Available online at www.sciencedirect.com
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An International Journal
computers & mathematics with applications
Computers and Mathematics with Applications 52 (2006) 497–508
www.elsevier.com/locate/camwa

Stability of Solution of Kuramoto-Sivashinsky-Korteweg-de Vries System

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(Received October 2005; accepted November 2005)

Abstract—A model consisting of a mixed Kuramoto-Sivashinsky-Korteweg-de Vries equation, linearly coupled to an extra linear dissipative equation has been proposed in [1] in order to describe the surface waves on multilayered liquid films and stability criteria are discussed using wave mode analysis. In this paper, we study the linear stability of solutions to the model from the viewpoint of energy estimate. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Kuramoto-Sivashinsky, Korteweg-de Vries, Linear stability.

1. INTRODUCTION

The Kortweg-de Vries (KdV) equation is a nonlinear partial differential equation of the third order, as follows,

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

which was first formulated as part of an analysis of shallow-water waves in canals. It has subsequently been found to be involved in a wide range of physical phenomena, especially those exhibiting shock waves, traveling waves, and solitons [2–5]. Certain theoretical physical phenomena in the quantum mechanics domain are explained by means of a KdV model. It is used in fluid dynamics, aerodynamics, and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behavior, and mass transport [2–5]. More recently, increasing attention has been paid to the problem of the existence and stability of traveling wave solutions

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This work was supported by the National Research Council of Thailand and the Thailand Research Fund.
†Supported in part by DoDEPSCOR N000014-02-1-0577 and WVU Faculty Development Fund.
‡This work was supported by the National Research Council of Thailand and the Eiesland Visiting Scholarship, West Virginia University.

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doi:10.1016/j.camwa.2005.11.038

in modeling signal transduction among neural cells [6,7]. Thus, the study of the KdV model can have a lot of implications in the fields of biology and medical science as well.

The Kuramoto-Sivashinsky (KS) equation is a well-known model of one-dimensional turbulence, which was derived in various physical contexts, including chemical-reaction waves, propagation of combustion fronts in gases, surface waves in a film of a viscous liquid flowing along an inclined plane, patterns in thermal convection, rapid solidification, and others [8–12]. It has the form,

$$u_t + uu_x = -\alpha u_{xx} - \gamma u_{xxxx}, \tag{2}$$

where $\alpha, \gamma > 0$ are constant coefficients accounting for the long-wave instability (gain) and short-wave dissipation, respectively.

An important one-dimensional generalized form of the KS equation that combines conservative and dissipative effects is a mixed Kuramoto-Sivashinsky-Korteweg-de Vries (KS-KdV) equation,

$$u_t + uu_x + u_{xxx} = -\alpha u_{xx} - \gamma u_{xxxx} \tag{3}$$

which was first introduced by Benney [13] and is often called the Benney equation. This equation finds various applications in plasma physics, hydrodynamics and other fields [14–16]. In particular, a subject of considerable interest was the study of solitary-pulse (SP) solutions of the above equations [17–19]. However, solitary-wave (SW) cannot be stable in the Benney equation proper, as the zero solution, which is a background on top of which SWs are to be found, is linearly unstable in this equation due to the presence of the linear gain, which is accounted for by the coefficient α.

A stabilized version of the Benney equation was proposed in [1] based on the KS-KdV equation for a real wave field $u(x,t)$, which is linearly coupled to an additional linear dissipative equation for an extra real wave field $v(x,t)$, that provides for the stabilization of the zero background. The model is as follows,

$$u_t + uu_x + u_{xxx} = -\alpha u_{xx} - \gamma u_{xxxx} + \varepsilon_1 v_x, \tag{4}$$
$$v_t + a_1 v_x = \Gamma v_{xx} + \varepsilon_2 u_x. \tag{5}$$

The system describes, for instance, the propagation of surface waves in a two-layer liquid film in the case when one layer is dominated by viscosity. Here $\alpha, \gamma > 0$ and $\Gamma > 0$ are constant coefficients accounting for the gain and loss in the u subsystem and loss in the v subsystem, respectively. a_1 is a group-velocity mismatch between the two waves fields. The coupling parameters $\varepsilon_1, \varepsilon_2$ are positive. The linear coupling via the first derivatives is the same as in known models of coupled internal waves propagating in multilayered fluids [20]. Then, the linear dissipative equation (5) implies that the substrate layer is essentially more viscous than the upper one [1]. In [1], the stability of SP solutions in the system of equations (4) and (5) was investigated by treating the gain and the dissipation constants α, γ, Γ in the model as small parameters (while the group-velocity mismatch a_1 needs not be small) and making use of the balance equation for the net momentum. It is found that the condition of the balance between the gain and dissipation may select two steady state solutions from their continuous family existing in absence of the dissipation and gain. When the zero solution is stable and two SP solutions are picked up by the balance equation for the momentum, the pulse with the larger value of the amplitude is stable in the indefinitely long system, while the other pulse is unstable, playing the role of a separatrix between attraction domains of the stable pulse and zero solution.

In this paper, we will study the linear stability of the solution to the following system from the viewpoint of energy estimate,

$$u_t + uu_x + u_{xxx} = -\alpha u_{xx} - \gamma u_{xxxx} + \varepsilon_1 v_x, \tag{6}$$
$$v_t + a_1 v_x = \Gamma v_{xx} + \varepsilon_2 u_x, \tag{7}$$

with the initial conditions,

$$u(x,0) = u_0, \qquad v(x,0) = v_0, \tag{8}$$

where $\alpha \equiv \alpha_0 + \alpha_1(x,t)$, $\gamma \equiv \gamma_0 + \gamma_1(x,t)$, $\Gamma \equiv \Gamma_0 + \Gamma_1(x,t)$ could be variations of positive constants α_0, γ_0 and Γ_0. In Section 2, we derive the energy estimate for a small perturbation of the travelling wave solutions in [1]. For completeness, we also derive the higher order estimates in Section 3. The question of existence of solutions to the problem is also of great interest and relevance, and thus is a subject for future studies.

2. STABILITY BY ENERGY ESTIMATE

In the following, we will consider the case that the solution (u,v) of the system (6)–(8) are Schwaltz fast-decaying smooth functions: $(u,v) \in S(R)$ and hence (u,v) with all their derivatives vanish as $|x| \to \infty$. Since $S(R)$ is dense in $L^2(R)$ as well as other Sobolev spaces used in the paper, this assumption is without loss of generality. Obviously, the energy estimates thus obtained in this paper is also valid for the periodic solutions discussed in [21], where the boundary terms from integration by parts all cancel out because of the periodicity assumption. This gives the linear stability of the periodic solutions in [21] from the viewpoint of energy estimate.

Consider the linearized equations for a small perturbation $(\tilde{u},\tilde{v})$ of (6) and (7) at a given solution (u^0,v^0), i.e.,

$$\begin{aligned} u &\sim u^0 + \epsilon\tilde{u}, \\ v &\sim v^0 + \epsilon\tilde{v}, \end{aligned} \tag{9}$$

$\epsilon \ll 1$, which can be written as follows,

$$\tilde{u}_t + a_2\tilde{u}_x + \tilde{u}_{xxx} = -\alpha\tilde{u}_{xx} - \gamma\tilde{u}_{xxxx} + \varepsilon_1\tilde{v}_x + \tilde{f}, \tag{10}$$

$$\tilde{v}_t + a_1\tilde{v}_x = \Gamma\tilde{v}_{xx} + \varepsilon_2\tilde{u}_x + \tilde{g}, \tag{11}$$

with

$$\tilde{u}(x,0) = \tilde{u}_0, \qquad \tilde{v}(x,0) = \tilde{v}_0, \tag{12}$$

where $a_2 = u^0$.

