JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **201,** 851–866 (1996)
ARTICLE NO. 0288

A Singular Boundary Value Problem for the One-Dimensional p-Laplacian*

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Received July 31, 1995

The singular boundary value problem

$$\begin{cases} (g(u'))' = -k(t)f(u), & 0 < t < 1, \\ u(0) = u(1) = 0 \end{cases}$$

is studied in this paper where $g(s) = |s|^{p-2}s$, $p > 1$. The singularity may appear at $u = 0$ and at $t = 0$ or $t = 1$ and the function f may be discontinuous. The authors prove that for any $p > 1$ and for any positive, nonincreasing function f and nonnegative measurable function k with some integrability conditions, the above-mentioned problem has a unique solution. Also, the properties of the solution are discussed in the paper.

1. INTRODUCTION

The boundary value problem for the one-dimensional p-Laplacian

$$\begin{cases} (g(u'))' = -k(t)f(u), & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $g(s) = |s|^{p-2}s$, $p > 1$, has been studied extensively. For details, see, for example, Refs. [1–5, 7]. The boundary value problem treated in the above-mentioned references is not able to possess singularity.

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0022-247X/96 $18.00

In [6], Taliaferro considered a particular case of (1.1) where $p = 2$, $f(u) = u^{-\lambda}$, $\lambda > 0$, and $k(t)$ is positive and continuous in $(0,1)$. The following theorem was established there.

TALIAFERRO'S THEOREM. *Assume that* $p = 2$, $f(u) = u^{-\lambda}$, $\lambda > 0$, *and* $k(t)$ *is positive and continuous in* $(0,1)$. *Then the following statements hold.*

(I) *The boundary value problem* (1.1) *has a unique solution* $u_\lambda(t)$ *if and only if*

$$\int_0^1 t(1-t)k(t)\,dt < +\infty. \tag{1.2}$$

(II) $\max\{u_\lambda(t);\ 0 \le t \le 1\} \le M$, *where* M *is the positive solution of the equation*

$$\left(\frac{M-1}{2}\right)\left(\frac{M+1}{2}\right)^{\lambda} = N$$

with

$$N := \max\left\{\int_0^{1/2} k(t)t\,dt, \int_{1/2}^1 k(t)(1-t)\,dt\right\}.$$

(III) $u_\lambda(t)$ *tends to* 1, *uniformly on compact subsets of* $(0,1)$, *as* $\lambda \to +\infty$.

(IV) $u'_\lambda(0+) = \lim_{t\downarrow 0} u'_\lambda(t)(u'_\lambda(1-) = \lim_{t\uparrow 1} u'_\lambda(t))$ *is finite if and only if*

$$\int_0^{1/2} k(t)t^{-\lambda}\,dt < +\infty \qquad \left(\int_{1/2}^1 k(t)(1-t)^{-\lambda}\,dt < +\infty\right). \tag{1.3}$$

The above particular case of (1.1) possesses singularity at $u = 0$ and is able to possess singularity at $t = 0$ and $t = 1$. The existence and uniqueness of the solution $u_\lambda(t)$ were obtained by means of the shooting method.

The aim of this paper is to extend the above-mentioned results. We adopt the following hypotheses:

(H1) $f(u)$ is positive, right continuous, nonincreasing in $(0, +\infty)$ and

$$f(0+) = \lim_{u\downarrow 0} f(u) = +\infty. \tag{1.4}$$

(H2) $k(t)$ is a nonnegative measurable function defined in $(0,1)$.

We will prove the following theorem.

THEOREM 1. *Assume that* (H1) *and* (H2) *are satisfied. Then the following statements hold.*

(I) *The boundary value problem* (1.1) *has a positive solution* $u(t)$ *if and only if*

$$0 < \int_0^{1/2} G\left(\int_s^{1/2} k(r)\,dr\right) ds + \int_{1/2}^1 G\left(\int_{1/2}^s k(r)\,dr\right) ds < +\infty, \quad (1.5)$$

where $G(x)$ *is the inverse function to* $g(s)$.

(II_a) *If for every* $\theta > 0$,

$$\int_0^{1/2} k(r) f(\theta r)\,dr < +\infty, \qquad \left(\int_{1/2}^1 k(r) f(\theta(1-r))\,dr < +\infty\right) \quad (1.6)$$

then $u'(0+)$ $(u'(1-))$ *is finite*.

(II_b) *If* $u'(0+)$ $(u'(1-))$ *is finite, then* (1.6) *holds for* $\theta \geq u'(0+)$ $(\theta \geq |u'(1)|)$.

