A Singular Boundary Value Problem for the One-Dimensional $p$-Laplacian*

Junyu Wang and Wenjie Gao

Department of Mathematics, Jilin University, Changchun 130023, People's Republic of China

Submitted by Hal L. Smith

Received July 31, 1995

The singular boundary value problem

$$
\begin{cases}
(g(u'))' = -k(t)f(u), & 0 < t < 1, \\
u(0) = u(1) = 0
\end{cases}
$$

is studied in this paper where $g(s) = |s|^{p-2}s$, $p > 1$. The singularity may appear at $u = 0$ and at $t = 0$ or $t = 1$ and the function $f$ may be discontinuous. The authors prove that for any $p > 1$ and for any positive, nonincreasing function $f$ and nonnegative measurable function $k$ with some integrability conditions, the above-mentioned problem has a unique solution. Also, the properties of the solution are discussed in the paper.

1. INTRODUCTION

The boundary value problem for the one-dimensional $p$-Laplacian

$$
\begin{cases}
(g(u'))' = -k(t)f(u), & 0 < t < 1, \\
u(0) = u(1) = 0
\end{cases}
$$

(1.1)

where $g(s) = |s|^{p-2}s$, $p > 1$, has been studied extensively. For details, see, for example, Refs. [1–5, 7]. The boundary value problem treated in the above-mentioned references is not able to possess singularity.

*The authors are partially supported by the NNSF of China.

†E-mail address: yjx@mail.jlu.edu.cn.

Copyright © 1996 by Academic Press, Inc.
All rights of reproduction in any form reserved.
In [6], Taliaferro considered a particular case of (1.1) where $p = 2$, $f(u) = u^{-\lambda}$, $\lambda > 0$, and $k(t)$ is positive and continuous in $(0,1)$. The following theorem was established there.

**Taliaferro’s Theorem.** Assume that $p = 2$, $f(u) = u^{-\lambda}$, $\lambda > 0$, and $k(t)$ is positive and continuous in $(0,1)$. Then the following statements hold.

1. The boundary value problem (1.1) has a unique solution $u_\lambda(t)$ if and only if
   \[ \int_0^1 t(1-t)k(t)\,dt < +\infty. \]  

2. $\max(u_\lambda(t); 0 \leq t \leq 1) \leq M$, where $M$ is the positive solution of the equation
   \[ \left( \frac{M - \frac{1}{2}}{2} \right) \left( \frac{M + \frac{1}{2}}{2} \right) = N \]
   with
   \[ N := \max \left\{ \int_0^{1/2} k(t)\,dt, \int_{1/2}^1 k(t)(1-t)\,dt \right\}. \]

3. $u_\lambda(t)$ tends to 1, uniformly on compact subsets of $(0,1)$, as $\lambda \to +\infty$.

4. $u_\lambda'(0+) = \lim_{t \to 0^+} u_\lambda'(t)u_\lambda'(1-) = \lim_{t \to 1^-} u_\lambda'(t)$ is finite if and only if
   \[ \int_0^{1/2} k(t)t^{-\lambda}\,dt < +\infty \quad \left( \int_{1/2}^1 k(t)(1-t)^{-\lambda}\,dt < +\infty \right). \]

The above particular case of (1.1) possesses singularity at $u = 0$ and is able to possess singularity at $t = 0$ and $t = 1$. The existence and uniqueness of the solution $u_\lambda(t)$ were obtained by means of the shooting method.

The aim of this paper is to extend the above-mentioned results. We adopt the following hypotheses:

1. $f(u)$ is positive, right continuous, nonincreasing in $(0, +\infty)$ and $f(0+) = \lim_{u \to 0^+} f(u) = +\infty$.

2. $k(t)$ is a nonnegative measurable function defined in $(0,1)$.
We will prove the following theorem.

**Theorem 1.** Assume that (H1) and (H2) are satisfied. Then the following statements hold.

1. The boundary value problem (1.1) has a positive solution \( u(t) \) if and only if
   \[
   0 < \int_0^{1/2} G \left( \int_0^{1/2} k(r) dr \right) ds + \int_0^1 G \left( \int_0^r k(r) dr \right) ds < +\infty, \quad (1.5)
   \]
   where \( G(x) \) is the inverse function to \( g(s) \).

   (1) If for every \( \theta > 0 \),
   \[
   \int_0^{1/2} k(r) f(\theta r) dr < +\infty, \quad \left( \int_0^{1/2} k(r) f(\theta(1-r)) dr < +\infty \right) \quad (1.6)
   \]
   then \( u'(0+) \) (\( u'(1-) \)) is finite.

   (1) If \( u'(0+) \) (\( u'(1-) \)) is finite, then (1.6) holds for \( \theta \geq u'(0+) \) (\( \theta \geq |u'(1)| \)).

