Stability radius of linear parameter-varying systems and applications

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Abstract
In this paper, we present a unifying approach to the problems of computing of stability radii of positive linear systems. First, we study stability radii of linear time-invariant parameter-varying differential systems. A formula for the complex stability radius under multi perturbations is given. Then, under hypotheses of positivity of the system matrices, we prove that the complex, real and positive stability radii of the system under multi perturbations (or affine perturbations) coincide and they are computed via simple formulae. As applications, we consider problems of computing of (strong) stability radii of linear time-invariant time-delay differential systems and computing of stability radii of positive linear functional differential equations under multi perturbations and affine perturbations. We show that for a class of positive linear time-delay differential systems, the stability radii of the system under multi perturbations (or affine perturbations) are equal to the strong stability radii. Next, we prove that the stability radii of a positive linear functional differential equation under multi perturbations (or affine perturbations) are equal to those of the associated linear time-invariant parameter-varying differential system. In particular, we get back some explicit formulas for these stability radii which are given recently in [P.H.A. Ngoc, Strong stability radii of positive linear time-delay systems, Internat. J. Robust Nonlinear Control 15 (2005) 459–472; P.H.A. Ngoc, N.K. Son, Stability radii of positive linear functional differential equations under multi perturbations, SIAM J. Control Optim. 43 (2005) 2278–2295]. Finally, we give two examples to illustrate the obtained results.
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1. Introduction

Problems of robust stability of linear dynamical systems have attracted a good deal of attention in control theory during the last twenty years. In the study of these problems, the notion of stability radius was proved to be an effective tool. In its simplest form, the stability radius of a given asymptotically stable system \( \dot{x}(t) = Ax(t) \) is the maximal \( \gamma > 0 \) for which all the systems of the form

\[
\dot{x}(t) = (A + D\Delta E)x(t), \quad \|\Delta\| < \gamma,
\]

are asymptotically stable. Here, \( \Delta \) is an unknown disturbance matrix, \( D \) and \( E \) are given matrices defining the structure of the perturbations. In this case, we say that the system matrix \( A \) is subjected to \textit{structured perturbations} and denote it by

\[
A \to A + D\Delta E. \tag{1}
\]

Depending upon whether complex or real disturbances \( \Delta \) are considered this maximal \( \gamma \) is called complex or real stability radius, respectively. The basic problem in the study of robustness of stability of the system is to characterize and compute these radii in terms of given matrices \( A, D, E \). It is important to note that these two stability radii are in general distinct. The analysis and computation of the complex stability radius for systems under structured perturbations have been done first in [14] and extended later in many subsequent papers (see [15] for a survey up till 1990) while the computation of the real stability radius being a much more difficult problem, some complicated solutions have been given only recently, see e.g. [20,33].

The situation is much simpler for a class of positive systems, see e.g. [9,23]. It has been shown in [16,34,35] that if \( A \) is a Metzler matrix (i.e., all off-diagonal entries of \( A \) are nonnegative) and \( D, E \) are nonnegative matrices, then the complex and the real stability radii coincide and can be computed directly by a simple formula. Then these results have been extended quite recently to positive continuous time-delay systems in [36–39] and to discrete time-delay systems in [18,26].

It is worth noticing that the notion of stability radius can be extended to various perturbation types [15]. Among perturbation types, two of the following

\[
A \to A + \sum_{i=1}^{N} D_i \Delta_i E_i \quad \text{(multi perturbation)}, \tag{2}
\]

\[
A \to A + \sum_{i=1}^{N} \delta_i B_i \quad \text{(affine perturbation)} \tag{3}
\]

are most well known in control theory and include perturbation types studied in the literature. The problem of computing of stability radii of positive linear systems without delay under multi perturbations has just been solved by ourselves only recently, see [27]. Furthermore, in our latest works, we considered the problem of computing of stability radii of positive linear functional differential equations under multi perturbations and affine perturbations [29].

In this paper, we study stability radius of linear time-invariant parameter-varying differential systems of the form

\[
\dot{x}(t) = A(z)x(t), \quad t \geq 0,
\]

\[
A(z) := A_0 + z_1 A_1 + \cdots + z_m A_m, \quad |z_i| \leq \alpha_i, \quad i \in \{0, 1, \ldots, m\}, \tag{4}
\]
where the system matrices $A_i, i \in \{0, 1, \ldots, m\}$, are subjected to the multi perturbations or affine perturbations. It is important to note that motivated by many applications in various areas, stability and robust stability problems of linear parameter-varying differential systems of the form (4) have attracted a lot of attention of many researchers during the last decades, see e.g. [2–8,10, 11,24,40,41]. Most of these papers focus on finding sufficient conditions for stability of systems of this class. In particular, in the recent papers [6,7], G. Chesi et al. dealt with the problem of computing the robust parametric margin of linear time-invariant parameter-varying differential systems. However, the stability radius problems for this class have not been studied yet so far and this paper tries to fill this gap.

