Classes of chromatically unique graphs

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Abstract

We prove that graphs obtained from complete equibipartite graphs by deleting some independent sets of edges are chromatically unique.

1. Preliminary definitions and results

In this paper we consider finite, undirected, simple and loopless graphs. Two graphs $G$ and $H$ are said to be chromatically equivalent if they have the same chromatic polynomial, i.e. $P_H(\lambda)=P_G(\lambda)$. A graph $G$ is said to be chromatically unique if $P_H(\lambda)=P_G(\lambda)$ implies that $H$ is isomorphic to $G$.

At present, any method for recognizing whether a given graph is chromatically unique is not known. For large graphs, it is extremely difficult to prove chromatic uniqueness. Some classes of chromatically unique graphs are well known. See, for instance, [1, 2, 5-7] and their references, where some families of such graphs are presented. Here, by solving an extremal problem for bipartite graphs, we prove that graphs obtained from complete equibipartite graphs by deleting some independent sets of edges are chromatically unique.

By a bipartite graph we mean such a graph whose vertex set can be partitioned into two nonempty sets $U$ and $V$, called colour classes, such that every edge of the graph...
joins an element of \( U \) with an element of \( V \). If colour classes of a bipartite graph have the same cardinality then we call it equibipartite. The reader is referred to [4] for further information.

Using the chromatic polynomial, the bipartite graphs can be characterized as follows.

**Proposition 1.1.** \( G \) is bipartite graph if and only if \( P_G(\lambda) \) is not divisible by \( \lambda - 2 \).

From this we immediately have the following proposition.

**Proposition 1.2.** If \( H \) is chromatically equivalent to \( G \) and \( G \) is bipartite, then \( H \) is bipartite.

Thus, the only candidates to be chromatically equivalent to a bipartite graph are bipartite graphs.

According to [3], for a given graph \( G \), the first four coefficients of its chromatic polynomial can be written as follows:

\[
P_G(\lambda) = \lambda^p - q\lambda^{p-1} + \left[ \binom{q}{2} - N_T(G) \right] \lambda^{p-2} + \left[ -\binom{q}{3} + (q-2)N_T(G) + N_Q(G) - 2N_K(G) \right] \lambda^{p-3} + \cdots,
\]

where \( p, q, N_T(G), N_Q(G) \) and \( N_K(G) \) denote the number of vertices, edges, triangles (cycles of order three), pure quadrilaterals (cycles of order four without chords) and complete graphs with four vertices of \( G \), respectively.

For a bipartite graph \( G \), we have \( N_T(G) - N_K(G) = 0 \). Thus, we have the following result.

**Proposition 1.3.** If \( H \) is chromatically equivalent to a bipartite graph \( G \), then \( H \) must be a bipartite graph having the same number of vertices, edges and pure quadrilaterals as \( G \).

Let \( G = (U, V; E) \) be a bipartite graph with \( |U| = m \) and \( |V| = n \). Such a graph will also be denoted by \( G_{m,n} \) to emphasize the cardinalities of colour classes of \( G \). Without loss of generality, we assume that \( m \leq n \).

For a given graph \( G_{m,n} \) with \( U = \{u_1, \ldots, u_m\}, V = \{v_1, \ldots, v_n\} \), let us denote by \( X(G_{m,n}) \) the \( m \times n \) matrix \( [x_{ij}] \) defined by \( x_{ij} = 1 \) if there exists an edge joining \( u_i \) and \( v_j \), and \( x_{ij} = 0 \) otherwise. The degrees of \( u_i \) and \( v_j \) are denoted by \( d_i \) and \( e_j \), respectively. The complementary graph \( \tilde{G}_{m,n} \) of \( G_{m,n} \) with respect to \( K_{m,n} \) is defined by its matrix \( X(\tilde{G}_{m,n}) = [\tilde{x}_{ij}] \) in the following way: \( \tilde{x}_{ij} = 1 - x_{ij} \). Degrees of vertices of \( \tilde{G}_{m,n} \) will be denoted by \( \tilde{d}_i \) and \( \tilde{e}_j \), respectively.
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Proposition 1.4 Salzberg et al. [6].

\[ N_Q(G_{m,n}) = \frac{1}{2} mn(m-1)(n-1) - \frac{1}{2} [(2m-1)(2n-1)\tilde{q} - (2m-1)\tilde{D} - (2n-1)\tilde{E} + 4\tilde{R} - \tilde{S} - 2\tilde{q}^2], \]

where \( \tilde{q} \) denotes the number of edges of \( \tilde{G}_{m,n} \), \( \tilde{D} = \sum_{i=1}^{m} \tilde{d}_{i}^2 \), \( \tilde{E} = \sum_{j=1}^{n} \tilde{e}_{j}^2 \), \( \tilde{R} = \sum_{i,j} \tilde{d}_{i} \tilde{e}_{j} \tilde{x}_{ij} \), \( \tilde{S} = \sum_{i,j} \tilde{x}_{ij} \tilde{x}_{ij}^t \tilde{x}_{i}^t \).