   (1) If
   \[
   \left( \frac{M-1}{2} \right) G \left( f \left( \frac{M+1}{2} \right) \right) = N;
   \]
   \[
   N := \max \left\{ \int_0^{1/2} G \left( \int_0^{1/2} k(r) dr \right) ds, \int_0^1 G \left( \int_0^r k(r) dr \right) ds \right\},
   \]
   has a positive solution \( M \) and \( u \) is the positive solution of (1.1), then \( u \leq M \).

   **Remark 1.** The existence of the positive solution will be obtained by means of the perturbation technique and the Schauder fixed point theorem.

   **Remark 2.** The condition (1.5) allows \( k(t) \) to be equal to zero on some open or closed subintervals of (0, 1). For example, the function
   \[
   k(t) = \begin{cases} 
   t^{-\alpha}, & 0 < t < 1/9, 0 < \alpha < p, \\
   0, & 1/9 \leq t \leq 8/9, \\
   (1-t)^{-\beta}, & 8/9 < t < 1, 0 < \beta < p
   \end{cases}
   \]
   satisfies the condition (1.5).
Remark 3. When \( f(u) = u^{-\lambda} \), \( \lambda > 0 \), (1.6) becomes
\[
\int_0^{1/2} G\left( \frac{1}{s} \int_s^{1/2} k(r) r^{-\lambda} \; dr \right) \; ds < +\infty,
\]
\[
\left( \int_0^{1/2} G\left( \frac{1}{s} \int_s^{1/2} k(r) (1-r)^{-\lambda} \; dr \right) \; ds < +\infty. \right)
\]
Therefore, \( u'(0+) \) (\( u'(1-) \)) is finite if and only if the above condition holds.

Remark 4. The claim (iii) in Taliaferro's Theorem is also true for \( f(u) = u^{-\lambda} \), the proof is the same as that in [6].

Remark 5. Our result shows that the function \( f \) may be discontinuous.

2. SOME PRIMARY RESULTS

Assume (1.5) and consider the "approximate" boundary value problem
\[
\begin{aligned}
(g(u'))' &= -k(t)f(u), \quad 0 < t < 1, \\
u(0) &= u(1) = h.
\end{aligned}
\]
(2.1)_h

A function \( u(t) \) is said to be a positive solution to the boundary value problem (2.1)_h with \( h \geq 0 \), if the following conditions are satisfied:
(i) \( u(t) \in C[0, 1] \cap C^1(0, 1) \);
(ii) \( u(t) > 0 \) in \( (0, 1) \), \( u(0) = u(1) = h \);
(iii) \( g(u'(t)) \) is locally absolutely continuous in \( (0, 1) \), and
(iv) \( (g(u'(t)))' = -k(t)f(u(t)) \) a.e. in \( (0, 1) \).

Lemma 1. For each fixed \( h \geq 0 \), the boundary value problem (2.1)_h has at most one positive solution.

Proof. Suppose that \( u_1(t) \) and \( u_2(t) \) are positive solutions to (2.1)_h.
If \( u_1(t) \neq u_2(t) \) on \([0, 1]\), then there would exist a \( t_0 \in (0, 1) \) at which \( u_1(t_0) > u_2(t_0) \) and hence there would exist an interval \((a, b)\) such that \( u_1(t) > u_2(t) \) in \((a, b)\) and \( u_1(a) - u_1(b) = u_2(b) - u_2(b) = 0 \). Let \( m = u_1(B) - u_2(B) \) be the positive maximum of \( u_1(t) - u_2(t) \) on \([a, b]\). Then \( B \in (a, b) \) and \( u_1(B) = u_2(B) \). Notice that for \( j = 1, 2 \),
\[
(g(u_j'(r)))' = -k(r)f(u_j(r)) \quad \text{a.e. in } (0, 1).
\]
Integrating both sides of this equality over \([s, B], a < s < B\), we get
\[
u_j'(s) = G\left( g(u_j'(B)) + \int_s^B k(r)f(u_j(r)) \; dr \right), \quad a < s \leq B.
\]
Integrating both sides of the above equality from $a$ to $B$, we obtain

$$u_j(B) - u_j(a) = \int_a^B G \left( g(u'_j(B)) + \int_a^B k(r)f(u_j(r)) \, dr \right) \, ds.$$  

Consequently, we are lead to a contradiction $0 < m = u_j(B) - u_j(a) \leq 0$. Here we have used the fact that $f(u)$ is nonincreasing in $u$. The proof of the lemma is complete.

To prove the existence of solution to (2.1)$_h$ with $h > 0$, we consider the boundary value problem

$$\begin{cases}
(g(u'))' = -k(t)f(w(t)), & 0 < t < 1, \\
u(0) = u(1) = h > 0.
\end{cases} \tag{2.2}_h$$

for any $w(t) \in D_h := \{w \in C[0,1]; w(t) \geq h\}$.

