9}$$

(2) *$|f'|$ is decreasing, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{8}\left[|f'(a)| + \left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{10}$$

Proof. It follows directly by Theorem 2.2. □

Remark 1. We note that the inequalities (9) and (10) are two new refinements of the trapezoid inequality for quasi-convex functions, and thus for convex functions.

Another similar result may be extended in the following theorem.

Theorem 2.3. *Let $f : I \subset [0, \infty) \to \mathbf{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|^{p/(p-1)}$ is an quasi-convex on $[a, b]$, for $p > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4(p+1)^{1/p}}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, \left|f'(b)\right|^{p/(p-1)}\right\}\right)^{(p-1)/p} + \left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{p/(p-1)}, \left|f'(a)\right|^{p/(p-1)}\right\}\right)^{(p-1)/p}\right]. \tag{11}$$

Proof. From Lemma 2.1 and using well known Hölder integral inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{b-a}{4}\left[\int_0^1 |(-t)|\left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|dt + \int_0^1 |t|\left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|dt\right] \\ &\le \frac{b-a}{4}\left[\left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^q dt\right)^{1/q} + \left(\int_0^1 t^p dt\right)^{1/p}\left(\int_0^1\left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|^q dt\right)^{1/q}\right] \\ &\le \frac{b-a}{4(p+1)^{1/p}}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(a)\right|^q\right\}\right)^{1/q} + \left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(b)\right|^q\right\}\right)^{1/q}\right]\end{aligned}$$

where $1/p + 1/q = 1$, which completes the proof. □

Corollary 2.2. *Let f as in Theorem 2.3, if in addition*

(1) $|f'|^{p/(p-1)}$ *is increasing, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4(p+1)^{1/p}}\left[\left|f'(b)\right| + \left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{12}$$

(2) $|f'|^{p/(p-1)}$ *is decreasing, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4(p+1)^{1/p}}\left[\left|f'(a)\right| + \left|f'\left(\frac{a+b}{2}\right)\right|\right]. \tag{13}$$

An improvement of the constants in Theorem 2.3 and a consolidation of this result with Theorem 2.2. are given in the following theorem.

Theorem 2.4. *Let $f : I^\circ \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|^q$ is an quasi-convex on $[a, b]$, $q \ge 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{8}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}} + \left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right]. \tag{14}$$

Proof. From Lemma 2.1 and using well known power mean inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{b-a}{4}\left[\int_0^1 |(-t)|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right.\\ &\quad\left.+\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]\\ &\le \frac{b-a}{4}\left[\left(\int_0^1 t\,dt\right)^{1-1/q}\left(\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q}\right.\\ &\quad\left.+\left(\int_0^1 t\,dt\right)^{1-1/q}\left(\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{1/q}\right]\\ &\le \frac{b-a}{8}\left[\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{1/q}\right.\\ &\quad\left.+\left(\sup\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{1/q}\right]\end{aligned}$$

which completes the proof. □

Remark 2. For $q = 1$ this reduces to Theorem 2.2. For $q = p/(p-1)(p > 1)$ we have an improvement of the constants in Theorem 2.3, since $4^p > p + 1$ if $p > 1$ and accordingly

$$\frac{1}{8} < \frac{1}{4(p+1)^{1/p}}.$$

Corollary 2.3. *Let f as in Theorem 2.4, if in addition*
(1) *$|f'|$ is increasing, then (9) holds.*
(2) *$|f'|$ is decreasing, then (10) holds.*

3. Applications to trapezoidal formula

Let d be a division of the interval $[a, b]$, i.e., $d : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and consider the Trapezoidal formula

$$T(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i). \tag{15}$$

It is well known that if the mapping $f : [a, b] \to \mathbf{R}$, is differentiable such that $f''(x)$ exists on (a, b) and $M = \max_{x\in(a,b)} \left|f''(x)\right| < \infty$, then

$$I = \int_a^b f(x)\,dx = T(f, d) + E(f, d), \tag{16}$$

where the approximation error $E(f, d)$ of the integral I by the Trapezoidal formula $T(f, d)$ satisfies

$$|E(f, d)| \le \frac{M}{12}\sum_{i=0}^{n-1}(x_{i+1} - x_i)^3. \tag{17}$$

It is clear that if the mapping f is not twice differentiable or the second derivative is not bounded on (a, b), then (17) cannot be applied.