2. The chromatic uniqueness of \( K_{m,m-r}K_2 \)

The graph \( G = K_{m,m-r}K_2 \), \( 0 \leq r \leq m \), has order \( 2m \) and size \( m^2 - r \). According to Proposition 1.3, a prospective candidate to be chromatically equivalent to \( G \) should be a bipartite graph \( G_{s,t} \) satisfying \( s + t = 2m \) and \( st \geq m^2 - r \). By solving these two expressions for \( s \) or \( t \), we obtain

\[ m - \sqrt{r} \leq s \leq t \leq m + \sqrt{r}. \]

Parametrizing \( s \) and \( t \), we have \( s = m - k \), \( t = m + k \) for \( 0 \leq k \leq \sqrt{r} \). This implies the following result.

Lemma 2.1. If \( H \) is chromatically equivalent to \( G = K_{m,m-r}K_2 \), then \( H = H_{m-k,m+k} \) for some \( k \), \( 0 \leq k \leq \sqrt{r} \). \( H \) can be obtained from the complete bipartite graph \( K_{m-k,m+k} \) by deleting \( r - k^2 \) edges.

Lemma 2.2. Let \( G = K_{m,m-r}K_2 \). Then

\[ N_Q(G) = \frac{1}{2} m^2 (m-1)^2 - 2r(m^2 - 4m + 3 - r). \]

Proof. For the graph \( \tilde{G} \), we have \( \tilde{q} = \tilde{D} = \tilde{E} = \tilde{R} = \tilde{S} = r \). By Proposition 1.4 and simple calculations we have the require formula. \( \square \)

Let us denote by \( S_{k}^{r} \) the graph \( K_{m-k,m+k}-(r-k^2)K_2 \), where \( 0 \leq k \leq \sqrt{r} \). It is easy to see that \( S_{k}^{r} \) is a candidate to be chromatically equivalent to \( G = K_{m,m-r}K_2 \).

Lemma 2.3.

\[ N_Q(S_{k}^{r}) = \frac{1}{2} [(m^2-k^2)((m-1)^2-k^2)-2r(k^2)(2(m^2-k^2)-4m+3-(r-k^2))]. \]

Proof. By the fact that \( \tilde{q} = \tilde{D} = \tilde{E} = \tilde{R} = \tilde{S} = r - k^2 \) for \( S_{k}^{r} \) and Proposition 1.4, the formula follows. \( \square \)

Lemma 2.4. Let \( G = K_{m,m-r}K_2 \). Then for \( k \geq 1 \) and \( m \geq 3 \),

\[ N_Q(S_{k}^{r}) > N_Q(G). \]
Proof. By Lemmas 2.2, 2.3, and the fact that $k^2 \leq r \leq m$, we have

$$N_Q(S) - N_Q(G) = \frac{1}{8} k^2 (2m^2 - 6m + 5 - k^2) > \frac{1}{8} k^2 (2m^2 - 7m + 5).$$

But $\frac{1}{8} k^2 (2m^2 - 7m + 5) > 0$ for $k \geq 1$ and $m \geq 3$. The lemma is proved. \[\square\]

Now we define on a bipartite graph $G_{m,n}$ (for simplicity, via its complement $\tilde{G}_{m,n}$) the removing and adding of an edge operation (in short, RA-operation).

Let there be given a bipartite graph $G_{m,n}$ satisfying the following condition:

\[(*) \quad 1 \leq \tilde{d}_i \leq m \leq n \text{ and } (\tilde{d}_i = 0, \tilde{d}_j \geq 2 \text{ or } \tilde{e}_i = 0, \tilde{e}_j \geq 2).\]

Let $\tilde{G}_{m,n}$ denote the graph obtained from $\tilde{G}_{m,n}$ by either deleting the edge $U_iU_j$ and adding the edge $U_iV_l$ for some $1 \leq i \leq n$ or deleting the edge $V_jU_l$ and adding the edge $V_lU_i$ for some $1 \leq l \leq m$, respectively. The graph $G_{m,n}$ will be called a result of the RA-operation on $G_{m,n}$.