**Lemma 2.** For each fixed $h > 0$ and each $w \in D_h$, the boundary value problem (2.2)$_h$ has a unique solution $u(t) \geq h$.

**Proof.** We only prove the existence since the proof of the uniqueness is very simple. Set for $0 < t < 1$

$$x(t) := \int_0^t G \left( \int_s^t k(r)f(w(r)) \, dr \right) \, ds - \int_t^1 G \left( \int_t^s k(r)f(w(r)) \, dr \right) \, ds.$$  

Clearly, $x(t)$ is continuous and nondecreasing in $(0,1)$ and $x(0+) < 0 < x(1-)$. Thus, $x(t)$ has zeros in $(0,1)$. Let $A$ be a zero of $x(t)$ in $(0,1)$. Then

$$\int_0^A G \left( \int_s^A k(r)f(w(r)) \, dr \right) \, ds = \int_A^1 G \left( \int_A^s k(r)f(w(r)) \, dr \right) \, ds. \tag{2.3}$$

Put

$$u(t) = (\Phi w)(t) := \begin{cases} 
    h + \int_0^t G \left( \int_s^A k(r)f(w(r)) \, dr \right) \, ds, & 0 \leq t \leq A, \\
    h + \int_t^1 G \left( \int_A^s k(r)f(w(r)) \, dr \right) \, ds, & A \leq t \leq 1.
\end{cases} \tag{2.4}_h$$

Then, $u$ is a well-defined differentiable function and

$$u'(t) = (\Phi w)'(t) = G \left( \int_t^A k(r)f(w(r)) \, dr \right), \quad 0 < t < 1.$$
It is obvious that \( u'(t) = (\Phi w)'(t) \) defined as above is continuous and nonincreasing in \((0, 1)\), \( u'(A) = 0 \), \( u(t) \in D_h \), and \((2.2)_h \) is satisfied for a.e. \( t \in (0, 1) \). This shows that \( u(t) \) is a solution of \((2.2)_h \) and a concave function defined on \([0, 1]\). The lemma is proven.

**Remark 6.** It is easy to show that \((2.3) \) and \((2.4)_h \) are independent of the choice of the zero \( \mathcal{A} \). Therefore, \( \Phi \) is a well defined map on \( D_h \).

**Lemma 3.** Let \( \Phi : D_h \to D_h \) be the mapping defined by \((2.3) \) and \((2.4)_h \), and \( w_1, w_2 \in D_h \). If \( w_1(t) \leq w_2(t) \) on \([0, 1]\), then \( (\Phi w_1)(t) \geq (\Phi w_2)(t) \) on \([0, 1]\).

**Proof.** The proof of this lemma is very similar to that of Lemma 1 and hence omitted here.

**Lemma 4.** For any \( w \in D_h \), we have
\[
\mathcal{A} \leq (\Phi w)(t) \leq (\Phi h)(t) \leq (\Phi h)(\mathcal{A}^*) \quad \text{on} \quad [0, 1],
\]
where \( \mathcal{A}^* \) is a zero of the function
\[
y(t) := \int_0^t G\left( \int_0^s k(r) \, dr \right) ds - \int_t^1 G\left( \int_t^r k(r) \, dr \right) ds, \quad 0 < t < 1.
\]

**Proof.** The lemma follows from Lemma 3 and the definition of \( \Phi \).

**Lemma 5.** \( \Phi(D_h) \) is equicontinuous on \([0, 1]\).

**Proof.** For any \( \epsilon > 0 \), from the continuity of \((\Phi h)(t) \) on \([0, 1]\), it follows that there is a \( \delta_1 \in (0, 1/4) \) such that
\[
(\Phi h)(2\delta_1), (\Phi h)(1 - 2\delta_1) < \epsilon + h.
\]
If \( (\Phi w)(A) < \epsilon + h \), then for any \( t_1, t_2 \in [0, 1] \)
\[
|(\Phi w)(t_1) - (\Phi w)(t_2)| \leq |(\Phi w)(A) - (\Phi w)(0)| < \epsilon.
\]
If \( (\Phi w)(A) \geq \epsilon + h \), then \( A \in [2\delta_1, 1 - 2\delta_1] \) and hence for \( t \in [\delta_1, 1 - \delta_1] \),
\[
|(\Phi w)'(t)| = \left| G\left( \int_t^A k(r) f(w(r)) \, dr \right) \right|
\]
\[
\leq G\left( \int_{\delta_1}^{1 - \delta_1} k(r) \, dr \right) G(f(h)) = L.
\]
Put \( \delta_2 = \epsilon/L \), then for \( t_1, t_2 \in [\delta_1, 1 - \delta_1] \), \( |t_1 - t_2| < \delta_2 \)
\[
|(\Phi w)(t_1) - (\Phi w)(t_2)| = |(\Phi w)'(\xi)||t_1 - t_2| < L\delta_2 = \epsilon,
\]