The organization of the paper is as follows. In the next section, we summarize some notations and preliminary results on nonnegative matrices which will be used in the sequel. In Section 3, a formula for the complex stability radius of linear time-invariant parameter-varying differential systems under multi perturbations is derived. Then, under hypotheses of positivity of the system matrices, we prove that the complex, real and positive stability radii of the system under multi perturbations coincide and a simple formula for their computation is established. Apart from that, we derive solution of a global optimization problem associated with the problem of computing of the stability radii of a linear time-invariant parameter-varying differential system under multi perturbations. In Section 4, the class of affine perturbations is considered and an explicit formula for computing the complex, real and positive stability radii of the system is given. In Section 5, we present two applications of the obtained results in the previous sections. First, as a particular case of the obtained results, we get back the formulae for the strong stability radii of the linear time-delay differential systems under the multi perturbations and affine perturbations of [28]. Then, we show that for a class of positive linear time-delay differential systems, the stability radii of the system under multi perturbations (or affine perturbations) are equal to those of the associated linear time-invariant parameter-varying differential system under multi perturbations. In particular, we obtain the explicit formulas for stability radii of a positive linear functional differential equation under multi perturbations (or affine perturbations) are equal to those of the associated linear time-invariant parameter-varying differential system. In particular, we prove that the stability radii of a positive linear time-delay differential systems, the stability radii of the system under multi perturbations (or affine perturbations) are equal to the strong stability radii. For positive linear functional differential equations, we prove that the stability radii of a positive linear functional differential equation under multi perturbations (or affine perturbations) are equal to those of the associated linear time-invariant parameter-varying differential system. In particular, we obtain the explicit formulas for stability radii of a positive linear functional differential equation under multi perturbations and affine perturbations which are the main results of [29]. Finally, in Section 6, we give two examples to illustrate the obtained results.

2. Preliminaries

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and $l, n, q$ be positive integers. For a complex number $s$, denote by $\Re s$ the real part of $s$. Inequalities between real matrices or vectors will be understood componentwise, i.e., for two real $l \times q$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, the inequality $A \geq B$ means $a_{ij} \geq b_{ij}$ for $i = 1, \ldots, l$, $j = 1, \ldots, q$. The set of all nonnegative $l \times q$-matrices is denoted by $\mathbb{R}^{l \times q}_{\geq 0}$. If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{K}^n$ and $P = (p_{ij}) \in \mathbb{K}^{l \times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. It is easy to see that $|CD| \leq |C||D|$. For any matrix $A \in \mathbb{K}^{n \times n}$ the spectral radius, spectral absicsssa of $A$ are denoted respectively by $\rho(A) = \max\{||\lambda||: \lambda \in \sigma(A)\}$, $\mu(A) = \max\{|\Re \lambda|: \lambda \in \sigma(A)\}$, where $\sigma(A) := \{z \in \mathbb{C}: \det(zI_n - A) = 0\}$ is the set of all eigenvalues of $A$. A norm $\| \cdot \|$ on $\mathbb{K}^n$ is said to be monotonic if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in \mathbb{K}^n$, see e.g. [1]. Every $p$-norm on $\mathbb{K}^n$, $1 \leq p \leq \infty$, is monotonic. Throughout the paper, the norm $\|M\|$ of a matrix $M \in \mathbb{K}^{l \times q}$ is always understood as the operator norm defined by $\|M\| = \max_{\|y\|=1} \|My\|$ where $\mathbb{K}^q$ and $\mathbb{K}^l$ are provided with some monotonic vector norms. Then, the operator norm $\| \cdot \|$ has
the following monotonicity property, see e.g. [17]:

\[ P \in \mathbb{K}^{l \times q}, \quad Q \in \mathbb{K}^{l \times q}, \quad |P| \leq Q \Rightarrow \|P\| \leq \||P|\| \leq \|Q\|. \] (5)

A matrix \( A \in \mathbb{R}^{n \times n} \) is called a Metzler matrix if all the off-diagonal entries of \( A \) are nonnegative. It is obvious that \( A \in \mathbb{R}^{n \times n} \) is a Metzler matrix if and only if \( tI_n + A \succeq 0 \), for some \( t \geq 0 \).

The next theorem summarizes some basic properties of Metzler matrices, see e.g. [17].

**Theorem 2.1.** Let \( A \in \mathbb{R}^{n \times n} \) be a Metzler matrix. Then

(i) **(Perron–Frobenius Theorem)** \( \mu(A) \) is an eigenvalue of \( A \) and there exists a nonnegative eigenvector \( x \geq 0 \), \( x \neq 0 \) such that \( Ax = \mu(A)x \).

(ii) Given \( \alpha \in \mathbb{R} \), there exists a nonzero vector \( x \geq 0 \) such that \( Ax \geq \alpha x \) if and only if \( \mu(A) \geq \alpha \).

(iii) \( (tI_n - A)^{-1} \) exists and is nonnegative if and only if \( t > \mu(A) \).