Let $\langle\cdot,\cdot\rangle$ denote the L^2 inner product in R, $\|\cdot\| \equiv \|\cdot\|_0$ denote the corresponding norm. $\|\cdot\|_k$ denote the k^{th}-order Sobolev norm, and $\|\cdot\|_k^2 \equiv \sum_{i=0}^k \|\partial_x^i\cdot\|^2$.

Choose a small constant δ_0 such that the variables $\alpha_1, \gamma_1, \Gamma_1$ in the coefficients of (10) and (11) satisfy

$$\sup_{t,x}\left(|\alpha_1| + |\gamma_1| + |\partial_x\gamma_1| + \left|\partial_x^2\gamma_1\right| + |\Gamma_1| + |\partial_x\Gamma_1|\right) \le \delta_0 \tag{13}$$

and

$$\frac{\delta_0}{2} < \gamma_0 - \sup_{t,x}|\gamma_1|, \qquad \frac{\delta_0}{2} < \Gamma_0 - \sup_t|\Gamma_1|. \tag{14}$$

For simplicity of notation, in what follows, we denote by C a constant depending only upon δ_0 and T. Then, we have the following zero order energy estimate.

THEOREM 1. *Let $(u,v) \in C^1([0,T];S(R))$ be the solution of (10) and (11), then (u,v) satisfies the following estimate,*

$$\partial_t\left(\|\tilde{u}\|^2 + \|\tilde{v}\|^2\right) + \|\tilde{u}\|_2^2 + \|\tilde{v}\|_1^2 \le C\left(\|\tilde{u}\|^2 + \|\tilde{v}\|^2 + \left\|\tilde{f}\right\|^2 + \|\tilde{g}\|^2\right) \tag{15}$$

and

$$\begin{aligned} &\sup_{0\le t\le T}\left(\|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2\right) + \int_0^T \left(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_1^2\right) dt \\ &\le C(T)\left(\|\tilde{u}_0\|^2 + \|\tilde{v}_0\|^2 + \int_0^T \left(\left\|\tilde{f}\right\|^2 + \|\tilde{g}\|^2\right) dt\right). \end{aligned} \tag{16}$$

PROOF. By taking L^2 inner product of (10) with $\tilde{u}$ and (11) with $\tilde{v}$, integrating by parts and then combining, we obtain

$$\begin{aligned}&\frac{1}{2}\partial_t\left(\|\tilde{u}\|^2+\|\tilde{v}\|^2\right)+\gamma_0\|\tilde{u}_{xx}\|^2+\Gamma_0\|\tilde{v}_x\|^2+\langle\gamma_1\tilde{u}_{xxxx},\tilde{u}\rangle+\langle\alpha_1\tilde{u}_{xx},\tilde{u}\rangle\\&\quad=\alpha_0\|\tilde{u}_x\|^2+\varepsilon_1\langle\tilde{v}_x,\tilde{u}\rangle+\varepsilon_2\langle\tilde{u}_x,\tilde{v}\rangle+\left\langle\tilde{f},\tilde{u}\right\rangle+\langle\tilde{g},\tilde{v}\rangle+\langle\Gamma_1\tilde{v}_{xx},\tilde{v}\rangle.\end{aligned}\tag{17}$$

By applying Cauchy-Schwarz inequality and Cauchy's inequality [22],

$$\langle a,b\rangle\le\beta\|a\|^2+\frac{1}{4\beta}\|b\|^2,$$

where β is an arbitrary positive constant, (17) becomes

$$\begin{aligned}&\frac{1}{2}\partial_t\left(\|\tilde{u}\|^2+\|\tilde{v}\|^2\right)+\gamma_0\|\tilde{u}_{xx}\|^2+\Gamma_0\|\tilde{v}_x\|^2\\&\le\alpha_0\|\tilde{u}_x\|^2+\beta\left(\|\tilde{u}_x\|^2+\|\tilde{v}_x\|^2+\|\tilde{u}\|^2+\|\tilde{v}\|^2+\left\|\tilde{f}\right\|^2+\|\tilde{g}\|^2\right)+\langle\Gamma_1\tilde{v}_{xx},\tilde{v}\rangle-\langle\alpha_1\tilde{u}_{xx},\tilde{u}\rangle\\&\quad-\langle\gamma_1\tilde{u}_{xxxx},\tilde{u}\rangle\end{aligned}\tag{18}$$

Here, we estimate $\|\tilde{u}_x\|^2$ by means of the inequality,

$$\|\tilde{u}_x\|^2\le\|\tilde{u}\|\,\|\tilde{u}_{xx}\|\le\beta\|\tilde{u}\|^2+\frac{1}{4\beta}\|\tilde{u}_{xx}\|^2.\tag{19}$$

Then,

$$\begin{aligned}&\frac{1}{2}\partial_t\left(\|\tilde{u}\|^2+\|\tilde{v}\|^2\right)+\gamma_0\|\tilde{u}_{xx}\|^2+\Gamma_0\|\tilde{v}_x\|^2\\&\le\beta\left(\|\tilde{u}\|^2+\|\tilde{v}\|^2+\left\|\tilde{f}\right\|^2+\|\tilde{g}\|^2+\|\tilde{u}_{xx}\|^2+\|\tilde{v}_x\|^2\right)\\&\quad+\langle\Gamma_1\tilde{v}_{xx},\tilde{v}\rangle-\langle\alpha_1\tilde{u}_{xx},\tilde{u}\rangle-\langle\gamma_1\tilde{u}_{xxxx},\tilde{u}\rangle.\end{aligned}\tag{20}$$

Hölder inequality and Cauchy's inequality [22] imply that

$$\begin{aligned}\langle\Gamma_1\tilde{v}_{xx},\tilde{v}\rangle&=-\langle(\Gamma_1\tilde{v})_x,\tilde{v}_x\rangle\\&\le|\langle\Gamma_1\tilde{v}_x,\tilde{v}_x\rangle|+|\langle\tilde{v}(\Gamma_1)_x,\tilde{v}_x\rangle|\\&\le\|\Gamma_1\|\,\|\tilde{v}_x\|^2+\|(\Gamma_1)_x\|\left(\beta\|\tilde{v}_x\|^2+\frac{1}{4\beta}\|\tilde{v}\|^2\right)\end{aligned}\tag{21}$$

and

$$\begin{aligned}-\langle\gamma_1\tilde{u}_{xxxx},\tilde{u}\rangle&=-\langle(\gamma_1\tilde{u})_{xx},\tilde{u}_{xx}\rangle\\&\le|\langle\gamma_1\tilde{u}_{xx},\tilde{u}_{xx}\rangle|+2|\langle(\gamma_1)_x\tilde{u}_x,\tilde{u}_{xx}\rangle|+|\langle(\gamma_1)_{xx}\tilde{u},\tilde{u}_{xx}\rangle|\\&\le\|\gamma_1\|\,\|\tilde{u}_{xx}\|^2+2\|(\gamma_1)_x\|\,\|\tilde{u}_x\|\,\|\tilde{u}_{xx}\|+\|(\gamma_1)_{xx}\|\,\|\tilde{u}\|\,\|\tilde{u}_{xx}\|\\&\le\|\gamma_1\|\,\|\tilde{u}_{xx}\|^2+\beta\left(\|(\gamma_1)_x\|+\|(\gamma_1)_{xx}\|\right)\|\tilde{u}\|^2\\&\quad+\beta\left(\|(\gamma_1)_x\|+\|(\gamma_1)_{xx}\|\right)\|\tilde{u}_{xx}\|^2,\end{aligned}\tag{22}$$

and also,

$$\begin{aligned}-\langle\alpha_1\tilde{u}_{xx},\tilde{u}\rangle&\le|\langle\alpha_1\tilde{u}_{xx},\tilde{u}\rangle|\\&\le\|\alpha_1\|\,\|\tilde{u}_{xx}\|\,\|\tilde{u}\|\\&\le\|\alpha_1\|\left(\beta\|\tilde{u}_{xx}\|^2+\frac{1}{4\beta}\|\tilde{u}\|^2\right).\end{aligned}\tag{23}$$