In the following, we shall propose some new estimates for the remainder term $E(f, d)$ in terms of the first derivative using the inequalities above.

Proposition 3.1. *Let $f : I^\circ \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|$ is quasi-convex on $[a, b]$, then in (16), for every division d of $[a, b]$, the following holds:*

$$\begin{aligned}|E(f, d)| &\le \frac{1}{8}\sum_{i=1}^{n-1}(x_{i+1} - x_i)\left[\sup\left\{\left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|, \left|f'(x_{i+1})\right|\right\}\right.\\ &\quad\left.\le \sup\left\{\left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|, \left|f'(x_i)\right|\right\}\right].\end{aligned} \tag{18}$$

Proof. Applying Theorem 2.2 on the subintervals $[x_i, x_{i+1}]$, $(i = 0, 1, \ldots, n-1)$ of the division d, we get

$$\left|(x_{i+1}-x_i)\frac{f(x_i)+f(x_{i+1})}{2} - \int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x\right| \le (x_{i+1}-x_i)\left[\sup\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, \left|f'(x_{i+1})\right|\right\} + \sup\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, \left|f'(x_i)\right|\right\}\right].$$

Summing over i from 0 to $n-1$ and taking into account that $\left|f'\right|$ is quasi-convex, we deduce that

$$\left|T(f,d) - \int_a^b f(x)\,\mathrm{d}x\right| \le \frac{1}{8}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left[\sup\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, \left|f'(x_{i+1})\right|\right\} + \sup\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|, \left|f'(x_i)\right|\right\}\right]$$

which completes the proof. □

Corollary 3.1. *Let $f : I^\circ \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L[a, b]$. If $\left|f'\right|$ is quasi-convex on $[a, b]$. then in (16), for every division d of $[a, b]$, we have*

(1) *$|f'|$ is increasing, then we have*

$$|E(f,d)| \le \frac{1}{8}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left(\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + \left|f'(x_{i+1})\right|\right). \tag{19}$$

(2) *$|f'|$ is decreasing, then we have*

$$|E(f,d)| \le \frac{1}{8}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left(\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + \left|f'(x_i)\right|\right). \tag{20}$$

Proof. The proof can be done similar to that of Proposition 3.1. and using Corollary 2.2. □

Proposition 3.2. *Let $f : I^\circ \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|^{p/(p-1)}$ is an quasi-convex on $[a, b]$, $p > 1$, then in (16), for every division d of $[a, b]$, the following holds:*

$$|E(f,d)| \le \frac{1}{4(p+1)^{1/p}}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left[\left(\sup\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^{p/(p-1)}, \left|f'(x_{i+1})\right|^{p/(p-1)}\right\}\right)^{(p-1)/p} + \left(\sup\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^{p/(p-1)}, \left|f'(x_i)\right|^{p/(p-1)}\right\}\right)^{(p-1)/p}\right]. \tag{21}$$

Proof. The proof can be done similar to that of Proposition 3.1. and using Theorem 2.3. □

Corollary 3.2. *Let $f : I^\circ \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L[a, b]$. If $\left|f'\right|^{p/(p-1)}$ is quasi-convex on $[a, b]$. then in (16), for every division d of $[a, b]$, we have*

(1) *$|f'|$ is increasing, then we have*

$$|E(f,d)| \le \frac{1}{4(p+1)^{1/p}}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left(\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + \left|f'(x_{i+1})\right|\right).$$

(2) *$|f'|$ is decreasing, then we have*

$$|E(f,d)| \le \frac{1}{4(p+1)^{1/p}}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left(\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + \left|f'(x_i)\right|\right).$$

Proof. The proof can be done similar to that of Proposition 3.2. and using Corollary 2.3. □