(iv) Given \( B \in \mathbb{R}^{n \times n}_+, \ C \in \mathbb{C}^{n \times n}. \) Then

\[ |C| \leq B \Rightarrow \mu(A + C) \leq \mu(A + B). \]

3. **Stability radii of linear time-invariant parameter-varying differential systems under multi perturbations**

Consider a linear time-invariant parameter-varying differential system of the form

\[ \dot{x}(t) = A(z)x(t) \quad t \geq 0, \] (6)

where the system matrix \( A(z) \) is affine parameter-varying of the form

\[ A(z) := A_0 + z_1 A_1 + \cdots + z_m A_m, \quad z := (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m. \] (7)

Here \( A_0, A_1, \ldots, A_m \in \mathbb{C}^{n \times n} \) are fixed matrices, and \( z_i, i \in \underline{m} := \{1, 2, \ldots, m\} \), are uncertain parameters. Throughout this paper, we assume that \( |z_i| \leq \alpha_i, i \in \underline{m}, \) where \( \alpha_i, i \in \underline{m}, \) are given positive numbers.

Denote by

\[ \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m, \]

\[ D_\alpha := \{ z = (z_1, z_2, \ldots, z_m): z_i \in \mathbb{C}, \ |z_i| \leq \alpha_i, \ i \in \underline{m} \}. \] (8)

**Definition 3.1.** The system (6)–(7) is called stable if for every \( z \in D_\alpha \), the system \( \dot{x}(t) = A(z)x(t), t \geq 0, \) is asymptotically stable in Lyapunov’s sense.

Let us define

\[ \mathbb{C}^- := \{ s \in \mathbb{C}: \Re s < 0 \}; \quad \mathbb{C}^+ := \{ s \in \mathbb{C}: \Re s \geq 0 \}. \] (9)

By the definition, it is obvious that the system (6)–(7) is stable if and only if

\[ \det H(s, z) \neq 0, \quad \forall s \in \mathbb{C}^+, \forall z \in D_\alpha. \] (10)

where

\[ H(s, z) := sI_n - A(z) = sI_n - (A_0 + z_1 A_1 + \cdots + z_m A_m). \] (11)
Remark 3.2.

(i) In particular, if $\alpha_i = 1$, $\forall i \in m$, then by the definition, the system (6)–(7) is stable if
\[
\det(sI_n - A_0 - z_1A_1 - \cdots - z_mA_m) \neq 0,
\forall s \in \mathbb{C}, \Re s \geq 0 \text{ and } \forall z_i \in \mathbb{C}, \ |z_i| \leq 1, \ i \in m.
\]
This means that the following linear time-delay differential system
\[
\dot{x}(t) := A_0x(t) + \sum_{i=1}^{m} A_ix(t - h_i), \ t \geq 0,
\]
is strongly delay-independently stable, see e.g. [2].

(ii) Recently, F.A. Bliman gave some sufficient conditions for the stability of the linear time-invariant parameter dependent systems of the form (6) in terms of solvability of simple linear matrix inequalities, see e.g. [2,5].

(iii) It follows from (10) that the system (6)–(7) is stable if and only if
\[
\mu(A_0 + z_1A_1 + \cdots + z_mA_m) < 0, \ \forall z \in D_\alpha.
\]
Then, by the continuity of the spectral abscissa $\mu(\cdot)$ in matrix $(\cdot)$, it is easy to see that if the system (6)–(7) is stable then a perturbed system of the form
\[
\dot{x}(t) = \left( A(z) + R(z) \right)x(t), \ t \geq 0,
\]
\[
R(z) = \Delta_0 + z_1\Delta_1 + \cdots + z_m\Delta_m, \ \Delta_i \in \mathbb{C}^{n \times n}, \ i \in m,
\]
remains stable if the size of perturbation matrices $\Delta_i, i \in m$, is small enough. This is the foundation for us to define the notion of stability radius of linear time-invariant parameter dependent systems of the form (6) under perturbations.

We now assume that the system (6)–(7) is stable and the system matrices $A_i, i \in m_0 := \{0, 1, \ldots, m\}$, are subjected to multi perturbations of the form
\[
A_i \rightarrow A_i + \sum_{j=1}^{N} D_{ij}\Delta_{ij}E, \ i \in m_0,
\]
where $D_{ij} \in \mathbb{C}^{n \times ij}, i \in m_0, j \in N := \{1, 2, \ldots, N\}, E \in \mathbb{C}^{q \times n}$ are given matrices determining the structure of the perturbations and $\Delta_{ij} \in \mathbb{K}^{ij \times q}, i \in m_0, j \in N$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) are unknown disturbances.

Thus the perturbed systems are described by
\[
\dot{x}(t) = \left( A(z) + \Delta(z) \right)x(t), \ t \geq 0,
\]
where $\Delta(z) = \sum_{j=1}^{N} D_{0j}\Delta_{0j}E + \sum_{i=1}^{m} z_i(\sum_{j=1}^{N} D_{ij}\Delta_{ij}E)$. Let us define
\[
H_\Delta(s, z) = sI_n - \left( A(z) + \Delta(z) \right), \ (s, z) \in \mathbb{C}^2.
\]
Denote by $\Delta := (\Delta_0, \Delta_1, \ldots, \Delta_m)$, where $\Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{in}) \in \mathbb{K}^{l_{i1} \times q} \times \ldots \times \mathbb{K}^{l_{iN} \times q}$, $i \in m_0$. The size of each perturbation $\Delta$ is measured by
\[
\gamma(\Delta) := \sum_{i=0}^{m} \sum_{j=1}^{N} \|\Delta_{ij}\|.
\]
**Definition 3.3.** Let the linear time-invariant parameter-varying differential system (6)–(7) be stable. The complex stability radius of the system (6)–(7) with respect to multi perturbations of the form (13) is defined by

\[
 r_C = \inf \{ \gamma(\Delta) : \Delta_{ij} \in \mathbb{C}^{l_{ij} \times q}, \forall i \in m_0, \forall j \in N, \ \det H_\Delta(s, z) = 0, \\
 \text{for some } (s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha \}.
\]

If the disturbance matrices \( \Delta_{ij} \) in (17) are restricted, respectively, to the spaces \( \mathbb{R}^{l_{ij} \times q} \), \( \mathbb{R}^{l_{ij} \times q}_+ \) \( i \in m_0, \ j \in N \), then we obtain the real, positive stability radius \( r_\mathbb{R}, r_+ \), respectively.