By applying (21), (22), and (23) to (20), we obtain

$$\begin{aligned}&\frac{1}{2}\partial_t\left(\|\tilde{u}\|^2+\|\tilde{v}\|^2\right)+\gamma_0\|\tilde{u}_{xx}\|^2+\Gamma_0\|\tilde{v}_x\|^2\\&\leq\beta\left(1+\|(\gamma_1)_x\|+\|(\gamma_1)_{xx}\|+\|\alpha_1\|\right)\|\tilde{u}\|^2+\beta\left(1+\|(\Gamma_1)_x\|\right)\|\tilde{v}\|^2+\beta\left(\left\|\tilde{f}\right\|^2+\|\tilde{g}\|^2\right)\\&\quad+\beta\left(1+\|(\gamma_1)_x\|+\|(\gamma_1)_{xx}\|+\|\alpha_1\|\right)\|\tilde{u}_{xx}\|^2+\beta\left(1+\|(\Gamma_1)_x\|\right)\|\tilde{v}_x\|^2\\&\quad+\|\Gamma_1\|\|\tilde{v}_x\|^2+\|\gamma_1\|\|\tilde{u}_{xx}\|^2.\end{aligned}\tag{24}$$

By using (13), (14), and (19), (24) becomes

$$\begin{aligned}&\frac{1}{2}\partial_t\left(\|\tilde{u}\|^2+\|\tilde{v}\|^2\right)+\|\tilde{u}_x\|^2+\frac{\delta_0}{2}\|\tilde{u}_{xx}\|^2+\frac{\delta_0}{2}\|\tilde{v}_x\|^2\\&\leq\beta\left(1+3\delta_0\right)\|\tilde{u}\|^2+\beta\left(1+\delta_0\right)\|\tilde{v}\|^2+\beta\left(\left\|\tilde{f}\right\|^2+\|\tilde{g}\|^2\right)+\beta\left(1+3\delta_0\right)\|\tilde{u}_{xx}\|^2\\&\quad+\beta\left(1+\delta_0\right)\|\tilde{v}_x\|^2.\end{aligned}$$

Since β is an arbitrary positive constant, we can choose β depending on δ_0 so that

$$\partial_t\left(\|\tilde{u}\|^2+\|\tilde{v}\|^2\right)+\|\tilde{u}_x\|^2+\|\tilde{u}_{xx}\|^2+\|\tilde{v}_x\|^2\leq C\left(\|\tilde{u}\|^2+\|\tilde{v}\|^2+\left\|\tilde{f}\right\|^2+\|\tilde{g}\|^2\right).\tag{25}$$

By adding $\|\tilde{u}\|^2$ and $\|\tilde{v}\|^2$ to both sides of (25), we obtain (15) which means

$$\partial_t\left(\|\tilde{u}\|^2+\|\tilde{v}\|^2\right)\leq C\left(\|\tilde{u}\|^2+\|\tilde{v}\|^2+\left\|\tilde{f}\right\|^2+\|\tilde{g}\|^2\right).\tag{26}$$

By applying Gronwall's inequality to (26), we obtain

$$\|\tilde{u}(t)\|^2+\|\tilde{v}(t)\|^2\leq e^{Ct}\left(\|\tilde{u}_0\|^2+\|\tilde{v}_0\|^2+C\int_0^t\left(\left\|\tilde{f}\right\|^2+\|\tilde{g}\|^2\right)ds\right),$$

which yields

$$\sup_{0\leq t\leq T}\left(\|\tilde{u}(t)\|^2+\|\tilde{v}(t)\|^2\right)\leq C(T)\left(\|\tilde{u}_0\|^2+\|\tilde{v}_0\|^2+\int_0^T\left(\left\|\tilde{f}\right\|^2+\|\tilde{g}\|^2\right)dt\right).\tag{27}$$

Then, (15) and (27) imply (16) and the proof is complete.

Theorem 1 implies that $\|\tilde{u}\|$ and $\|\tilde{v}\|$ can be bounded by a constant depending on the initial condition $(\tilde{u}_0,\tilde{v}_0)$, hence, for a sufficiently small $\epsilon>0$, the solution (u^0,v^0) is stable.

3. HIGHER-ORDER ESTIMATES

We now choose a small constant δ_k such that the variable $\alpha_1,\gamma_1,\Gamma_1$ in equations (10) and (11) satisfy

$$\sup_t\left(\|\alpha_1\|_{C^k}+\|\gamma_1\|_{C^{k+2}}+\|\Gamma_1\|_{C^{k+1}}\right)\leq\delta_k.\tag{28}$$

We denote by C_k a constant depending only upon δ_0, δ_k, and T. Then, we have the following k-order energy estimate.

THEOREM 2. *For integer $k \geq 1$ and let $(u, v) \in C^1([0,T]; S(R))$ be the solution of (10) and (11), then (u, v) satisfies the following k^{th}-order energy estimate,*

$$\partial_t \left(\|\tilde{u}\|_k^2 + \|\tilde{v}\|_k^2 \right) + \|\tilde{u}\|_{k+2}^2 + \|\tilde{v}\|_{k+1}^2 \leq C_k \left(\|\tilde{u}\|_k^2 + \|\tilde{v}\|_k^2 + \left\|\tilde{f}\right\|_k^2 + \|\tilde{g}\|_k^2 \right) \tag{29}$$

and

$$\begin{aligned} &\sup_{0\leq t\leq T} \left(\|\tilde{u}(t)\|_k^2 + \|\tilde{v}(t)\|_k^2 \right) + \int_0^T \left(\|\tilde{u}\|_{k+2}^2 + \|\tilde{v}\|_{k+1}^2 \right) dt \\ &\leq C_k(T) \left(\|\tilde{u}_0\|_k^2 + \|\tilde{v}_0\|_k^2 + \in t_0^T \left(\left\|\tilde{f}\right\|_k^2 + \|\tilde{g}\|_k^2 \right) dt \right). \end{aligned} \tag{30}$$

PROOF. We will prove this theorem by mathematical induction. For $k = 1$, first we have to prove that

$$\partial_t \left(\|\tilde{u}\|_1^2 + \|\tilde{v}\|_1^2 \right) + \|\tilde{u}\|_3^2 + \|\tilde{v}\|_2^2 \leq C_1 \left(\|\tilde{u}\|_1^2 + \|\tilde{v}\|_1^2 + \left\|\tilde{f}\right\|_1^2 + \|\tilde{g}\|_1^2 \right).$$