where $\xi$ lies between $t_1$ and $t_2$. Set $\delta = \min(\delta_1, \delta_2)$. Then for $t_1, t_2 \in [0, 1]$, $|t_1 - t_2| < \delta$,

$$ |(\Phi w)(t_1) - (\Phi w)(t_2)| < \epsilon. $$

This shows that $\Phi(D_h)$ is equicontinuous on $[0, 1]$.

**Lemma 6.** The mapping $\Phi$ is continuous on $D_h$ if the function $f$ is continuous in its variable.

**Proof.** Assume that $(w_{j})_{j=0}^{n} \subset D_h$ and $w_{j}(t)$ converges to $w_{0}(t)$ uniformly on $[0, 1]$. By Lemma 5, it follows that $(\Phi w_{j}(t))_{j=1}^{n}$ is uniformly bounded and equicontinuous on $[0, 1]$. The Arzela–Ascoli Theorem tells us that there exist uniformly convergent subsequences in $(\Phi w_{j}(t))_{j=1}^{n}$. Let $((\Phi w_{j}(t)))_{j=1}^{n}$ be a subsequence which converges to $\nu(t)$ uniformly on $[0, 1]$ and $(A_{j})_{j=1}^{n}$ converges to $A$. Then there exists an $H > h$ such that

$$ h \leq w_{j}(t) \leq H \quad \text{on} \quad [0, 1], $$

and hence

$$ (\Phi H)(t) \leq (\Phi w_{j})(t) \leq (\Phi h)(t) \quad \text{on} \quad [0, 1]. $$

Put

$$ [a, b] = \{t \in [0, 1] : (\Phi h)(t) = \max(\Phi H)(t) > h\}. $$

Then $[a, b] \subset (0, 1)$ and $(A_{j}) \subset [a, b]$ where $A_{j}$ is the maximum point of $(\Phi w_{j})$ in $(0, 1)$. Thus,

$$ (\Phi w_{j})(A_{j}) = h + \int_{a}^{A_{j}} G\left(\int_{s}^{A_{j}} k(r) f(w_{j}(r)) \, dr\right) \, ds $$

$$ \leq h + \int_{0}^{b} G\left(\int_{s}^{b} k(r) \, dr\right) \, ds G(f(h)), $$

$$ (\Phi w_{j})(A_{j}) = h + \int_{A_{j}}^{b} G\left(\int_{A_{j}}^{r} k(r) f(w_{j}(r)) \, dr\right) \, ds $$

$$ \leq h + \int_{a}^{1} G\left(\int_{a}^{r} k(r) \, dr\right) \, ds G(f(h)). $$
Notice that
\[
(\Phi_w)(t) = \begin{cases} 
  h + \int_0^t G \left( \int_s^x k(r) f(w_j(r)) \, dr \right) \, ds, & 0 \leq t \leq A_j, \\
  h + \int_1^t G \left( \int_{A_j}^x k(r) f(w_j(r)) \, dr \right) \, ds, & A_j \leq t \leq 1.
\end{cases}
\]

Inserting $w_{j(n)}$ and $A_{j(n)}$ into the above and then letting $n \to \infty$, we obtain
\[
v(t) = \begin{cases} 
  h + \int_0^1 G \left( \int_s^x k(r) f(w_0(r)) \, dr \right) \, ds, & 0 \leq t \leq \bar{A}, \\
  h + \int_1^1 G \left( \int_{\bar{A}}^x k(r) f(w_0(r)) \, dr \right) \, ds, & \bar{A} \leq t \leq 1,
\end{cases}
\]
and
\[
v(\bar{A}) - h = \int_0^{\bar{A}} G \left( \int_s^{\bar{A}} k(r) f(w_0(r)) \, dr \right) \, ds
= \int_\bar{A}^1 G \left( \int_\bar{A}^x k(r) f(w_0(r)) \, dr \right) \, ds.
\]

Here we have applied Lebesgue’s Dominated Convergence Theorem since $f(w) \leq f(h)$. From the definition of $\Phi$, we know that $v(t) = (\Phi w)(t)$ on $[0, 1]$. This shows that each subsequence of $((\Phi w_j)(t))$ uniformly converges to $(\Phi w_0)(t)$. Therefore, the sequence $((\Phi w_j)(t))$ itself uniformly converges to $(\Phi w_0)(t)$. This means that $\Phi$ is continuous at $w_0 \in D_h$. Therefore $\Phi$ is continuous on $D_h$ since $w_0 \in D_h$ is arbitrary.