(III) *If*

$$\left(\frac{M-1}{2}\right) G\left(f\left(\frac{M+1}{2}\right)\right) = N;$$

$$N := \max\left\{\int_0^{1/2} G\left(\int_s^{1/2} k(r)\,dr\right) ds, \int_{1/2}^1 G\left(\int_{1/2}^s k(r)\,dr\right) ds\right\},$$

has a positive solution M *and* u *is the positive solution of* (1.1), *then* $u \leq M$.

Remark 1. The existence of the positive solution will be obtained by means of the perturbation technique and the Schauder fixed point theorem.

Remark 2. The condition (1.5) allows $k(t)$ to be equal to zero on some open or closed subintervals of $(0, 1)$. For example, the function

$$k(t) = \begin{cases} t^{-\alpha}, & 0 < t < 1/9,\ 0 < \alpha < p, \\ 0, & 1/9 \leq t \leq 8/9, \\ (1-t)^{-\beta}, & 8/9 < t < 1,\ 0 < \beta < p \end{cases}$$

satisfies the condition (1.5).

Remark 3. When $f(u) = u^{-\lambda}$, $\lambda > 0$, (1.6) becomes

$$\int_0^{1/2} G\left(\int_s^{1/2} k(r) r^{-\lambda}\, dr\right) ds < +\infty,$$

$$\left(\int_0^{1/2} G\left(\int_{1/2}^{s} k(r)(1-r)^{-\lambda}\, dr\right) ds < +\infty\right).$$

Therefore, $u'(0+)$ $(u'(1-))$ is finite if and only if the above condition holds.

Remark 4. The claim (III) in Taliaferro's Theorem is also true for $f(u) = u^{-\lambda}$, the proof is the same as that in [6].

Remark 5. Our result shows that the function f may be discontinuous.

2. SOME PRIMARY RESULTS

Assume (1.5) and consider the "approximate" boundary value problem

$$\begin{cases} (g(u'))' = -k(t) f(u), & 0 < t < 1, \\ u(0) = u(1) = h. \end{cases} \tag{2.1$_h$}$$

A function $u(t)$ is said to be a positive solution to the boundary value problem $(2.1)_h$ with $h \geq 0$, if the following conditions are satisfied:

(i) $u(t) \in C[0,1] \cap C^1(0,1)$;

(ii) $u(t) > 0$ in $(0,1)$, $u(0) = u(1) = h$;

(iii) $g(u'(t))$ is locally absolutely continuous in $(0,1)$, and

(iv) $(g(u'(t)))' = -k(t) f(u(t))$ a.e. in $(0,1)$.

LEMMA 1. *For each fixed $h \geq 0$, the boundary value problem $(2.1)_h$ has at most one positive solution.*

Proof. Suppose that $u_1(t)$ and $u_2(t)$ are positive solutions to $(2.1)_h$. If $u_1(t) \not\equiv u_2(t)$ on $[0,1]$, then there would exist a $t_0 \in (0,1)$ at which $u_1(t_0) > u_2(t_0)$ and hence there would exist an interval (a,b) such that $u_1(t) > u_2(t)$ in (a,b) and $u_1(a) - u_2(a) = u_1(b) - u_2(b) = 0$. Let $m = u_1(B) - u_2(B)$ be the positive maximum of $u_1(t) - u_2(t)$ on $[a,b]$. Then $B \in (a,b)$ and $u_1'(B) = u_2'(B)$. Notice that for $j = 1, 2$,

$$\big(g\big(u_j'(r)\big)\big)' = -k(r) f\big(u_j(r)\big) \qquad \text{a.e. in } (0,1).$$

Integrating both sides of this equality over $[s, B]$, $a < s < B$, we get

$$u_j'(s) = G\left(g\big(u_j'(B)\big) + \int_s^B k(r) f\big(u_j(r)\big)\, dr\right), \qquad a < s \leq B.$$

Integrating both sides of the above equality from a to B, we obtain

$$u_j(B) - u_j(a) = \int_a^B G\left(g\left(u_j'(B)\right) + \int_s^B k(r) f\left(u_j(r)\right) dr\right) ds.$$

Consequently, we are lead to a contradiction $0 < m = u_1(B) - u_2(B) \leq 0$. Here we have used the fact that $f(u)$ is nonincreasing in u. The proof of the lemma is complete.