By taking L^2 inner product of (10) with $\tilde{u}_{xx}$, (11) with $\tilde{v}_{xx}$, integrating by parts and combining, we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t \left(\|\tilde{u}_x\|^2 + \|\tilde{v}_x\|^2 \right) + \gamma_0 \|\tilde{u}_{xxx}\|^2 + \Gamma_0 \|\tilde{v}_{xx}\|^2 \\ &= \alpha_0 \|\tilde{u}_{xx}\|^2 + \varepsilon_1 \langle \tilde{v}_{xx}, \tilde{u}_x \rangle + \varepsilon_2 \langle \tilde{u}_{xx}, \tilde{v}_x \rangle + \left\langle \tilde{f}_x, \tilde{u}_x \right\rangle + \langle \tilde{g}_x, \tilde{v}_x \rangle + \langle \alpha_1 \tilde{u}_{xx}, \tilde{u}_{xx} \rangle \\ &\quad + \langle \gamma_1 \tilde{u}_{xxxx}, \tilde{u}_{xx} \rangle - \langle \Gamma_1 \tilde{v}_{xx}, \tilde{v}_{xx} \rangle. \end{aligned} \tag{31}$$

By applying Cauchy-Schwarz inequality and Cauchy's inequality to (31), we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t \left(\|\tilde{u}_x\|^2 + \|\tilde{v}_x\|^2 \right) + \gamma_0 \|\tilde{u}_{xxx}\|^2 + \Gamma_0 \|\tilde{v}_{xx}\|^2 \\ &\leq \alpha_0 \|\tilde{u}_{xx}\|^2 + \beta \left(\|\tilde{u}_x\|^2 + \|\tilde{v}_x\|^2 + \left\|\tilde{f}_x\right\|^2 + \|\tilde{g}_x\|^2 + \|\tilde{u}_{xx}\|^2 + \|\tilde{v}_{xx}\|^2 \right) + \langle \alpha_1 \tilde{u}_{xx}, \tilde{u}_{xx} \rangle \\ &\quad + \langle \gamma_1 \tilde{u}_{xxxx}, \tilde{u}_{xx} \rangle - \langle \Gamma_1 \tilde{v}_{xx}, \tilde{v}_{xx} \rangle, \end{aligned} \tag{32}$$

where β is an arbitrary positive constant.

Since Hölder inequality and Cauchy's inequality imply that

$$-\langle \Gamma_1 \tilde{v}_{xx}, \tilde{v}_{xx} \rangle \leq \|\Gamma_1\| \|\tilde{v}_{xx}\|^2, \tag{33}$$

$$\langle \alpha_1 \tilde{u}_{xx}, \tilde{u}_{xx} \rangle \leq \|\alpha_1\| \|\tilde{u}_{xx}\|^2, \tag{34}$$

and

$$\begin{aligned} \langle \gamma_1 \tilde{u}_{xxxx}, \tilde{u}_{xx} \rangle &= -\langle (\gamma_1 \tilde{u}_{xx})_x, \tilde{u}_{xxx} \rangle \\ &\leq |\langle \gamma_1 \tilde{u}_{xxx}, \tilde{u}_{xxx} \rangle| + |\langle (\gamma_1)_x \tilde{u}_{xx}, \tilde{u}_{xxx} \rangle| \\ &\leq \|\gamma_1\| \|\tilde{u}_{xxx}\|^2 + \|(\gamma_1)_x\| \|\tilde{u}_{xx}\| \|\tilde{u}_{xxx}\| \\ &\leq \|\gamma_1\| \|\tilde{u}_{xxx}\|^2 + \|(\gamma_1)_x\| \left(\beta \|\tilde{u}_{xx}\|^2 + \frac{1}{4\beta} \|\tilde{u}_{xxx}\|^2 \right), \end{aligned} \tag{35}$$

here, we estimate $\|\tilde{u}_{xx}\|^2$ by means of the inequality,

$$\|\tilde{u}_{xx}\|^2 \leq \|\tilde{u}_x\| \|\tilde{u}_{xxx}\| \leq \beta \|\tilde{u}_x\|^2 + \frac{1}{4\beta} \|\tilde{u}_{xxx}\|^2.$$

Then, by using (33), (34), and (35), (32) becomes

$$\begin{aligned}&\frac{1}{2}\partial_t\left(\|\tilde{u}_x\|^2+\|\tilde{v}_x\|^2\right)+\gamma_0\|\tilde{u}_{xxx}\|^2+\Gamma_0\|\tilde{v}_{xx}\|^2\\ &\le\beta\left(1+\|\alpha_1\|+\|(\gamma_1)_x\|\right)\|\tilde{u}_x\|^2+\beta\|\tilde{v}_x\|^2+\beta\left(\left\|\tilde{f}_x\right\|^2+\|\tilde{g}_x\|^2\right)+\beta\|\tilde{v}_{xx}\|^2\\ &\quad+\|\Gamma_1\|\|\tilde{v}_{xx}\|^2+\beta\left(1+\|\alpha_1\|+\|(\gamma_1)_x\|\right)\|\tilde{u}_{xxx}\|^2+\|\gamma_1\|\|\tilde{u}_{xxx}\|^2.\end{aligned}\tag{36}$$

By using (13) and (14), we get

$$\begin{aligned}&\frac{1}{2}\partial_t\left(\|\tilde{u}_x\|^2+\|\tilde{v}_x\|^2\right)+\frac{\delta_0}{2}\|\tilde{u}_{xxx}\|^2+\frac{\delta_0}{2}\|\tilde{v}_{xx}\|^2\\ &\quad\le\beta\left(1+2\delta_0\right)\|\tilde{u}_x\|^2+\beta\|\tilde{v}_x\|^2+\beta\left(\left\|\tilde{f}_x\right\|^2+\|\tilde{g}_x\|^2\right)+\beta\|\tilde{v}_{xx}\|^2+\beta\left(1+2\delta_0\right)\|\tilde{u}_{xxx}\|^2.\end{aligned}$$

Since β is an arbitrary positive constant, we can choose β depending on δ_0 so that

$$\partial_t\left(\|\tilde{u}_x\|^2+\|\tilde{v}_x\|^2\right)+\|\tilde{u}_{xxx}\|^2+\|\tilde{v}_{xx}\|^2\le C_1\left(\|\tilde{u}_x\|^2+\|\tilde{v}_x\|^2+\left\|\tilde{f}_x\right\|^2+\|\tilde{g}_x\|^2\right).\tag{37}$$

From Theorem 1, we have (15) and (16). Combining (15) and (37), we get

$$\partial_t\left(\|\tilde{u}\|_1^2+\|\tilde{v}\|_1^2\right)+\|\tilde{u}\|_3^2+\|\tilde{v}\|_2^2\le C_1\left(\|\tilde{u}\|_1^2+\|\tilde{v}\|_1^2+\left\|\tilde{f}\right\|_1^2+\|\tilde{g}\|_1^2\right).\tag{38}$$