It is clear from Definition 3.3 that

\[
 0 < r_C \leq r_\mathbb{R} \leq r_+ \leq +\infty.
\]

With the system (6)–(7) and perturbation structure (13), we associate the transfer matrices which are defined by

\[
 G_{ij}(s, z) := EH(s, z)^{-1} D_{ij}, \quad i \in m_0, \ j \in N.
\]

Since the system (6)–(7) is stable, the matrices \( G_{ij}(s, z), i \in m_0, \ j \in N \) are well defined for \( (s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha \).

**Lemma 3.4.** Let the system (6)–(7) be stable. For \( i_0 \in m \) and \( j_0 \in N \) fixed, we have

(i) If \( G_{i_0,j_0}(s, z) \neq 0 \) for some \( (s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha \), then there exists a complex perturbation \( \Delta = (\Delta_0, \Delta_1, \ldots, \Delta_m) \) such that

\[
 \gamma(\Delta) = \frac{1}{\|G_{i_0,j_0}(s, z)\|} \quad \text{and} \quad \det H_\Delta(s, z) = 0.
\]

(ii) If \( G_{i_0,j_0}(s, z) \neq 0 \) for some \( (s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha, z = (z_1, z_2, \ldots, z_m) \) and \( z_{i_0} \neq 0 \) then there exists a complex perturbation \( \Delta = (\Delta_0, \Delta_1, \ldots, \Delta_m) \) such that

\[
 \gamma(\Delta) = \frac{1}{\|G_{i_0,j_0}(s, z)\| |z_{i_0}|} \quad \text{and} \quad \det H_\Delta(s, z) = 0.
\]

(iii) If \( G_{ij}(0, \alpha) \in \mathbb{R}^{q \times l_{ij}} \) for every \( i \in m, \ j \in N \) and \( \max_{i \in m_0, j \in N} \|G_{ij}(0, \alpha)\| \alpha_i \neq 0 \) then there exists a nonnegative perturbation \( \Delta = (\Delta_0, \Delta_1, \ldots, \Delta_m) \) such that

\[
 \gamma(\Delta) = \frac{1}{\max_{i \in m_0, j \in N} \|G_{ij}(0, \alpha)\| \alpha_i} \quad \text{and} \quad \det H_\Delta(0, \alpha) = 0,
\]

where \( \alpha_0 := 1 \).

**Proof.** The proofs of (i) and (ii) are similar. Therefore, we omit that of (i) here and give the proof of (ii).

(ii) By the definition of operator norm, there exists a vector \( u_0 \in \mathbb{C}^{l_{i_0,j_0}}, \|u_0\| = 1 \) such that

\[
 \|G_{i_0,j_0}(s, z)\| = \|G_{i_0,j_0}(s, z)u_0\|.
\]

Then, by the Hahn–Banach Theorem, there exists \( y_0^* \in (\mathbb{C}^q)^* \), \( \|y_0^*\| = 1 \) (where \( \|y_0^*\| \) is the dual norm) satisfying \( y_0^* G_{i_0,j_0}(z) u_0 = \|G_{i_0,j_0}(s, z)\| \). Let us define

\[
 \bar{\Delta}_{i_0,j_0} := \|G_{i_0,j_0}(s, z)\|^{-1} u_0 y_0^* \in \mathbb{C}^{l_{i_0,j_0} \times q}.
\]

It is clear that \( \|\bar{\Delta}_{i_0,j_0}\| = \|G_{i_0,j_0}(s, z)\|^{-1} \) and \( \bar{\Delta}_{i_0,j_0} G_{i_0,j_0}(s, z) u_0 = u_0 \). Setting \( x_0 := (sI_n - A_0 - z_1 A_1 - \cdots - z_m A_m)^{-1} D_{i_0,j_0} u_0 \), we have \( \bar{\Delta}_{i_0,j_0} E x_0 = \bar{\Delta}_{i_0,j_0} G_{i_0,j_0}(s, z) u_0 = u_0 \). Thus \( x_0 \neq 0 \) and...
Let the linear time-invariant parameter-varying differential system (6)–(7) be stable. Then

\[ r_C = \frac{1}{\max_{i \in m_0, j \in N} \{ \max_{(s, z) \in C^+ \times D_\alpha} \| G_{ij}(s, z) \| \| z_i \| \}} \]

where \( z_0 := 1 \).