To prove the existence of solution to $(2.1)_h$ with $h > 0$, we consider the boundary value problem

$$\begin{cases} (g(u'))' = -k(t) f(w(t)), & 0 < t < 1, \\ u(0) = u(1) = h > 0. \end{cases} \tag{2.2$_h$}$$

for any $w(t) \in D_h := \{w \in C[0,1];\ w(t) \geq h\}$.

LEMMA 2. *For each fixed $h > 0$ and each $w \in D_h$, the boundary value problem $(2.2)_h$ has a unique solution $u(t) \geq h$.*

Proof. We only prove the existence since the proof of the uniqueness is very simple. Set for $0 < t < 1$

$$x(t) := \int_0^t G\left(\int_s^t k(r) f(w(r))\, dr\right) ds - \int_t^1 G\left(\int_t^s k(r) f(w(r))\, dr\right) ds.$$

Clearly, $x(t)$ is continuous and nondecreasing in $(0,1)$ and $x(0+) < 0 < x(1-)$. Thus, $x(t)$ has zeros in $(0,1)$. Let A be a zero of $x(t)$ in $(0,1)$. Then

$$\int_0^A G\left(\int_s^A k(r) f(w(r))\, dr\right) ds = \int_A^1 G\left(\int_A^s k(r) f(w(r))\, dr\right) ds. \tag{2.3}$$

Put

$$u(t) = (\Phi w)(t) := \begin{cases} h + \int_0^t G\left(\int_s^A k(r) f(w(r))\, dr\right) ds, & 0 \leq t \leq A, \\ h + \int_t^1 G\left(\int_A^s k(r) f(w(r))\, dr\right) ds, & A \leq t \leq 1. \end{cases} \tag{2.4$_h$}$$

Then, u is a well-defined differentiable function and

$$u'(t) = (\Phi w)'(t) = G\left(\int_t^A k(r) f(w(r))\, dr\right), \qquad 0 < t < 1.$$

It is obvious that $u'(t) = (\Phi w)'(t)$ defined as above is continuous and nonincreasing in $(0,1)$, $u'(A) = 0$, $u(t) \in D_h$, and $(2.2)_h$ is satisfied for a.e. $t \in (0,1)$. This shows that $u(t)$ is a solution of $(2.2)_h$ and a concave function defined on $[0,1]$. The lemma is proven.

Remark 6. It is easy to show that (2.3) and $(2.4)_h$ are independent of the choice of the zero A. Therefore, Φ is a well defined map on D_h.

LEMMA 3. *Let* $\Phi\colon D_h \mapsto D_h$ *be the mapping defined by* (2.3) *and* $(2.4)_h$, *and* $w_1, w_2 \in D_h$. *If* $w_1(t) \le w_2(t)$ *on* $[0,1]$, *then* $(\Phi w_1)(t) \ge (\Phi w_2)(t)$ *on* $[0,1]$.

Proof. The proof of this lemma is very similar to that of Lemma 1 and hence omitted here.

LEMMA 4. *For any* $w \in D_h$, *we have*

$$h \le (\Phi w)(t) \le (\Phi h)(t) \le (\Phi h)(A^*) \qquad \text{on } [0,1],$$

where A^* *is a zero of the function*

$$y(t) := \int_0^t G\left(\int_s^t k(r)\,dr\right) ds - \int_t^1 G\left(\int_t^s k(r)\,dr\right) ds, \qquad 0 < t < 1.$$

Proof. The lemma follows from Lemma 3 and the definition of Φ.

LEMMA 5. $\Phi(D_h)$ *is equicontinuous on* $[0,1]$.

Proof. For any $\epsilon > 0$, from the continuity of $(\Phi h)(t)$ on $[0,1]$, it follows that there is a $\delta_1 \in (0, 1/4)$ such that

$$(\Phi h)(2\delta_1),\ (\Phi h)(1 - 2\delta_1) < \epsilon + h.$$

If $(\Phi w)(A) < \epsilon + h$, then for any $t_1, t_2 \in [0,1]$

$$|(\Phi w)(t_1) - (\Phi w)(t_2)| \le |(\Phi w)(A) - (\Phi w)(0)| < \epsilon.$$

If $(\Phi w)(A) \ge \epsilon + h$, then $A \in [2\delta_1, 1 - 2\delta_1]$ and hence for $t \in [\delta_1, 1 - \delta_1]$,

$$|(\Phi w)'(t)| = \left| G\left(\int_t^A k(r) f(w(r))\,dr\right)\right|$$
$$\le G\left(\int_{\delta_1}^{1-\delta_1} k(r)\,dr\right) G(f(h)) = L.$$

Put $\delta_2 = \epsilon/L$, then for $t_1, t_2 \in [\delta_1, 1 - \delta_1]$, $|t_1 - t_2| < \delta_2$

$$|(\Phi w)(t_1) - (\Phi w)(t_2)| = |(\Phi w)'(\xi)|\,|t_1 - t_2| < L\delta_2 = \epsilon,$$

where ξ lies between t_1 and t_2. Set $\delta = \min\{\delta_1, \delta_2\}$. Then for $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$,

$$|(\Phi w)(t_1) - (\Phi w)(t_2)| < \epsilon.$$

This shows that $\Phi(D_h)$ is equicontinuous on $[0,1]$.