Lemma 2.5. Let $G_{m,n}$ satisfy the condition $(*)$ and for some pair $(U_i, U_j)$ of vertices let $G_{m,n}'$ be a result of the RA-operation on $G_{m,n}$. Then

$$N_Q(G_{m,n}) \leq N_Q(G_{m,n}').$$

Proof. Consider the case when $\tilde{d}_i = 0$ and $\tilde{d}_j \geq 2$. Let $\tilde{q}', \tilde{D}', \tilde{E}', \tilde{R}', \tilde{S}'$ denote the corresponding parameters of $G_{m,n}'$. By the definition of RA-operation, we can compute that

$$\tilde{q}' = \tilde{q}, \quad \tilde{D}' = \tilde{D} - \tilde{d}_j^2 + 1 + (\tilde{d}_j - 1)^2 = \tilde{D} - 2\tilde{d}_j + 2, \quad \tilde{E}' = \tilde{E},$$

$$\tilde{R}' = \tilde{R} - \tilde{e}_j\tilde{d}_j - \tilde{d}_j \sum_{s \in \mathcal{P}} \tilde{e}_s + \tilde{e}_j + (\tilde{d}_j - 1) \sum_{s \in \mathcal{P}} \tilde{e}_s = \tilde{R} - \tilde{e}_j(\tilde{d}_j - 1) - \sum_{s \in \mathcal{P}} \tilde{e}_s,$$

where $\mathcal{P} = \{s: s \neq l, \tilde{x}_{js} = 1\}$, and

$$\tilde{S}' = \tilde{S} - 2\left(\frac{\tilde{d}_j}{2}\right)^2 - 4C = \tilde{S} - 2\tilde{d}_j + 2 - 4C,$$

$C$ being the number of pure quadrilaterals destroyed by deleting the edge $U_jV_l$.

Note that, by adding the edge $U_iV_l$, the number of pure quadrilaterals do not increase.

$$N_Q(G_{m,n}) - N_Q(G_{m,n}') = m\tilde{d}_j - m - \sum_{s \in \mathcal{P}} \tilde{e}_s - \tilde{e}_j(\tilde{d}_j - 1) + 4C \geq (m - \tilde{e}_j)(\tilde{d}_j - 1) - \sum_{s \in \mathcal{P}} \tilde{e}_s \geq m - \tilde{e}_j - \sum_{s \in \mathcal{P}} \tilde{e}_s \geq 0.$$

Similarly, the lemma holds when $\tilde{e}_i = 0$ and $\tilde{e}_j \geq 2$. \[\square\]

The above given result leads us to the next lemma.
Lemma 2.6. For any graph $G_{m-k,m+k}$ with $q=r-k^2$, $k^2 < r < m$, there exists a finite sequence of RA-operations which leads from a graph $G_{m-k,m+k}$ to the graph $K_{m-k,m+k}-(r-k^2)K_2 = S'_k$.

Theorem 2.7. For $m \geq 3$, the graph $G = K_{m,m} - rK_2$, $0 \leq r \leq m$, is chromatically unique.

Proof. Suppose that $G_{m-k,m+k}$ is a candidate to be chromatically equivalent to $G$. Thus, by Lemmas 2.5 and 2.6, we have

(a) $N_Q(G_{m-k,m+k}) \geq N_Q(K_{m-k,m+k}-(r-k^2)K_2) = N_Q(S'_k)$.

But, by Lemma 2.4 and (a), for $k \geq 1$, we have

(b) $N_Q(G_{m-k,m+k}) \geq N_Q(S'_k) > N_Q(G)$.

Let $k = 0$. For the graph $G_{0,m}^0$, which leads to $G = K_{m,m} - rK_2$ by performing the last RA-operation, we have

(c) $N_Q(G_{0,m}^0) > N_Q(S'_0) = N_Q(G)$.

By (c) and Lemma 2.5, we obtain

(d) $N_Q(G_{m,n}) \geq N_Q(G_{m,n}^0) > N_Q(G)$.

Thus, (b), (d) and Proposition 1.3 imply that $G = K_{m,m} - rK_2$, $0 \leq r \leq m$, is chromatically unique. $\square$

References