Since

$$\partial_t\left(\|\tilde{u}\|_1^2+\|\tilde{v}\|_1^2\right)\le C_1\left(\|\tilde{u}\|_1^2+\|\tilde{v}\|_1^2+\left\|\tilde{f}\right\|_1^2+\|\tilde{g}\|_1^2\right).\tag{39}$$

then, by applying Gronwall's inequality to (39), we get

$$\|\tilde{u}(t)\|_1^2+\|\tilde{v}(t)\|_1^2\le e^{C_1t}\left(\|\tilde{u}_0\|_1^2+\|\tilde{v}_0\|_1^2+C_1\int_0^t\left(\left\|\tilde{f}\right\|_1^2+\|\tilde{g}\|_1^2\right)ds\right)$$

which yields,

$$\sup_{0\le t\le T}\left(\|\tilde{u}(t)\|_1^2+\|\tilde{v}(t)\|_1^2\right)\le C_1(T)\left(\|\tilde{u}_0\|_1^2+\|\tilde{v}_0\|_1^2+\int_0^T\left(\left\|\tilde{f}\right\|_1^2+\|\tilde{g}\|_1^2\right)dt\right)\tag{40}$$

and hence, (38) and (40) imply that

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\tilde{u}(t)\|_1^2+\|\tilde{v}(t)\|_1^2\right)+\int_0^T\left(\|\tilde{u}\|_3^2+\|\tilde{v}\|_2^2\right)dt\\ &\qquad\le C_1(T)\left(\|\tilde{u}_0\|_1^2+\|\tilde{v}_0\|_1^2+\int_0^T\left(\left\|\tilde{f}\right\|_1^2+\|\tilde{g}\|_1^2\right)dt\right).\end{aligned}$$

Therefore, this theorem is true for $k=1$.

Next, suppose that this theorem is true for a positive integer j, $1<j<k+1$, we will show that this theorem is also true for $j=k+1$.

First, suppose that

$$\partial_t\left(\|\tilde{u}\|_j^2+\|\tilde{v}\|_j^2\right)+\|\tilde{u}\|_{j+2}^2+\|\tilde{v}\|_{j+1}^2\le C_j\left(\|\tilde{u}\|_j^2+\|\tilde{v}\|_j^2+\left\|\tilde{f}\right\|_j^2+\|\tilde{g}\|_j^2\right),\tag{41}$$

for a positive integer j, $1 < j < k+1$. We have to show that

$$\partial_t \left(\|\tilde{u}\|_{k+1}^2 + \|\tilde{v}\|_{k+1}^2 \right) + \|\tilde{u}\|_{k+3}^2 + \|\tilde{v}\|_{k+2}^2 \le C_{k+1} \left(\|\tilde{u}\|_{k+1}^2 + \|\tilde{v}\|_{k+1}^2 + \left\|\tilde{f}\right\|_{k+1}^2 + \|\tilde{g}\|_{k+1}^2 \right).$$

By taking L^2 inner product of (10) with $\partial_x^{2(k+1)}\tilde{u}$, (11) with $\partial_x^{2(k+1)}\tilde{v}$, integrating by parts and combining, we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t \left(\left\|\partial_x^{k+1}\tilde{u}\right\|^2 + \left\|\partial_x^{k+1}\tilde{v}\right\|^2 \right) + \gamma_0 \left\|\partial_x^{k+3}\tilde{u}\right\|^2 + \Gamma_0 \left\|\partial_x^{k+2}\tilde{v}\right\|^2 \\ &= \alpha_0 \left\|\partial_x^{k+2}\tilde{u}\right\|^2 + \varepsilon_1 \left\langle \partial_x^{k+2}\tilde{v}, \partial_x^{k+1}\tilde{u} \right\rangle + \varepsilon_2 \left\langle \partial_x^{k+2}\tilde{u}, \partial_x^{k+1}\tilde{v} \right\rangle + \left\langle \partial_x^{k+1}\tilde{f}, \partial_x^{k+1}\tilde{u} \right\rangle \\ &+ \left\langle \partial_x^{k+1}\tilde{g}, \partial_x^{k+1}\tilde{v} \right\rangle + \left\langle \partial_x^{k}\left(\alpha_1\tilde{u}_{xx}\right), \partial_x^{k+2}\tilde{u} \right\rangle + \left\langle \partial_x^{k}\left(\gamma_1\tilde{u}_{xxxx}\right), \partial_x^{k+2}\tilde{u} \right\rangle + \left\langle \partial_x^{k+1}\left(\Gamma_1\tilde{v}_{xx}\right), \partial_x^{k+1}\tilde{v} \right\rangle. \end{aligned} \tag{42}$$

Here we estimate $\|\partial_x^{k+2}\tilde{u}\|^2$ by means of the inequality

$$\left\|\partial_x^{k+2}\tilde{u}\right\|^2 \le \beta \left\|\partial_x^{k+1}\tilde{u}\right\|^2 + \frac{1}{4\beta} \left\|\partial_x^{k+3}\tilde{u}\right\|^2$$

and by applying Cauchy-Schwarz inequality and Cauchy's inequality to (42), we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t \left(\left\|\partial_x^{k+1}\tilde{u}\right\|^2 + \left\|\partial_x^{k+1}\tilde{v}\right\|^2) + \gamma_0 \left\|\partial_x^{k+3}\tilde{u}\right\|^2 + \Gamma_0 \left\|\partial_x^{k+2}\tilde{v}\right\|^2 \right) \\ &\le \beta \left(\left\|\partial_x^{k+1}\tilde{u}\right\|^2 + \left\|\partial_x^{k+2}\tilde{v}\right\|^2 + \left\|\partial_x^{k+1}\tilde{v}\right\|^2 + \left\|\partial_x^{k+3}\tilde{u}\right\|^2 + \left\|\partial_x^{k+1}\tilde{f}\right\|^2 + \left\|\partial_x^{k+1}\tilde{g}\right\|^2 \right) \\ &+ \left\langle \partial_x^{k}\left(\alpha_1\tilde{u}_{xx}\right), \partial_x^{k+2}\tilde{u} \right\rangle + \left\langle \partial_x^{k}\left(\gamma_1\tilde{u}_{xxxx}\right), \partial_x^{k+2}\tilde{u} \right\rangle + \left\langle \partial_x^{k+1}\left(\Gamma_1\tilde{v}_{xx}\right), \partial_x^{k+1}\tilde{v} \right\rangle. \end{aligned} \tag{43}$$

Considering that

$$\left\langle \partial_x^k(\alpha_1\tilde{u}_{xx}), \partial_x^{k+2}\tilde{u} \right\rangle \le \left|\left\langle \partial_x^k(\alpha_1\tilde{u}_{xx}), \partial_x^{k+2}\tilde{u} \right\rangle\right|$$

and since

$$\partial_x^k(\alpha_1\tilde{u}_{xx}) = \alpha_1\partial_x^{k+2}\tilde{u} + \binom{k}{1}(\alpha_1)_x\partial_x^{k+1}\tilde{u} + \binom{k}{2}(\alpha_1)_{xx}\partial_x^k\tilde{u} + \cdots + \tilde{u}_{xx}\partial_x^k\alpha_1,$$

where $\binom{k}{1}, \binom{k}{2}, \dots, \binom{k}{k-1}$ are the binomial coefficients, then Hölder inequality impiles that

$$\begin{aligned} \left\langle \partial_x^k\left(\alpha_1\tilde{u}_{xx}\right), \partial_x^{k+2}\tilde{u} \right\rangle \le \|\alpha_1\| \left\|\partial_x^{k+2}\tilde{u}\right\|^2 &+ \binom{k}{1} \|(\alpha_1)_x\| \left\|\partial_x^{k+1}\tilde{u}\right\| \left\|\partial_x^{k+2}\tilde{u}\right\| \\ &+ \binom{k}{2} \|(\alpha_1)_{xx}\| \left\|\partial_x^k\tilde{u}\right\| \left\|\partial_x^{k+2}\tilde{u}\right\| + \cdots + \left\|\partial_x^k\alpha_1\right\| \|\tilde{u}_{xx}\| \left\|\partial_x^{k+2}\tilde{u}\right\|. \end{aligned}$$