LEMMA 6. *The mapping Φ is continuous on D_h if the function f is continuous in its variable.*

Proof. Assume that $\{w_j\}_{j=0}^{\infty} \subset D_h$ and $w_j(t)$ converges to $w_0(t)$ uniformly on $[0,1]$. By Lemma 5, it follows that $\{(\Phi w_j)(t)\}_{j=1}^{\infty}$ is uniformly bounded and equicontinuous on $[0,1]$. The Arzela–Ascoli Theorem tells us that there exist uniformly convergent subsequences in $\{(\Phi w_j)(t)\}_{j=1}^{\infty}$. Let $\{(\Phi w_{j(n)})(t)\}_{n=1}^{\infty}$ be a subsequence which converges to $v(t)$ uniformly on $[0,1]$ and $\{A_{j(n)}\}_{n=1}^{\infty}$ converges to $\overline{A}$. Then there exists an $H > h$ such that

$$h \le w_j(t) \le H \qquad \text{on } [0,1],$$

and hence

$$(\Phi H)(t) \le (\Phi w_j)(t) \le (\Phi h)(t) \qquad \text{on } [0,1].$$

Put

$$[a,b] = \{t \in [0,1];\ (\Phi h)(t) \ge \max(\Phi H)(t) > h\}.$$

Then $[a,b] \subset (0,1)$ and $\{A_j\} \subset [a,b]$ where A_j is the maximum point of $(\Phi w_j)(t)$ in $(0,1)$. Thus,

$$\begin{aligned}(\Phi w_j)(A_j) &= h + \int_0^{A_j} G\left(\int_s^{A_j} k(r) f\big(w_j(r)\big)\, dr\right) ds \\ &\le h + \int_0^b G\left(\int_s^b k(r)\, dr\right) ds G(f(h)),\\ (\Phi w_j)(A_j) &= h + \int_{A_j}^1 G\left(\int_{A_j}^s k(r) f\big(w_j(r)\big)\, dr\right) ds \\ &\le h + \int_a^1 G\left(\int_a^s k(r)\, dr\right) ds G(f(h)).\end{aligned}$$

Notice that

$$(\Phi w_j)(t) = \begin{cases} h + \int_0^t G\left(\int_s^{A_j} k(r) f(w_j(r))\,dr\right) ds, & 0 \le t \le A_j, \\ h + \int_t^1 G\left(\int_{A_j}^s k(r) f(w_j(r))\,dr\right) ds, & A_j \le t \le 1. \end{cases}$$

Inserting $w_{j(n)}$ and $A_{j(n)}$ into the above and then letting $n \to \infty$, we obtain

$$v(t) = \begin{cases} h + \int_0^t G\left(\int_s^{\bar{A}} k(r) f(w_0(r))\,dr\right) ds, & 0 \le t \le \bar{A}, \\ h + \int_t^1 G\left(\int_{\bar{A}}^s k(r) f(w_0(r))\,dr\right) ds, & \bar{A} \le t \le 1, \end{cases}$$

and

$$\begin{aligned} v(\bar{A}) - h &= \int_0^{\bar{A}} G\left(\int_s^{\bar{A}} k(r) f(w_0(r))\,dr\right) ds \\ &= \int_{\bar{A}}^1 G\left(\int_{\bar{A}}^s k(r) f(w_0(r))\,dr\right) ds. \end{aligned}$$

Here we have applied Lebesgue's Dominated Convergence Theorem since $f(w_j) \le f(h)$. From the definition of Φ, we know that $v(t) \equiv (\Phi w_0)(t)$ on $[0, 1]$. This shows that each subsequence of $\{(\Phi w_j)(t)\}$ uniformly converges to $(\Phi w_0)(t)$. Therefore, the sequence $\{(\Phi w_j)(t)\}$ itself uniformly converges to $(\Phi w_0)(t)$. This means that Φ is continuous at $w_0 \in D_h$. Therefore Φ is continuous on D_h since $w_0 \in D_h$ is arbitrary.