**Lemma 7.** Assume that $f$ is continuous. Then, for each fixed $h > 0$, the boundary value problem (2.1)$_h$ has a unique solution $u(t; h) \geq h$.

**Proof.** Lemmas 4, 5, and 6 imply that the mapping $\Phi$ is a compact continuous mapping from $D_h$ to $D_h$. The Schauder fixed point theorem tells us that $\Phi$ has at least one fixed point in $D_h$. Let $u(t; h)$ be a fixed point of $\Phi$ in $D_h$. Then
\[
u(t; h) = \begin{cases} 
  h + \int_0^t G \left( \int_s^x k(r) f(u(r; h)) \, dr \right) \, ds, & 0 \leq t \leq \bar{A}, \\
  h + \int_t^1 G \left( \int_{\bar{A}}^x k(r) f(u(r; h)) \, dr \right) \, ds, & \bar{A} \leq t \leq 1,
\end{cases}
\]
and 
\[ u(A; h) = h + \int_0^x G \left( \int_r^x k(r)f(u(r; h)) \, dr \right) \, ds = h + \int_0^1 G \left( \int_r^x k(r)f(u(r; h)) \, dr \right) \, ds. \]

It is easy to check that the function \( u(t; h) \) is a solution of (2.1) with \( h > 0 \).

**Lemma 8.** If \( h_1 > h_2 > 0 \), then
\[ 0 \leq u(t; h_1) - u(t; h_2) \leq h_1 - h_2. \] (2.5)

**Proof.** The proof of (2.5) is very similar to that of Lemma 1 and hence omitted here.

To prove our result, we need

**Lemma 9.** Let \( f_1, f_2 \) be two functions satisfying (H1) such that \( f_1 \leq f_2 \). If \( u_1 \) and \( u_2 \) are two solutions of problem (1.1) corresponding to \( f_1 \) and \( f_2 \), respectively, then \( u_1 \leq u_2 \).

**Proof.** The proof is similar to that of Lemma 1, so we omit the details.

The next lemma asserts that the continuity of \( f \) in Lemma 7 is not necessary in obtaining the existence of solutions of (2.1) for \( h > 0 \). We have

**Lemma 10.** Assume (H1) and (H2). Then the boundary value problem
(2.1) has a (unique) solution \( u(t; h) \geq h \) for each given \( h \in (0, 1] \).

**Proof.** Put
\[ f^h(u) := \begin{cases} f(u), & \text{if } u \geq h > 0, \\ f(h), & \text{otherwise,} \end{cases} \]
\[ f(u; e) := \frac{1}{e} \int_u^{u+e} f^h(s) \, ds, \quad F(u; e) := \frac{1}{e} \int_u^{u-e} f^h(s) \, ds \text{ on } \Omega, \]
\[ f_n(u) := f \left( u; \frac{1}{n} \right), \quad F_n(u) := F \left( u; \frac{1}{n} \right), \quad n = 1, 2, 3, \ldots, \]

where \( \Omega := (-\infty, +\infty) \times (0, +\infty) \). Then \( f_n(u), F_n(u), n = 1, 2, \ldots, \) are all nonnegative, nonincreasing, and continuous on \([0, +\infty)\),
\[ f_n(u) \leq f_{n+1}(u) \leq f^h(u) \leq F_{n+1}(u) \leq F_n(u) \quad \text{on } [0, +\infty), \]
\[ f^h(u) = \lim_{n \to \infty} f_n(u) = \lim_{n \to \infty} F_n(u) \quad \text{a.e. on } (-\infty, +\infty) \]
because for almost all \((u, \epsilon) \in \Omega,\)
\[
\frac{\partial f(u; \epsilon)}{\partial u} \leq 0, \quad \frac{\partial f(u; \epsilon)}{\partial \epsilon} \leq 0, \quad \frac{\partial F(u; \epsilon)}{\partial u} \leq 0, \quad \frac{\partial F(u; \epsilon)}{\partial \epsilon} \geq 0.
\]

Lemma 7 asserts that for fixed \(h > 0,\) the boundary value problem (2.1) with \(f_n\) (resp. \(F_n\)) in the place of \(f\) has a unique positive solution \(u_n(t; h)\) (resp. \(U_n(t; h)\)) satisfying
\[
u_n(t; h) = \begin{cases} 
  h + \int_0^t G \left( \int_s^{A_n} k(r) f_n(u_n(r; h)) \, dr \right) \, ds, & 0 \leq t \leq A_n, \\
  h + \int_t^1 G \left( \int_s^{r} k(r) f_n(u_n(r; h)) \, dr \right) \, ds, & A_n \leq t \leq 1,
\end{cases}
\]
with \(A_n \in (0, 1).\)

A similar equality holds for \(U_n(t; h)\) with \(F_n\) and \(B_n\) in place of \(f_n\) and \(A_n,\) respectively.