Since $\|\alpha_1\|_{k+1} \le \delta_{k+1}$, Cauchy's inequality implies that

$$\left\langle \partial_x^k(\alpha_1\tilde{u}_{xx}), \partial_x^{k+2}\tilde{u} \right\rangle \le \beta C_{k+1} \left(\left\|\partial_x^{k+2}\tilde{u}^2\right\| + \left\|\partial_x^{k+1}\tilde{u}^2\right\| + \cdots + \|\tilde{u}_{xx}\|^2 \right),$$

where β is an arbitrary positive constant. From (41),

$$\|\tilde{u}\|_{j+2}^2 \le C_j \left(\|\tilde{u}\|_j^2 + \|\tilde{v}\|_j^2 + \left\|\tilde{f}\right\|_j^2 + \|\tilde{g}\|_j^2 \right), \qquad 1 < j < k+1.$$

Thus,

$$\left\langle \partial_x^k\left(\alpha_1\tilde{u}_{xx}\right), \partial_x^{k+2}\tilde{u} \right\rangle \le \beta C_{k+1} \left(\|\tilde{u}\|_k^2 + \|\tilde{v}\|_k^2 + \left\|\tilde{f}\right\|_k^2 + \|\tilde{g}\|_k^2 \right). \tag{44}$$

Considering that

$$\left\langle \partial_x^k(\gamma_1\tilde{u}_{xxxx}), \partial_x^{k+2}\tilde{u}\right\rangle \leq \left|\left\langle \partial_x^{k-1}(\gamma_1\tilde{u}_{xxxx}), \partial_x^{k+3}\tilde{u}\right\rangle\right|$$

and since

$$\partial_x^{k-1}\left(\gamma_1\tilde{u}_{xxxx}\right) = \gamma_1\partial_x^{k+3}\tilde{u} + \binom{k-1}{1}\left(\gamma_1\right)_x\partial_x^{k+2}\tilde{u} + \binom{k-1}{2}\left(\gamma_1\right)_{xx}\partial_x^{k+1}\tilde{u} + \cdots + \tilde{u}_{xxxx}\partial_x^{k-1}\gamma_1,$$

where $\binom{k-1}{1}, \binom{k-1}{2}, \ldots \binom{k-1}{k-2}$ are the binomial coefficients, then Hölder inequality impiles that

$$\begin{aligned}\left\langle \partial_x^k\left(\gamma_1\tilde{u}_{xxxx}\right), \partial_x^{k+2}\tilde{u}\right\rangle \leq \|\gamma_1\|\left\|\partial_x^{k+3}\tilde{u}\right\|^2 &+ \binom{k-1}{1}\|(\gamma_1)_x\|\left\|\partial_x^{k+2}\tilde{u}\right\|\left\|\partial_x^{k+3}\tilde{u}\right\| \\ &+\binom{k-1}{2}\|(\gamma_1)_{xx}\|\left\|\partial_x^{k+1}\tilde{u}\right\|\left\|\partial_x^{k+3}\tilde{u}\right\| \\ &+\cdots+\left\|\partial_x^{k-1}\gamma_1\right\|\|\tilde{u}_{xxxx}\|\left\|\partial_x^{k+3}\tilde{u}\right\|.\end{aligned}$$

Since $\|\gamma_1\|_{k+1} \leq \delta_{k+1}$, Cauchy's inequality implies that

$$\begin{aligned}&\left\langle \partial_x^k\left(\gamma_1\tilde{u}_{xxxx}\right), \partial_x^{k+2}\tilde{u}\right\rangle \\ &\quad\leq \|\gamma_1\|\left\|\partial_x^{k+3}\tilde{u}\right\|^2 + \beta C_{k+1}\left\|\partial_x^{k+3}\tilde{u}\right\|^2 + \beta C_{k+1}\left(\left\|\partial_x^{k+2}\tilde{u}\right\|^2 + \left\|\partial_x^{k+1}\tilde{u}\right\|^2 + \cdots + \|\tilde{u}_{xxxx}\|^2\right).\end{aligned}$$

From (41), $\|\tilde{u}\|_{j+2}^2 \leq C_j(\|\tilde{u}\|_j^2 + \|\tilde{v}\|_j^2 + \|\tilde{f}\|_j^2 + \|\tilde{g}\|_j^2)$, $1 < j < k+1$ and therefore,

$$\begin{aligned}\left\langle \partial_x^k\left(\gamma_1\tilde{u}_{xxxx}\right), \partial_x^{k+2}\tilde{u}\right\rangle \leq \|\gamma_1\|\left\|\partial_x^{k+3}\tilde{u}\right\|^2 &+ \beta C_{k+1}\left\|\partial_x^{k+3}\tilde{u}\right\|^2 \\ +\beta C_{k+1}&\left(\|\tilde{u}\|_k^2 + \|\tilde{v}\|_k^2 + \left\|\tilde{f}\right\|_k^2 + \|\tilde{g}\|_k^2\right).\end{aligned} \tag{45}$$

From

$$\left\langle \partial_x^{k+1}\left(\Gamma_1\tilde{v}_{xx}\right), \partial_x^{k+1}\tilde{v}\right\rangle \leq \left|\left\langle \partial_x^k\left(\Gamma_1\tilde{v}_{xx}\right), \partial_x^{k+2}\tilde{v}\right\rangle\right|,$$

since

$$\partial_x^k\left(\Gamma_1\tilde{v}_{xx}\right) = \Gamma_1\partial_x^{k+2}\tilde{v} + \binom{k}{1}\left(\Gamma_1\right)_x\partial_x^{k+1}\tilde{v} + \binom{k}{2}\left(\Gamma_1\right)_{xx}\partial_x^k\tilde{v} + \cdots + \tilde{v}_{xx}\partial_x^k\Gamma_1,$$

where $\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{k-1}$ are the binomial coefficients, then Hölder inequality impiles that

$$\begin{aligned}\left\langle \partial_x^{k+1}\left(\Gamma_1\tilde{v}_{xx}\right), \partial_x^{k+1}\tilde{v}\right\rangle \leq \|\Gamma_1\|\left\|\partial_x^{k+2}\tilde{v}\right\|^2 &+ \binom{k}{1}\|(\Gamma_1)_x\|\left\|\partial_x^{k+1}\tilde{v}\right\|\left\|\partial_x^{k+2}\tilde{v}\right\| \\ &+\binom{k}{2}\|(\Gamma_1)_{xx}\|\left\|\partial_x^k\tilde{v}\right\|\left\|\partial_x^{k+2}\tilde{v}\right\| + \cdots + \left\|\partial_x^k\Gamma_1\right\|\|\tilde{v}_{xx}\|\left\|\partial_x^{k+2}\tilde{v}\right\|.\end{aligned}$$