LEMMA 7. *Assume that f is continuous. Then, for each fixed $h > 0$, the boundary value problem* $(2.1)_h$ *has a* (*unique*) *solution* $u(t; h) \ge h$.

Proof. Lemmas 4, 5, and 6 imply that the mapping Φ is a compact continuous mapping from D_h to D_h. The Schauder fixed point theorem tells us that Φ has at least one fixed point in D_h. Let $u(t; h)$ be a fixed point of Φ in D_h. Then

$$u(t; h) = \begin{cases} h + \int_0^t G\left(\int_s^{\bar{A}} k(r) f(u(r; h))\,dr\right) ds, & 0 \le t \le \bar{A}, \\ h + \int_t^1 G\left(\int_{\bar{A}}^s k(r) f(u(r; h))\,dr\right) ds, & \bar{A} \le t \le 1, \end{cases}$$

and

$$u(\bar{A};h) = h + \int_0^{\bar{A}} G\left(\int_s^{\bar{A}} k(r) f(u(r;h))\,dr\right) ds$$
$$= h + \int_{\bar{A}}^1 G\left(\int_{\bar{A}}^s k(r) f(u(r;h))\,dr\right) ds.$$

It is easy to check that the function $u(t;h)$ is a solution of $(2.1)_h$ with $h > 0$.

LEMMA 8. *If $h_1 > h_2 > 0$, then*

$$0 \le u(t;h_1) - u(t;h_2) \le h_1 - h_2. \tag{2.5}$$

Proof. The proof of (2.5) is very similar to that of Lemma 1 and hence omitted here.

To prove our result, we need

LEMMA 9. *Let f_1, f_2 be two functions satisfying* (H1) *such that $f_1 \le f_2$. If u_1 and u_2 are two solutions of problem* (1.1) *corresponding to f_1 and f_2, respectively, then $u_1 \le u_2$.*

Proof. The proof is similar to that of Lemma 1, so we omit the details.

The next lemma asserts that the continuity of f in Lemma 7 is not necessary in obtaining the existence of solutions of $(2.1)_h$ for $h > 0$. We have

LEMMA 10. *Assume* (H1) *and* (H2). *Then the boundary value problem $(2.1)_h$ has a* (*unique*) *solution $u(t;h) \ge h$ for each given $h \in (0,1]$.*

Proof. Put

$$f^h(u) := \begin{cases} f(u), & \text{if } u \ge h > 0, \\ f(h), & \text{otherwise}, \end{cases}$$

$$f(u;\epsilon) := \frac{1}{\epsilon}\int_u^{u+\epsilon} f^h(s)\,ds, \qquad F(u;\epsilon) := \frac{1}{\epsilon}\int_{u-\epsilon}^{u} f^h(s)\,ds \text{ on } \Omega,$$

$$f_n(u) := f\left(u;\frac{1}{n}\right), \qquad F_n(u) := F\left(u;\frac{1}{n}\right),\ n = 1,2,3,\dots,$$

where $\Omega := (-\infty, +\infty) \times (0, +\infty)$. Then $f_n(u)$, $F_n(u)$, $n = 1,2,\dots,$ are all nonnegative, nonincreasing, and continuous on $[0, +\infty)$,

$$f_n(u) \le f_{n+1}(u) \le f^h(u) \le F_{n+1}(u) \le F_n(u) \qquad \text{on } [0, +\infty),$$

$$f^h(u) = \lim_{n\to\infty} f_n(u) = \lim_{n\to\infty} F_n(u) \qquad \text{a.e. on } (-\infty, +\infty)$$

because for almost all $(u, \epsilon) \in \Omega$,

$$\frac{\partial f(u;\epsilon)}{\partial u} \le 0, \qquad \frac{\partial f(u;\epsilon)}{\partial \epsilon} \le 0, \qquad \frac{\partial F(u;\epsilon)}{\partial u} \le 0, \qquad \frac{\partial F(u;\epsilon)}{\partial \epsilon} \ge 0.$$

Lemma 7 asserts that for fixed $h > 0$, the boundary value problem $(2.1)_h$ with f_n (resp. F_n) in the place of f has a unique positive solution $u_n(t;h)$ (resp. $U_n(t;h)$) satisfying

$$u_n(t;h) = \begin{cases} h + \int_0^t G\left(\int_s^{A_n} k(r) f_n(u_n(r;h))\,dr\right) ds, & 0 \le t \le A_n, \\ h + \int_t^1 G\left(\int_{A_n}^s k(r) f_n(u_n(r;h))\,dr\right) ds, & A_n \le t \le 1, \end{cases}$$

with $A_n \in (0,1)$.