**Proof.** The proof is similar to those of Theorem 3.8 of [18], Theorem 3.3 of [27], Theorem 1 of [36] and is omitted here. \( \square \)

**Remark 3.6.** By Lemma 3.4 and the argument used in the proof of the above theorem, we can show that \( r_C = +\infty \) if and only if

\[ \max_{i \in m_0, j \in N} \max_{(s, z) \in C^+ \times D_\alpha} \| G_{ij}(s, z) \| \| z_i \| = 0. \]

Hence, the formula (23) still holds in this case.

It is important to note that Theorem 3.5 reduces the computation of the complex stability radius of (6)–(7) to solving the following global problems:

\[ \max_{(s, z) \in C^+ \times D_\alpha} \| G_{ij}(s, z) \| \| z_i \|, \quad i \in m_0, \ j \in N. \]

The following theorem gives a solution of these problems in a special case in which \( A_0 \) is a Metzler matrix, \( A_i, i \in m \); \( D_{ij}, i \in m_0, \ j \in N \), and \( E \) are nonnegative matrices.
**Theorem 3.7.** Let $A_0 \in \mathbb{R}^{n \times n}$ be a Metzler matrix, $A_i \in \mathbb{R}^{n \times n}_+$, $i \in \mathbb{m}$, and $D \in \mathbb{R}^{n \times l}_+$, $E \in \mathbb{R}^{q \times n}_+$. Assume that
\[
\det H(s, z) \neq 0, \quad \forall (s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha.
\]
Let $G(s, z) := E H(s, z)^{-1} D$, we have
\[
\max_{(s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha} \| G(s, z) \| = \| G(0, \alpha) \|. \tag{24}
\]

**Proof.** From the assumption $\det H(s, z) \neq 0, \forall (s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha$, it follows that $\mu(A_0 + z_1 A_1 + \cdots + z_m A_m) < 0, \forall z \in \mathcal{D}_\alpha$. Fix $z \in \mathcal{D}_\alpha$, then we can represent the following
\[
H(s, z)^{-1} x = (sI_n - A(z))^{-1} x = \int_0^{+\infty} e^{-s\theta} e^{A(z)\theta} x d\theta, \quad x \in \mathbb{C}^n, \tag{25}
\]
for every $s \in \mathbb{C}, \Re s \geq 0$, see [25, p. 8], [32]. Since $A_0$ is a Metzler matrix, there exists a real number $\alpha_0 > 0$ such that $(A + \alpha_0 I_n) \geq 0$. For every $\theta \geq 0$, we have
\[
eq \alpha_0 \theta \mid e^{A(z)\theta} \mid = \left| e^{\alpha \alpha_0 \theta} e^{A(z)\theta} \right| = \left| e^{\alpha \alpha_0 \theta} e^{(A_0 + \alpha_0 I_n + z_1 A_1 + \cdots + z_m A_m)\theta} \right|
\]
\[
\leq e^{((\alpha_0 I_n + \alpha_1 A_1 + \cdots + \alpha_m A_m)\theta)} = e^{\alpha \alpha_0 \theta} e^{A(\alpha)\theta}.
\]
This implies
\[
eq \alpha_0 \theta \mid e^{A(z)\theta} \mid \leq e^{A(\alpha)\theta}, \quad \text{for every } \theta \geq 0, \quad z \in \mathcal{D}_\alpha. \tag{26}
\]
Taking (25), (26) into account, we get
\[
\| G(s, z) x \| = \| E H(s, z)^{-1} D x \| \leq E \| H(s, z)^{-1} D x \|
\]
\[
\leq E \int_0^{+\infty} | e^{-s\theta} e^{A(z)\theta} D x | d\theta \leq E \int_0^{+\infty} e^{-\Re s \theta} e^{A(\alpha)\theta} D | x | d\theta
\]
\[
= G(\Re s, \alpha) | x | \leq E H(0, \alpha) D | x | = G(0, \alpha) | x |, \tag{27}
\]
for every $(s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha$. By monotonicity property of the vector norm and definition of operator norm, we get
\[
\| G(s, z) \| \leq \| G(0, \alpha) \|, \quad (s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha,
\]
as to be shown. \hfill \Box

**Remark 3.8.** Under the assumptions of Theorem 3.7, it is now easy to see that
\[
\max_{(s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha} \| G(s, z) \| | z_i | = \| G(0, \alpha) \| \alpha_i,
\]
for every $i \in \mathbb{m}$.

Moreover, from the argument of the proof of Theorem 3.7, we have the following.

**Proposition 3.9.** Let the linear time-invariant parameter-varying differential system (6)–(7) be stable and $A_0 \in \mathbb{R}^{n \times n}$ be a Metzler matrix, $A_i \in \mathbb{R}^{n \times n}_+$, $i \in \mathbb{m}$. Then
\[
H(s, z)^{-1} x \leq H(\Re s, \alpha)^{-1} | x | \leq H(0, \alpha)^{-1} | x |, \quad x \in \mathbb{C}^n, \tag{28}
\]
for every $(s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha$. In particular, $H(0, \alpha)^{-1}$ is a nonnegative matrix.
As noted in the introduction section, the problem of computation of the real stability radius \( r_R \) is much more difficult and has been so far solved only for linear systems without parameters with the complicated solutions, see, e.g. [20,33]. We note that, by the definition, \( r_C \leq r_R \), so \( r_C \) can be accepted as the lower bound for \( r_R \). Unfortunately, as shown in many previous papers, even in the case of systems without parameters, these stability radii can be arbitrarily distinct. Therefore, it is an interesting problem to find classes of systems of the form (6)–(7) for which these stability coincide and they can be computed by a simple formula. The following theorem gives a solution of this problem.