Since $\|\Gamma_1\|_{k+1} \leq \delta_{k+1}$, Cauchy's inequality implies that

$$\begin{aligned}\left\langle \partial_x^{k+1}(\Gamma_1\tilde{v}_{xx}), \partial_x^{k+1}\tilde{v}\right\rangle \leq \|\Gamma_1\|\left\|\partial_x^{k+2}\tilde{v}\right\|^2 &+ \beta C_{k+1}\left\|\partial_x^{k+2}\tilde{v}\right\|^2 \\ &+ \beta C_{k+1}\left(\left\|\partial_x^{k+1}\tilde{v}\right\|^2 + \left\|\partial_x^k\tilde{v}\right\|^2 + \cdots + \|\tilde{v}_{xx}\|^2\right).\end{aligned}$$

From (41), $\|\tilde{v}\|_{j+1}^2 \leq C_j(\|\tilde{u}\|_j^2 + \|\tilde{v}\|_j^2 + \|\tilde{f}\|_j^2 + \|\tilde{g}\|_j^2)$, $1 < j < k+1$, so that one has

$$\begin{aligned}\left\langle \partial_x^{k+1}\left(\Gamma_1\tilde{v}_{xx}\right), \partial_x^{k+1}\tilde{v}\right\rangle \leq \|\Gamma_1\|\left\|\partial_x^{k+2}\tilde{v}\right\|^2 &+ \beta C_{k+1}\left\|\partial_x^{k+2}\tilde{v}\right\|^2 \\ +\beta C_{k+1}&\left(\|\tilde{u}\|_k^2 + \|\tilde{v}\|_k^2 + \left\|\tilde{f}\right\|_k^2 + \|\tilde{g}\|_k^2\right).\end{aligned} \tag{46}$$

By using (44), (45), and (46), (43) becomes

$$\begin{aligned} &\frac{1}{2}\partial_t\left(\left\|\partial_x^{k+1}\tilde{u}\right\|^2+\left\|\partial_x^{k+1}\tilde{v}\right\|^2\right)+\gamma_0\left\|\partial_x^{k+3}\tilde{u}\right\|^2+\Gamma_0\left\|\partial_x^{k+2}\tilde{v}\right\|^2\Big) \\ &\leq\beta\left(\left\|\partial_x^{k+1}\tilde{u}\right\|^2+\left\|\partial_x^{k+2}\tilde{v}\right\|^2+\left\|\partial_x^{k+1}\tilde{v}\right\|^2+\left\|\partial_x^{k+3}\tilde{u}\right\|^2+\left\|\partial_x^{k+1}\tilde{f}\right\|^2+\left\|\partial_x^{k+1}\tilde{g}\right\|^2\right) \\ &+\beta C_{k+1}\left(\|\tilde{u}\|_k^2+\|\tilde{v}\|_k^2+\left\|\tilde{f}\right\|_k^2+\|\tilde{g}\|_k^2\right)+\|\gamma_1\|\left\|\partial_x^{k+3}\tilde{u}\right\|^2+\beta C_{k+1}\left\|\partial_x^{k+3}\tilde{u}\right\|^2\Big) \\ &\qquad+\|\Gamma_1\|\left\|\partial_x^{k+2}\tilde{v}\right\|^2+\beta C_{k+1}\left\|\partial_x^{k+2}\tilde{v}\right\|^2. \end{aligned} \tag{47}$$

From (14) and since β is an arbitrary positive constant, we can choose β depending on δ_0, δ_k so that

$$\begin{aligned} &\partial_t\left(\left\|\partial_x^{k+1}\tilde{u}\right\|^2+\left\|\partial_x^{k+1}\tilde{v}\right\|^2\right)+\left\|\partial_x^{k+3}\tilde{u}\right\|^2+\left\|\partial_x^{k+2}\tilde{v}\right\|^2 \\ &\leq C_{k+1}\left(\|\tilde{u}\|_{k+1}^2+\|\tilde{v}\|_{k+1}^2+\left\|\tilde{f}\right\|_{k+1}^2+\|\tilde{g}\|_{k+1}^2\right). \end{aligned} \tag{48}$$

Combining (41) (for $j = k$) and (48), we get

$$\partial_t\left(\|\tilde{u}\|_{k+1}^2+\|\tilde{v}\|_{k+1}^2\right)+\|\tilde{u}\|_{k+3}^2+\|\tilde{v}\|_{k+2}^2\leq C_{k+1}\left(\|\tilde{u}\|_{k+1}^2+\|\tilde{v}\|_{k+1}^2+\left\|\tilde{f}\right\|_{k+1}^2+\|\tilde{g}\|_{k+1}^2\right). \tag{49}$$

Since

$$\partial_t\left(\|\tilde{u}\|_{k+1}^2+\|\tilde{v}\|_{k+1}^2\right)\leq C_{k+1}\left(\|\tilde{u}\|_{k+1}^2+\|\tilde{v}\|_{k+1}^2+\left\|\tilde{f}\right\|_{k+1}^2+\|\tilde{g}\|_{k+1}^2\right), \tag{50}$$

then by applying Gronwall's inequality to (50), we get

$$\|\tilde{u}(t)\|_{k+1}^2+\|\tilde{v}(t)\|_{k+1}^2\leq e^{C_{k+1}t}\left(\|\tilde{u}_0\|_{k+1}^2+\|\tilde{v}_0\|_{k+1}^2+C_{k+1}\int_0^t\left(\left\|\tilde{f}\right\|_{k+1}^2+\|\tilde{g}\|_{k+1}^2\right)ds\right).$$

Therefore,

$$\begin{aligned} &\sup_{0\leq t\leq T}\left(\|\tilde{u}(t)\|_{k+1}^2+\|\tilde{v}(t)\|_{k+1}^2\right) \\ &\leq C_{k+1}(T)\left(\|\tilde{u}_0\|_{k+1}^2+\|\tilde{v}_0\|_{k+1}^2+\int_0^T\left(\left\|\tilde{f}\right\|_{k+1}^2+\|\tilde{g}\|_{k+1}^2\right)dt\right). \end{aligned} \tag{51}$$

Then, (49) and (51) imply that

$$\begin{aligned} &\sup_{0\leq t\leq T}\left(\|\tilde{u}(t)\|_{k+1}^2+\|\tilde{v}(t)\|_{k+1}^2\right)+\int_0^T\left(\|\tilde{u}\|_{k+3}^2+\|\tilde{v}\|_{k+2}^2\right)dt \\ &\leq C_{k+1}(T)\left(\|\tilde{u}_0\|_{k+1}^2+\|\tilde{v}_0\|_{k+1}^2+\int_0^T\left(\left\|\tilde{f}\right\|_{k+1}^2+\|\tilde{g}\|_{k+1}^2\right)dt\right), \end{aligned}$$

and hence, this theorem is true for k+1 and the proof is complete.