A similar equality holds for $U_n(t;h)$ with F_n and B_n in place of f_n and A_n, respectively.

Lemma 9 tells us that

$$0 < h \le u_n(t;h) \le u_{n+1}(t;h) \le U_{n+1}(t;h) \le U_n(t;h) \qquad \text{on } [0,1].$$

Whence it follows that there are continuous functions $u(t;h)$ and $U(t;h)$ such that

$$u(t;h) := \lim_{n\to\infty} u_n(t;h) \quad \text{and} \quad U(t;h) := \lim_{n\to\infty} U_n(t;h) \qquad \text{uniformly on } [0,1],$$

$$u_n(t;h) \le u(t;h) \le U(t;h) \le U_n(t;h) \qquad \text{on } [0,1].$$

Consequently, we have

$$\begin{aligned} u(t;h) &\ge u_n(t;h) \\ &= \begin{cases} h + \int_0^t G\left(\int_s^{A_n} k(r) f_n(u_n(r;h))\,dr\right) ds, & 0 \le t \le A_n, \\ h + \int_t^1 G\left(\int_{A_n}^s k(r) f_n(u_n(r;h))\,dr\right) ds, & A_n \le t \le 1, \end{cases} \\ &\ge \begin{cases} h + \int_0^t G\left(\int_s^{A_n} k(r) f_n(u(r;h))\,dr\right) ds, & 0 \le t \le A_n, \\ h + \int_t^1 G\left(\int_{A_n}^s k(r) f_n(u(r;h))\,dr\right) ds, & A_n \le t \le 1, \end{cases} \end{aligned}$$

and

$$U(t;h) \le U_n(t;h)$$

$$= \begin{cases} h + \int_0^t G\left(\int_s^{B_n} k(r) F_n(U_n(r;h))\, dr\right) ds, & 0 \le t \le B_n, \\ h + \int_t^1 G\left(\int_{B_n}^s k(r) F_n(U_n(r;h))\, dr\right) ds, & B_n \le t \le 1, \end{cases}$$

$$\le \begin{cases} h + \int_0^t G\left(\int_s^{B_n} k(r) F_n(U(r;h))\, dr\right) ds, & 0 \le t \le B_n, \\ h + \int_t^1 G\left(\int_{B_n}^s k(r) F_n(U(r;h))\, dr\right) ds, & B_n \le t \le 1. \end{cases}$$

Without loss of generality, we may assume that $A_n \to A$ and $B_n \to B$ for some $A, B \in [0,1]$. Letting $n \to \infty$ in the above, we obtain

$$U(t;h) \le \begin{cases} h + \int_0^t G\left(\int_s^{B} k(r) f(U(r;h))\, dr\right) ds, & 0 \le t \le B, \\ h + \int_t^1 G\left(\int_{B}^s k(r) f(U(r;h))\, dr\right) ds, & B \le t \le 1, \end{cases}$$

$$\le \begin{cases} h + \int_0^t G\left(\int_s^{B} k(r) f(u(r;h))\, dr\right) ds, & 0 \le t \le B, \\ h + \int_t^1 G\left(\int_{B}^s k(r) f(u(r;h))\, dr\right) ds, & B \le t \le 1. \end{cases}$$

Here we have used the Dominated Convergence Theorem.

Since U_n takes its maximum at B_n, a simple observation shows that

$$\max\left\{h + \int_0^A G\left(\int_s^A k(r) f(u(r;h))\, dr\right) ds, \right.$$
$$\left. h + \int_A^1 G\left(\int_A^s k(r) f(u(r;h))\, dr\right) ds\right\}$$
$$\le u(A;h) \le \lim_{n\to\infty} U_n(A_n;h) \le \lim_{n\to\infty} U_n(B_n;h)$$
$$\le \min\left\{h + \int_0^B G\left(\int_s^B k(r) f(u(r;h))\, dr\right) ds, \right.$$
$$\left. h + \int_B^1 G\left(\int_B^s k(r) f(u(r;h))\, dr\right) ds\right\},$$

where we write $f(u(t;h))$ instead of $f^h(u(t;h))$ since $u(t;h) \geq h$. These equalities and the nonnegativity of the integrands imply that

$$\begin{aligned}
&\int_0^A G\left(\int_s^A k(r)f(u(r;h))\,dr\right)ds \\
&\quad = \int_A^1 G\left(\int_A^s k(r)f(u(r;h))\,dr\right)ds \\
&\quad = \int_0^B G\left(\int_s^B k(r)f(u(r;h))\,dr\right)ds \\
&\quad = \int_B^1 G\left(\int_B^s k(r)f(u(r;h))\,dr\right)ds.
\end{aligned} \tag{2.6}$$