Lemma 9 tells us that
\[
0 < h \leq u_n(t; h) \leq u_{n+1}(t; h) \leq U_n(t; h) \leq U_n(t; h) \quad \text{on } [0, 1].
\]

Whence it follows that there are continuous functions \(u(t; h)\) and \(U(t; h)\) such that
\[
u(t; h) := \lim_{n \to \infty} u_n(t; h) \quad \text{and} \quad U(t; h) := \lim_{n \to \infty} U_n(t; h)

\text{uniformly on } [0, 1],
\]

\[
u_n(t; h) \leq u(t; h) \leq U(t; h) \leq U_n(t; h) \quad \text{on } [0, 1].
\]

Consequently, we have
\[
u(t; h) \geq u_n(t; h)
\]
\[
v(t; h) = \begin{cases} 
  h + \int_0^t G \left( \int_s^{A_n} k(r) f_n(u_n(r; h)) \, dr \right) \, ds, & 0 \leq t \leq A_n, \\
  h + \int_t^1 G \left( \int_s^{r} k(r) f_n(u_n(r; h)) \, dr \right) \, ds, & A_n \leq t \leq 1,
\end{cases}
\]
\[
v(t; h) \geq \begin{cases} 
  h + \int_0^t G \left( \int_s^{A_n} k(r) f_n(u(r; h)) \, dr \right) \, ds, & 0 \leq t \leq A_n, \\
  h + \int_t^1 G \left( \int_s^{r} k(r) f_n(u(r; h)) \, dr \right) \, ds, & A_n \leq t \leq 1,
\end{cases}
\]
and

\[ U(t; h) \leq U_n(t; h) \]

\[
\begin{aligned}
&= \left\{ h + \int_0^t G \left( \int_{s}^{t} k(r) F_n(u_n(r; h)) \, dr \right) \, ds, \quad 0 \leq t \leq B_n, \\
&\quad + \int_t^1 G \left( \int_{s}^{t} k(r) F_n(u_n(r; h)) \, dr \right) \, ds, \quad B_n \leq t \leq 1, \\
&\leq \left\{ h + \int_0^t G \left( \int_{s}^{t} k(r) F_n(u_n(r; h)) \, dr \right) \, ds, \quad 0 \leq t \leq B_n, \\
&\quad + \int_t^1 G \left( \int_{s}^{t} k(r) F_n(U_n; h) \, dr \right) \, ds, \quad B_n \leq t \leq 1. \\
\end{aligned}
\]

Without loss of generality, we may assume that \( A_n \to A \) and \( B_n \to B \) for some \( A, B \in [0, 1] \). Letting \( n \to \infty \) in the above, we obtain

\[
U(t; h) \leq \left\{ h + \int_0^t G \left( \int_{s}^{t} k(r) f(u(r; h)) \, dr \right) \, ds, \quad 0 \leq t \leq B, \\
\quad + \int_t^1 G \left( \int_{s}^{t} k(r) f(u(r; h)) \, dr \right) \, ds, \quad B \leq t \leq 1, \\
\leq \left\{ h + \int_0^t G \left( \int_{s}^{t} k(r) f(u(r; h)) \, dr \right) \, ds, \quad 0 \leq t \leq B, \\
\quad + \int_t^1 G \left( \int_{s}^{t} k(r) f(u(r; h)) \, dr \right) \, ds, \quad B \leq t \leq 1. \\
\right.
\]

Here we have used the Dominated Convergence Theorem.

Since \( U_n \) takes its maximum at \( B_n \), a simple observation shows that

\[
\max \left\{ h + \int_0^A G \left( \int_{s}^{A} k(r) f(u(r; h)) \, dr \right) \, ds, \\
\quad h + \int_A^1 G \left( \int_{s}^{A} k(r) f(u(r; h)) \, dr \right) \, ds \right\}
\]

\[
\leq u(A; h) \leq \lim_{n \to \infty} U_n(A_n; h) \leq \lim_{n \to \infty} U_n(B_n; h)
\]

\[
\leq \min \left\{ h + \int_0^B G \left( \int_{s}^{B} k(r) f(u(r; h)) \, dr \right) \, ds, \\
\quad h + \int_B^1 G \left( \int_{s}^{B} k(r) f(u(r; h)) \, dr \right) \, ds \right\},
\]
where we write $f(u(t; h))$ instead of $f^h(u(t; h))$ since $u(t; h) \geq h$. These equalities and the nonnegativity of the integrands imply that

$$
\int_0^A G \left( \int_s^A k(r) f(u(r; h)) \, dr \right) \, ds = \int_0^A G \left( \int_s^A k(r) f(u(r; h)) \, dr \right) \, ds
$$

By using these inequalities, we can easily conclude that

$$
u(t; h) = \begin{cases} 
  h + \int_0^t G \left( \int_s^A k(r) f(u(r; h)) \, dr \right) \, ds, & 0 \leq t \leq A, \\
  h + \int_t^1 G \left( \int_s^A k(r) f(u(r; h)) \, dr \right) \, ds, & A \leq t \leq 1,
\end{cases}
$$
on $[0, 1]$. Therefore, this equality and (2.6) show that $u(t; h)$ is a positive solution to the boundary value problem (2.1)$_h$ with $h > 0$.