THEOREM 3. *A Priori* ESTIMATE IN t-DIRECTION. *Let $(u,v)\in C^1([0,T];S(R))$ be the solution of (10) and (11), then (u,v) satisfies the following energy estimate*

$$\|\tilde{u}_t\|^2+\|\tilde{v}_t\|^2+\partial_t\left(\|\tilde{u}_{xx}\|^2+\|\tilde{v}_x\|^2\right)\leq C\left(\|\tilde{u}\|_2^2+\|\tilde{v}\|_2^2+\left\|\tilde{f}\right\|_2^2+\|\tilde{g}\|_2^2\right). \tag{52}$$

PROOF. By taking L^2 inner product of (10) with $\tilde{u}_t$, (11) with $\tilde{v}_t$, integrating by parts and combining, we obtain

$$\begin{aligned}
&\|\tilde{u}_t\|^2 + \|\tilde{v}_t\|^2 + \frac{\gamma_0}{2}\partial_t \|\tilde{u}_{xx}\|^2 + \frac{\Gamma_0}{2}\partial_t \|\tilde{v}_x\|^2 \\
&= \frac{\alpha_0}{2}\partial_t \|\tilde{u}_x\|^2 + \varepsilon_1 \langle \tilde{v}_x, \tilde{u}_t\rangle + \varepsilon_2 \langle \tilde{u}_x, \tilde{v}_t\rangle + \left\langle \tilde{f}, \tilde{u}_t\right\rangle + \langle \tilde{g}, \tilde{v}_t\rangle - \langle \alpha_1 \tilde{u}_{xx}, \tilde{u}_t\rangle \\
&\quad - \langle \gamma_1 \tilde{u}_{xxxx}, \tilde{u}_t\rangle + \langle \Gamma_1 \tilde{v}_{xx}, \tilde{v}_t\rangle .
\end{aligned} \tag{53}$$

By applying Cauchy-Schwarz inequality and Cauchy's inequality, (53) becomes

$$\begin{aligned}
&\|\tilde{u}_t\|^2 + \|\tilde{v}_t\|^2 + \frac{\gamma_0}{2}\partial_t \|\tilde{u}_{xx}\|^2 + \frac{\Gamma_0}{2}\partial_t \|\tilde{v}_x\|^2 \\
&\le \beta \left(\|\tilde{u}_t\|^2 + \|\tilde{v}_t\|^2 + \|\tilde{u}_x\|^2 + \|\tilde{v}_x\|^2 + \left\|\tilde{f}\right\|^2 + \|\tilde{g}\|^2 \right) + \frac{\alpha_0}{2}\partial_t \|\tilde{u}_x\|^2 \\
&\quad - \langle \alpha_1 \tilde{u}_{xx}, \tilde{u}_t\rangle - \langle \gamma_1 \tilde{u}_{xxxx}, \tilde{u}_t\rangle + \langle \Gamma_1 \tilde{v}_{xx}, \tilde{v}_t\rangle .
\end{aligned} \tag{54}$$

Hölder inequality and Cauchy's inequality then imply that

$$\begin{aligned}
- \langle \alpha_1 \tilde{u}_{xx}, \tilde{u}_t\rangle &\le |\langle \alpha_1 \tilde{u}_{xx}, \tilde{u}_t\rangle| \\
&\le \|\alpha_1\| \|\tilde{u}_{xx}\| \|\tilde{u}_t\| \\
&\le \|\alpha_1\| \left(\beta \|\tilde{u}_{xx}\|^2 + \frac{1}{4\beta} \|\tilde{u}_t\|^2 \right)
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
- \langle \gamma_1 \tilde{u}_{xxxx}, \tilde{u}_t\rangle &\le |\langle \gamma_1 \tilde{u}_{xxxx}, \tilde{u}_t\rangle| \\
&\le \|\gamma_1\| \|\tilde{u}_{xxxx}\| \|\tilde{u}_t\| \\
&\le \|\gamma_1\| \left(\beta \|\tilde{u}_{xxxx}\|^2 + \frac{1}{4\beta} \|\tilde{u}_t\|^2 \right),
\end{aligned} \tag{56}$$

and also,

$$\begin{aligned}
- \langle \Gamma_1 \tilde{v}_{xx}, \tilde{v}_t\rangle &\le |\langle \Gamma_1 \tilde{v}_{xx}, \tilde{v}_t\rangle| \\
&\le \|\Gamma_1\| \|\tilde{v}_{xx}\| \|\tilde{v}_t\| \\
&\le \|\Gamma_1\| \left(\beta \|\tilde{v}_{xx}\|^2 + \frac{1}{4} \|\tilde{v}_t\|^2 \right).
\end{aligned} \tag{57}$$

By using (55), (56), and (57), (54) becomes

$$\begin{aligned}
&\|\tilde{u}_t\|^2 + \|\tilde{v}_t\|^2 + \frac{\gamma_0}{2}\partial_t \|\tilde{u}_{xx}\|^2 + \frac{\Gamma_0}{2}\partial_t \|\tilde{v}_x\|^2 \\
&\le \beta (1 + \|\alpha_1\| + \|\gamma_1\|) \|\tilde{u}_t\|^2 + \beta (1 + \|\Gamma_1\|) \|\tilde{v}_t\|^2 + \beta \left(\|\tilde{u}_x\|^2 + \|\tilde{v}_x\|^2 + \left\|\tilde{f}\right\|^2 + \|\tilde{g}\|^2 \right) \\
&\quad + \beta \|\alpha_1\| \|\tilde{u}_{xx}\|^2 + \beta \|\gamma_1\| \|\tilde{u}_{xxxx}\|^2 + \beta \|\Gamma_1\| \|\tilde{v}_{xx}\|^2 + \frac{\alpha_0}{2}\partial_t \|\tilde{u}_x\|^2 .
\end{aligned}$$

By using (13), one obtains

$$\begin{aligned}
&\|\tilde{u}_t\|^2 + \|\tilde{v}_t\|^2 + \frac{\gamma_0}{2}\partial_t \|\tilde{u}_{xx}\|^2 + \frac{\Gamma_0}{2}\partial_t \|\tilde{v}_x\|^2 \\
&\le \beta (1 + 2\delta_0) \|\tilde{u}_t\|^2 + \beta (1 + \delta_0) \|\tilde{v}_t\|^2 + \beta \left(\|\tilde{u}_x\|^2 + \|\tilde{v}_x\|^2 + \left\|\tilde{f}\right\|^2 + \|\tilde{g}\|^2 \right) \\
&\quad + \beta\delta_0 \|\tilde{u}_{xx}\|^2 + \beta\delta_0 \|\tilde{u}_{xxxx}\|^2 + \beta\delta_0 \|\tilde{v}_{xx}\|^2 + \frac{\alpha_0}{2}\partial_t \|\tilde{u}_x\|^2 .
\end{aligned} \tag{58}$$

From Theorem 1, we have

$$\|\tilde{u}_{xx}\|^2 \le C \left(\|\tilde{u}\|^2 + \|\tilde{v}\|^2 + \left\|\tilde{f}\right\|^2 + \|\tilde{g}\|^2 \right). \tag{59}$$

From Theorem 2 (for $k = 2$), we have

$$\|\tilde{u}_{xxxx}\|^2 \le C_2 \left(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_2^2 + \left\|\tilde{f}\right\|_2^2 + \|\tilde{g}\|_2^2 \right). \tag{60}$$

Also, from Theorem 2 (for $k = 1$), we have

$$\|\tilde{v}_{xx}\|^2, \partial_t \|\tilde{u}_x\|^2 \le C_1 \left(\|\tilde{u}\|_1^2 + \|\tilde{v}\|_1^2 + \left\|\tilde{f}\right\|_1^2 + \|\tilde{g}\|_1^2 \right). \tag{61}$$

Therefore, by using (59), (60), (61) and since β is an arbitrary positive constant, we can choose β depending on δ_0 so that (58) becomes

$$\|\tilde{u}_t\|^2 + \|\tilde{v}_t\|^2 + \partial_t \left(\|\tilde{u}_{xx}\|^2 + \|\tilde{v}_x\|^2 \right) \le C \left(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_2^2 + \left\|\tilde{f}\right\|_2^2 + \|\tilde{g}\|_2^2 \right)$$

and the proof is complete.

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