By using these inequalities, we can easily conclude that

$$u(t;h) = \begin{cases} h + \int_0^t G\left(\int_s^A k(r)f(u(r;h))\,dr\right)ds, & 0 \leq t \leq A, \\ h + \int_t^1 G\left(\int_A^s k(r)f(u(r;h))\,dr\right)ds, & A \leq t \leq 1, \end{cases}$$

on $[0,1]$. Therefore, this equality and (2.6) show that $u(t;h)$ is a positive solution to the boundary value problem $(2.1)_h$ with $h > 0$.

The proof of Lemma 10 is complete.

LEMMA 11. *The boundary value problem* $(2.1)_0$ *has a* (*unique*) *positive solution* $u(t;0)$ *if* (H1) *and* (H2) *hold*.

Proof. Inequality (2.5) implies that as $h \downarrow 0$, $\{u(t;h)\}$ is nonincreasing in h. We may assume that $u(t;h) \to u(t;0)$ uniformly on $[0,1]$. We now prove that the function is the unique solution to $(2.1)_0$.

Without loss of generalities, we may choose a sequence $\{h_n\}_{n=1}^\infty$, $h_n \downarrow 0$ such that $A_n := A(h_n)$ is monotonically increasing (the proof is similar if A_n is monotonically decreasing) and $A_n \to A^*$ where A_n is a maximum point of $u(t;h_n)$ in $(0,1)$. From the previous proof, we know that

$$u(t;h_n) = \begin{cases} h_n + \int_0^t G\left(\int_s^{A_n} k(r)f(u(r;h_n))\,dr\right)ds, & 0 \leq t \leq A_n, \\ h_n + \int_t^1 G\left(\int_{A_n}^s k(r)f(u(r;h_n))\,dr\right)ds, & A_n \leq t \leq 1, \end{cases} \tag{2.7}$$

and

$$\begin{aligned} u(A_n, h_n) - h_n &= \int_0^{A_n} G\left(\int_s^{A_n} k(r) f(u(r; h_n))\, dr\right) ds, \\ &= \int_{A_n}^1 G\left(\int_{A_n}^s k(r) f(u(r; h_n))\, dr\right) ds. \end{aligned} \tag{2.8}$$

Then, the Monotone Convergence Theorem implies that

$$u(t, 0) = \int_0^t G\left(\int_s^{A^*} k(r) f(u(r; 0))\, dr\right) ds, \qquad 0 \le t \le A^*, \tag{2.9}$$

here we have used the fact that f is right continuous.

By Fatou's Theorem,

$$\int_t^1 G\left(\int_{A^*}^s k(r) f(u(r; 0))\, dr\right) ds \le u(t, 0) < \infty, \qquad A^* \le t \le 1.$$

Therefore, the function

$$G\left(\int_{A^*}^s k(r) f(u(r; 0))\, dr\right)$$

is integrable over $[A^*, 1]$. Notice that for any integers $n > N$,

$$G\left(\int_{A_n}^s k(r) f(u(r; h_n))\, dr\right) \le G\left(\int_{A_N}^s k(r) f(u(r; 0))\, dr\right)$$

if the right hand side exists. Lebesgue's Dominated Convergence Theorem will show that

$$\begin{aligned} u(t, 0) &= \lim_{n \to \infty} \int_t^1 G\left(\int_{A_n}^s k(r) f(u(r; h_n))\, dr\right) ds \\ &= \int_t^1 G\left(\int_{A^*}^s k(r) f(u(r; 0))\, dr\right) ds \end{aligned} \tag{2.10}$$

for $A^* \le t \le 1$ if we can prove that for some sufficiently large N,

$$\int_{A_N}^1 G\left(\int_{A^N}^s k(r) f(u(r; 0))\, dr\right) < \infty. \tag{2.11}$$

If $k \equiv 0$, (2.11) is trivial. We may assume that the set $\{k(t) > 0;\ t \in [0, 1]\}$ has positive measure. We claim that there exists a positive number δ

independent of n such that $u(A_n, h_n) \geq \delta$. Furthermore, $A^* \in (0,1)$ and $u(A^*, 0) = \max_{t \in [0,1]} u(t, 0)$.