**Theorem 3.10.** Let the linear time-invariant parameter-varying differential system (6)–(7) be stable and \( A_0 \in \mathbb{R}^{n \times n} \) be a Metzler matrix, \( A_i \in \mathbb{R}^{n \times n}_+ \), \( i \in m_0 \). Assume that the system matrices \( A_i, i \in m_0 \), are subjected to the multi perturbations of the form

\[
D_{ij} \in \mathbb{R}^{n \times l_{ij}} \quad \text{for all} \quad i \in m_0, j \in N_0, E \in \mathbb{R}^{q \times n}_+.
\]

Then

\[
r_C = r_R = r_+ = \frac{1}{\max_{i \in m_0, j \in N} \|G_{ij}(0, \alpha)\| \alpha_i}, \tag{29}
\]

where \( \alpha_0 := 1 \).

**Proof.** Suppose that \( r_C < +\infty \), as otherwise, there is nothing to show. It follows from Theorem 3.7, Remark 3.8 and (23) that

\[
r_C = \frac{1}{\max_{i \in m_0, j \in N} \|G_{ij}(0, \alpha)\| \alpha_i}. \tag{30}
\]

On the other hand, from the assumption \( D_{ij} \in \mathbb{R}^{n \times l_{ij}} \) for all \( i \in m_0, j \in N_0 \) and \( E \in \mathbb{R}^{q \times n}_+ \), by Proposition 3.9, we get \( G_{ij}(0, \alpha) \in \mathbb{R}^{q \times l_{ij}}_+ \), for every \( i \in m_0, j \in N_0 \). Taking Remark 3.6 into account, we have \( \max_{i \in m_0, j \in N} \|G_{ij}(0, \alpha)\| \neq 0 \). By Lemma 3.4(iii) and the definition of the positive stability radius, we conclude

\[
r_+ \leq \frac{1}{\max_{i \in m_0, j \in N} \|G_{ij}(0, \alpha)\| \alpha_i}. \tag{31}
\]

Finally, (29) follows from (30), (31) and the inequalities \( r_C \leq r_R \leq r_+ \). \( \square \)

We close this section with a remark that in a similar way, we can obtain similar results for the complex, real, positive stability radii of the linear time-invariant parameter-varying differential system (6)–(7) under multi perturbations of the form:

\[
A_i \to A_i + \sum_{j=1}^{N} D_{ij} E_{ij}, \quad i \in m_0, \ j \in N. \tag{32}
\]

4. Stability radii of linear time-invariant parameter-varying differential systems under affine perturbations

In this section, we study the stability radii of system (6)–(7) under affine perturbations of the following form:

\[
A_i \to A_i + \sum_{j=1}^{N} \delta_{ij} B_{ij}, \quad i \in m_0 := \{0, 1, 2, \ldots, m\}. \tag{33}
\]
Here \( B_{ij} \in \mathbb{R}^{n \times n}, \ i \in m_0, \ j \in N := \{1,2,\ldots,N\} \) are given and \( \delta_{ij}, \ i \in m_0, \ j \in N \) are unknown scalar parameters. Thus the perturbed systems are described by

\[
\dot{x}(t) = (A(z) + \delta(z))x(t), \quad t \geq 0, \quad \delta(z) := \sum_{j=1}^{N} \delta_{0j}B_{0j} + \sum_{i=1}^{m} z_i \left( \sum_{j=1}^{N} \delta_{ij}B_{ij} \right).
\] (34)

Denote by \( \delta := (\delta_{ij})_{i \in m_0, j \in N} \) and the size of each perturbation \( \delta \) is measured by

\[
||\delta|| := \max_{i \in m_0, j \in N} |\delta_{ij}|.
\] (35)

It is important to note that for the problems of computing of stability radii of linear systems without parameters, the class of affine perturbations of this type has been considered first in [16, 17] for linear invariant-time systems with no time delays in \( \mathbb{R}^n \), then in [18,38] for time-delay systems in \( \mathbb{R}^n \) and only recently in [29] for linear functional differential equations in \( \mathbb{R}^n \). In this section, we consider this class of perturbations for the problem of computing of stability radii of linear time-invariant parameter-varying differential systems.