The proof of Lemma 10 is complete.

**Lemma 11.** The boundary value problem (2.1)$_0$ has a (unique) positive solution $u(t; 0)$ if (H1) and (H2) hold.

**Proof.** Inequality (2.5) implies that as $h \downarrow 0$, $(u(t; h))$ is nonincreasing in $h$. We may assume that $u(t; h) \to u(t; 0)$ uniformly on $[0, 1]$. We now prove that the function is the unique solution to (2.1)$_0$.

Without loss of generality, we may choose a sequence $(h_{n})_{n=1}^\infty$, $h_n \downarrow 0$ such that $A_n := A(h_n)$ is monotonically increasing (the proof is similar if $A_n$ is monotonically decreasing) and $A_n \to A^*$ where $A_n$ is a maximum point of $u(t; h_n)$ in $(0, 1)$. From the previous proof, we know that

$$
u(t; h_n) = \begin{cases} 
  h_n + \int_0^t G \left( \int_s^{A_n} k(r) f(u(r; h_n)) \, dr \right) \, ds, & 0 \leq t \leq A_n, \\
  h_n + \int_t^1 G \left( \int_s^{A_n} k(r) f(u(r; h_n)) \, dr \right) \, ds, & A_n \leq t \leq 1,
\end{cases}
$$

(2.7)
and
\[
    u(A_n, h_n) - h_n = \int_0^{A*} G \left( \int_s^{A*} k(r) f(u(r; h_n)) \, dr \right) \, ds,
\]
\[
    = \int_0^1 G \left( \int_{A_n}^{s} k(r) f(u(r; h_n)) \, dr \right) \, ds. \tag{2.8}
\]

Then, the Monotone Convergence Theorem implies that
\[
    u(t, 0) = \int_0^t G \left( \int_s^{A*} k(r) f(u(r; 0)) \, dr \right) \, ds, \quad 0 \leq t \leq A*, \tag{2.9}
\]
here we have used the fact that \( f \) is right continuous.

By Fatou’s Theorem,
\[
    \int_0^1 G \left( \int_{A^*}^s k(r) f(u(r; 0)) \, dr \right) \, ds \leq u(t, 0) < \infty, \quad A^* \leq t \leq 1.
\]

Therefore, the function
\[
    G \left( \int_{A^*}^s k(r) f(u(r; 0)) \, dr \right)
\]
is integrable over \([A^*, 1]\). Notice that for any integers \( n > N \),
\[
    G \left( \int_{A_n}^{s} k(r) f(u(r; h_n)) \, dr \right) \leq G \left( \int_{A_N}^{s} k(r) f(u(r; 0)) \, dr \right)
\]
if the right hand side exists. Lebesgue’s Dominated Convergence Theorem will show that
\[
    u(t, 0) = \lim_{n \to \infty} \int_0^1 G \left( \int_s^{A_n} k(r) f(u(r; h_n)) \, dr \right) \, ds
\]
\[
    = \int_0^1 G \left( \int_{A^*}^s k(r) f(u(r; 0)) \, dr \right) \, ds \tag{2.10}
\]
for \( A^* \leq t \leq 1 \) if we can prove that for some sufficiently large \( N \),
\[
    \int_{A_N}^{1} G \left( \int_{A^*}^s k(r) f(u(r; 0)) \, dr \right) < \infty. \tag{2.11}
\]

If \( k = 0 \), (2.11) is trivial. We may assume that the set \( \{k(t) > 0; \ t \in [0, 1]\} \)
has positive measure. We claim that there exists a positive number \( \delta \)
independent of \( n \) such that \( u(A_n, h_n) \geq \delta \). Furthermore, \( A^* \in (0, 1) \) and 
\[ u(A^*, 0) = \max_{t \in [0,1]} u(t, 0). \]

We first prove the claim.