We first prove the claim.

If, on the contrary, there were a subsequence of A_n denoted again by A_n such that $u(A_n, h_n) \to 0$, then by (2.7) and (H1),

$$u(A_n; h_n) > \nu(A_n) G\big(f(u(A_n; h_n))\big),$$

where for any $A \in [0,1]$, $\nu(A)$ is defined as

$$\nu(A) = \max\left\{\int_0^A G\left(\int_s^A k(r)\, dr\right) ds, \int_A^1 G\left(\int_A^s k(r)\, dr\right) ds\right\}.$$

It is obvious that $\nu(A_n) \to \nu(A^*) > 0$ by the assumption and (1.5). This leads to a contradiction to the uniform boundedness of $u(t, h_n)$ since $G(f(u(A_n; h_n))) \to +\infty$. The other parts of the conclusions in the claim follow easily.

Since $A^* \in (0,1)$, $u(t, h_n) \to u(t, 0)$ uniformly on $[0,1]$ and $u(t,0)$ is continuous, we can find an $\epsilon_0 > 0$ such that $u(t, h_n) > (1/2)\delta$ for $t \in [A^* - \epsilon_0, A^* + \epsilon_0] \subset (0,1)$. Therefore, $f(u(t, h_n))$ is uniformly bounded on $[A^* - \epsilon_0, A^* + \epsilon_0]$, (2.11) then can be shown easily.

As a consequence of (2.9) and (2.10) we have that

$$\begin{aligned} u(A^*, 0) &= \int_0^{A^*} G\left(\int_s^{A^*} k(r) f(u(r;0))\, dr\right) ds, \\ &= \int_{A^*}^1 G\left(\int_{A^*}^s k(r) f(u(r;0))\, dr\right) ds. \end{aligned} \tag{2.12}$$

It is easy to verify that $u(t, 0)$ is a solution of $(2.1)_0$.

This, together with Lemma 1, implies the conclusion of the lemma.

3. PROOF OF THEOREM 1

Now, we give the proof of Theorem 1.

Proof of Theorem 1. If u is a positive solution of the problem (1.1), then there must be a point $A \in (0,1)$ such that u takes its maximum and hence $u'(A) = 0$. Integrating the equation over (s, A), we get

$$g(u'(s)) = \int_s^A k(t) f(u(t))\, dt.$$

Therefore,

$$u'(s) = G\left(\int_s^A k(t) f(u(t))\, dt\right). \tag{3.1}$$

We then get

$$\begin{aligned} u(A) &= \int_0^A G\left(\int_s^A k(t) f(u(t))\, dt\right) ds \\ &\geq \int_0^A G\left(\int_s^A k(t)\, dt\right) ds G(f(A)), \end{aligned}$$

which implies that

$$\int_0^{1/2} G\left(\int_s^{1/2} k(t)\, dt\right) ds < \infty.$$

The other part in (1.5) can be derived in a similar way. Therefore the necessity of the Statement (I) is proven.

The sufficiency of Statement (I) follows from Lemma 9.

Notice that if $u(t)$ is a solution of (1.1), then it must satisfy (2.9)–(2.10) (by replacing $u(t, 0)$ with $u(t)$) and (3.1) and hence $u(t)$ is nondecreasing in $(0, A)$ where A is a maximum point of u and $u'(t)$ is nonincreasing for $t \in (0, A)$.

Since $u' \not\equiv 0$ on $(0, A)$, we may assume $u'(r_0) > 0$ for some $r_0 \in (0, A)$; then the Mean Value Theorem implies that $u(r) = u'(\xi)r \geq u'(r_0)r =: \theta r$ for $r \in (0, r_0)$. Hence

$$\int_s^{r_0} k(t) f(u(t))\, ds \leq \int_s^{r_0} k(t) f(\theta t)\, dt,$$

which, together with (1.6) and (3.1), implies the boundedness of $u'(0+)$. The other parts in Statement (II_a) can be shown similarly.

Statement (II_b) can be proven with a similar argument by replacing $u'(r_0)$ with $u'(0+)$, so we omit the details.

The proof of Statement (III) in Theorem 1 is the same as that of Theorem 2 in [6], so we omit the details.

The proof of Theorem 1 is complete.

Remark 7. From the proof of the above lemmas and theorems, we know that the condition $f(0+) = +\infty$ is not necessary. In fact, if f is a bounded nonincreasing function, the proof will be much easier since it suffices to use Lebesgue's Dominated Convergence Theorem in taking the limit in (2.7) and (2.8).

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