Let us denote

\[
H_\delta(s, z) = sI_n - \left( A_0 + \sum_{j=1}^{N} \delta_{0j}B_{0j} \right) - \sum_{i=1}^{m} z_i \left( A_i + \sum_{j=1}^{N} \delta_{ij}B_{ij} \right), \quad (s, z) \in \mathbb{C} \times \mathcal{D}_\alpha.
\] (36)

Assume that the system (6)–(7) is stable. We define the complex, real stability radius of the system (6)–(7) under affine perturbations (33) by setting, for \( \mathcal{K} = \mathbb{C}, \mathcal{K} = \mathbb{R} \), respectively,

\[
r_{\mathcal{K}}^a = \inf \{ ||\delta||: \delta_{ij} \in \mathcal{K}, \ i \in m_0, \ j \in N, \ \det H_\delta(s, z) = 0 \text{ for some } (s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha \}.
\] (37)

Similarly, the positive stability radius \( r_{+}^a \) is obtained by restricting, in the above definition, the disturbances \( \delta := (\delta_{ij})_{i \in m_0, j \in N} \) to be nonnegative. It is clear that

\[
0 < r_{\mathcal{K}}^a \leq r_{\mathcal{R}}^a \leq r_{+}^a \leq +\infty.
\] (38)

**Theorem 4.1.** Let the linear time-invariant parameter-varying differential system (6)–(7) be stable and \( A_0 \in \mathbb{R}^{n \times n} \) be a Metzler matrix, \( A_i \in \mathbb{R}^{n \times n}, \ i \in m \). Suppose that the system matrices \( A_i, \ i \in m \), are subjected to affine perturbations of the form (33) where \( B_{ij} \in \mathbb{R}^{n \times n}, \ i \in m_0, \ j \in N \). Then

\[
r_{\mathcal{C}}^a = r_{\mathcal{R}}^a = r_{+}^a = \frac{1}{\mu((-A_0 - \alpha_1 A_1 - \cdots - \alpha_m A_m)^{-1}(\sum_{j=1}^{N} B_{0j} + \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{N} B_{ij}))}.
\] (39)

**Proof.** We first show that

\[
r_{+}^a = \frac{1}{\mu((-A_0 - \alpha_1 A_1 - \cdots - \alpha_m A_m)^{-1}(\sum_{j=1}^{N} B_{0j} + \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{N} B_{ij}))}.
\] (40)

Assume that \( r_{+}^a < +\infty \). Let \( \delta = (\delta_{ij})_{i \in m_0, j \in N} \) be an arbitrary nonnegative perturbation such that \( \det H_\delta(s, z) = 0 \) for some \( (s, z) \in \mathbb{C}^+ \times \mathcal{D}_\alpha \). This implies

\[
H_\delta(s, z)x_0 = \left( sI_n - \left( A_0 + \sum_{j=1}^{N} \delta_{0j}B_{0j} \right) - \sum_{i=1}^{m} z_i \left( A_i + \sum_{j=1}^{N} \delta_{ij}B_{ij} \right) \right)x_0 = 0,
\]
for some \( x_0 \in \mathbb{C}^n, x_0 \neq 0 \). Since the system (6)–(7) is stable, it follows that

\[
H(s, z)^{-1} \left( \sum_{j=1}^N \delta_{0j} B_{0j} + \sum_{i=1}^m z_i \left( \sum_{j=1}^N \delta_{ij} B_{ij} \right) \right) x_0 = x_0.
\]

Then, it follows from Proposition 3.9 that

\[
|x_0| = \left| H(s, z)^{-1} \left( \sum_{j=1}^N \delta_{0j} B_{0j} + \sum_{i=1}^m z_i \left( \sum_{j=1}^N \delta_{ij} B_{ij} \right) \right) x_0 \right| 
\leq H(\Re s, \alpha)^{-1} \left| \left( \sum_{j=1}^N \delta_{0j} B_{0j} + \sum_{i=1}^m z_i \left( \sum_{j=1}^N \delta_{ij} B_{ij} \right) \right) x_0 \right| 
\leq H(0, \alpha)^{-1} \left| \left( \sum_{j=1}^N \delta_{0j} B_{0j} + \sum_{i=1}^m z_i \left( \sum_{j=1}^N \delta_{ij} B_{ij} \right) \right) x_0 \right| 
\leq \|\delta\| H(0, \alpha)^{-1} \left( \sum_{j=1}^N B_{0j} + \sum_{i=1}^m \alpha_i \sum_{j=1}^N B_{ij} \right) |x_0|.
\]

Therefore

\[
H(0, \alpha)^{-1} \left( \sum_{j=1}^N B_{0j} + \sum_{i=1}^m \alpha_i \sum_{j=1}^N B_{ij} \right) |x_0| \geq \frac{1}{\|\delta\|} |x_0|, \quad x_0 \neq 0.
\]

Since the matrix \( H(0, \alpha)^{-1} \left( \sum_{j=1}^N B_{0j} + \sum_{i=1}^m \alpha_i \sum_{j=1}^N B_{ij} \right) \) is nonnegative, by Theorem 2.1(ii),

\[
\mu \left( H(0, \alpha)^{-1} \left( \sum_{j=1}^N B_{0j} + \sum_{i=1}^m \alpha_i \sum_{j=1}^N B_{ij} \right) \right) \geq \frac{1}{\|\delta\|} > 0.
\]

(41)

Hence,

\[
\|\delta\| \geq \frac{1}{\mu \left( H(0, \alpha)^{-1} \left( \sum_{j=1}^N B_{0j} + \sum_{i=1}^m \alpha_i \sum_{j=1}^N B_{ij} \right) \right)}.
\]

From the definition of \( r^a_+ \), we have

\[
\frac{1}{\mu \left( H(0, \alpha)^{-1} \left( \sum_{j=1}^N B_{0j} + \sum_{i=1}^m \alpha_i \sum_{j=1}^N B_{ij} \right) \right)}.
\]