Proposition 3.3. *Let $f : I^\circ \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is an quasi-convex on $[a, b]$, $q \geq 1$, then in (16), for every division d of $[a, b]$, the following holds:*

$$|E(f, d)| \leq \frac{1}{8}\sum_{i=1}^{n-1}(x_{i+1}-x_i)\left[\left(\sup\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q, \left|f'(x_{i+1})\right|^q\right\}\right)^{1/q} + \left(\sup\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^{1/q}, \left|f'(x_i)\right|^q\right\}\right)^{1/q}\right]. \tag{22}$$

Proof. The proof can be done similar to that of Proposition 3.1. and using Theorem 2.4. □

Corollary 3.3. *Let f as in Proposition 3.3, if in addition*

(1) *$|f'|$ is increasing, then (19) holds.*
(2) *$|f'|$ is decreasing, then (20) holds.*

Proof. The proof can be done similar to that of Proposition 3.3. and using Corollary 2.3. □

4. Applications to special means

We shall consider the means for arbitrary real numbers $\alpha, \beta (\alpha \neq \beta)$. We take

(1) Arithmetic mean:

$$A(\alpha, \beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\ \beta \in \mathbf{R}.$$

(2) Logarithmic mean:

$$L(\alpha, \beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|,\ \alpha, \beta \neq 0,\ \alpha, \beta \in \mathbf{R}.$$

(3) Generalized log-mean:

$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n \in \mathbf{N},\ \alpha, \beta \in \mathbf{R},\ \alpha \neq \beta.$$

Now, using the results of Section 2, we give some applications to special means of real numbers.

Proposition 4.1. *Let $a, b \in \mathbf{R}$, $a < b$ and $n \in \mathbf{N}$, $n \geq 2$. Then, we have*

$$\left|L_n^n(a, b) - A\left(a^n, b^n\right)\right| \leq n\left(\frac{b-a}{8}\right)\left\{\sup\left(\left|\frac{a+b}{2}\right|^{n-1}, |a|^{n-1}\right) + \sup\left(\left|\frac{a+b}{2}\right|^{n-1}, |b|^{n-1}\right)\right\}.$$

Proof. The assertion follows from Theorem 2.2 applied to the quasi-convex mapping $f(x) = x^n$, $x \in \mathbf{R}$. □

Proposition 4.2. *Let $a, b \in \mathbf{R}$, $a < b$ and $0 \notin [a, b]$. Then, for all $p > 1$, we have*

$$\left|L^{-1}(a, b) - A\left(a^{-1}, b^{-1}\right)\right| \leq \frac{(b-a)}{4(p+1)^{1/p}}\left\{\left[\sup\left(\left|\frac{a+b}{2}\right|^{-\frac{2p}{p-1}}, |a|^{-\frac{2p}{p-1}}\right)\right]^{\frac{p-1}{p}} + \left[\sup\left(\left|\frac{a+b}{2}\right|^{-\frac{2p}{p-1}}, |b|^{-\frac{2p}{p-1}}\right)\right]^{\frac{p-1}{p}}\right\}.$$

Proof. The assertion follows from Theorem 2.3 applied to the quasi-convex mapping $f(x) = 1/x$, $x \in [a, b]$. □

Proposition 4.3. *Let* $a, b \in \mathbf{R}$, $a < b$ *and* $n \in \mathbf{N}$, $n \geq 2$. *Then, for all* $q \geq 1$, *we have*

$$\left|L_s^s(a,b) - A^s(a,b)\right| \leq n\left(\frac{b-a}{8}\right)\left\{\left[\sup\left(\left|\frac{a+b}{2}\right|^{(n-1)q}, |a|^{(n-1)q}\right)\right]^{1/q} + \left[\sup\left(\left|\frac{a+b}{2}\right|^{(n-1)q}, |b|^{(n-1)q}\right)\right]^{1/q}\right\}.$$

Proof. The assertion follows from Theorem 2.4 applied to the quasi-convex mapping $f(x) = x^n$, $x \in \mathbf{R}$. □

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