If, on the contrary, there were a subsequence of \( A_n \) denoted again by \( A_n \) such that \( u(A_n, h_n) \to 0 \), then by (2.7) and (H1),

\[ u(A_n; h_n) > \nu(A_n) G(f(u(A_n; h_n))), \]

where for any \( A \in [0, 1], \nu(A) \) is defined as

\[ \nu(A) = \max \left\{ \int_0^A G \left( \int_s^A k(r) \, dr \right) \, ds, \int_A^1 G \left( \int_s^A k(r) \, dr \right) \, ds \right\}. \]

It is obvious that \( \nu(A_n) \to \nu(A^*) > 0 \) by the assumption and (1.5). This
leads to a contradiction to the uniform boundedness of \( u(t, h_n) \) since 
\( G(f(u(A_n; h_n))) \to +\infty \). The other parts of the conclusions in the claim
follow easily.

Since \( A^* \in (0, 1), u(t, h_n) \to u(t, 0) \) uniformly on \([0, 1]\) and \( u(t, 0) \)
is continuous, we can find an \( \epsilon_0 > 0 \) such that \( u(t, h_n) > (1/2)\delta \) for \( t \in [A^* - \epsilon_0, A^* + \epsilon_0] \subset (0, 1) \). Therefore, \( f(u(t, h_n)) \) is uniformly bounded on \([A^* - \epsilon_0, A^* + \epsilon_0]\), (2.11) then can be shown easily.

As a consequence of (2.9) and (2.10) we have that

\[ u(A^*, 0) = \int_0^{A^*} G \left( \int_s^{A^*} k(r) f(u(r; 0)) \, dr \right) \, ds, \tag{2.12} \]

It is easy to verify that \( u(t, 0) \) is a solution of (2.1). This,
together with Lemma 1, implies the conclusion of the lemma.

3. PROOF OF THEOREM 1

Now, we give the proof of Theorem 1.

\begin{proof}
If \( u \) is a positive solution of the problem (1.1), then
there must be a point \( A \in (0, 1) \) such that \( u \) takes its maximum and hence 
\( u'(A) = 0 \). Integrating the equation over \((s, A)\), we get

\[ g(u'(s)) = \int_s^A k(t) f(u(t)) \, dt. \]
Therefore,

$$u'(s) = G \left( \int_{s}^{A} k(t)f(u(t)) \, dt \right). \tag{3.1}$$

We then get

$$u(A) = \int_{0}^{A} G \left( \int_{s}^{A} k(t)f(u(t)) \, dt \right) \, ds$$

$$\geq \int_{0}^{A} G \left( \int_{s}^{A} k(t) \, dt \right) \, ds G(f(A)),$$

which implies that

$$\int_{0}^{1/2} G \left( \int_{s}^{1/2} k(t) \, dt \right) \, ds < \infty.$$

The other part in (1.5) can be derived in a similar way. Therefore the necessity of the Statement (I) is proven.

The sufficiency of Statement (I) follows from Lemma 9.

Notice that if \( u(t) \) is a solution of (1.1), then it must satisfy (2.9)–(2.10) (by replacing \( u(t,0) \) with \( u(t) \)) and (3.1) and hence \( u(t) \) is nondecreasing in \((0,A)\) where \( A \) is a maximum point of \( u \) and \( u'(t) \) is nonincreasing for \( t \in (0,A) \).

Since \( u' \neq 0 \) on \((0,A)\), we may assume \( u'(r_0) > 0 \) for some \( r_0 \in (0,A) \); then the Mean Value Theorem implies that \( u(r) = u'(r_0)r \geq u'(r_0)r = \theta r \) for \( r \in (0,r_0) \). Hence

$$\int_{r}^{r_0} k(t)f(u(t)) \, ds \leq \int_{r}^{r_0} k(t)f(\theta t) \, dt,$$

which, together with (1.6) and (3.1), implies the boundedness of \( u'(0+) \). The other parts in Statement (II) can be shown similarly.

Statement (II,\(a)\) can be proven with a similar argument by replacing \( u'(r_0) \) with \( u'(0+) \), so we omit the details.

The proof of Statement (III) in Theorem 1 is the same as that of Theorem 2 in [6], so we omit the details.

The proof of Theorem 1 is complete.

Remark 7. From the proof of the above lemmas and theorems, we know that the condition \( f(0+) = +\infty \) is not necessary. In fact, if \( f \) is a bounded nonincreasing function, the proof will be much easier since it suffices to use Lebesgue's Dominated Convergence Theorem in taking the limit in (2.7) and (2.8).
REFERENCES

2. M. Del Pino, M. Elgueta, and R. Manasevich, Sturm’s comparison theorem and a Hartman’s type oscillation criterion for \( (M^{p-1} u')' + c(t)|u|^{p-2} u \), \( p > 1 \), preprint.
5. D. O’Regan, Some general existence principles and results for \( \phi(y')' = q(t)f(t, y, y') \), \( 0 < t < 1 \), *SIAM J. Math. Anal.* 24 (1993), 648–668.