To prove the inverse inequality, by Perron–Frobenius Theorem (Theorem 2.1(i)), there exists a nonzero vector \( x_1 \in \mathbb{R}^{n \times n} \) such that

\[
\left( H(0, \alpha)^{-1} \left( \sum_{j=1}^N B_{0j} + \sum_{i=1}^m \alpha_i \sum_{j=1}^N B_{ij} \right) \right) x_1 = \mu_0 x_1,
\]

where \( \mu_0 := \mu \left( H(0, \alpha)^{-1} \left( \sum_{j=1}^N B_{0j} + \sum_{i=1}^m \alpha_i \sum_{j=1}^N B_{ij} \right) \right) \). This gives

\[
\left( 0I_n - \left( A_0 + \sum_{i=1}^N \frac{1}{\mu_0} B_{0j} \right) - \sum_{i=1}^m \alpha_i \left( A_i + \sum_{j=1}^N \frac{1}{\mu_0} B_{ij} \right) \right) x_1 = 0, \quad x_1 \neq 0.
\]
Therefore, the nonnegative perturbation \( \delta^* \) defined by \( \delta^*_{ij} = 1/\mu_0, \ i \in \mathbb{N}_0, \ j \in \mathbb{N} \), satisfies \( \det H_{\delta^*}(0, \alpha) = 0 \). By the definition of \( r^a_+ \), we conclude that
\[
r^a_+ \leq \frac{1}{\mu_0}.
\]
Thus, we obtain
\[
r^a_+ = \frac{1}{\mu((A_0 - \alpha_1 A_1 - \cdots - \alpha_m A_m)^{-1}(\sum_{j=1}^{N} B_{0j} + \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{N} B_{ij}))).
\]
It is important to note that by the above argument, if \( r_+^a = +\infty \), then
\[
\mu\left((A_0 - \alpha_1 A_1 - \cdots - \alpha_m A_m)^{-1}\left(\sum_{j=1}^{N} B_{0j} + \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{N} B_{ij}\right)\right) = 0.
\]
So the equality (40) is obvious in this case.

We are now ready to show that \( r^a_C = r^a_R = r^a_+ \). Suppose \( r^a_C < +\infty \) and \( \delta = (\delta_{ij})_{i \in \mathbb{N}_0, j \in \mathbb{N}} \) is an arbitrary destabilizing complex perturbation. That is \( \det H_{\delta}(s, z) = 0 \) for some \( (s, z) \in \mathbb{C}^+ \times D_\alpha \).

By a similar argument as in the beginning, we get
\[
\mu_* := \mu\left(H(0, \alpha)^{-1}\left(\sum_{j=1}^{N} |\delta_{0j}| B_{0j} + \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{N} |\delta_{ij}| B_{ij}\right)\right) \geq 1.
\]

Since \( H(0, \alpha)^{-1}\left(\sum_{j=1}^{N} |\delta_{0j}| B_{0j} + \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{N} |\delta_{ij}| B_{ij}\right) \) is a nonnegative matrix, by Perron–Frobenius Theorem,
\[
\left(H(0, \alpha)^{-1}\left(\sum_{j=1}^{N} |\delta_{0j}| B_{0j} + \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{N} |\delta_{ij}| B_{ij}\right)\right)x = \mu_* x,
\]
for some nonzero vector \( x \in \mathbb{R}^{n_n \times 1} \). This gives
\[
(0I_n - \left(A_0 + \sum_{j=1}^{N} |\delta_{0j}| B_{0j} \right) - \sum_{i=1}^{m} \alpha_i \left(A_i + \sum_{j=1}^{N} |\delta_{ij}| B_{ij}\right))x = 0.
\]
It means that the nonnegative perturbation \( \delta' = \left(\frac{|\delta_{ij}|}{\mu_*}, i \in \mathbb{N}_0, j \in \mathbb{N}\right) \), satisfies \( \det H_{\delta'}(0, \alpha) = 0 \). By the definition of \( r^a_C, r^a_+ \), we get
\[
r^a_+ \leq r^a_C.
\]
In combining with the inequalities \( r^a_C \leq r^a_R \leq r^a_+ \), this implies \( r^a_C = r^a_R = r^a_+ \). This completes our proof. \( \square \)

5. Applications

In this section, we present two applications of the obtained results in the previous ones. First, as a particular case, from Theorems 3.5, 3.10, 4.1, we get back the formulae for the strong stability radii of linear time-delay differential systems under the multi perturbations and affine perturbations (see e.g. [28]). Then, using the obtained results in Sections 3 and 4, we show that for a class of positive linear time-delay differential systems, the stability radii and the strong
stability radii of the system coincide. In the second one, we prove that stability radii of a positive linear functional differential equation under multi perturbations (or affine perturbations) are equal to those of the associated linear time-invariant parameter-varying differential system. In particular, we get back the explicit formulae for the stability radii of a positive linear functional differential equation under multi perturbations and affine perturbations which are given recently in [29].

5.1. Stability radius and strong stability radius of linear time-delay differential systems

For simplicity, we consider the linear time-invariant time-delay differential system of the form

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \quad t \geq 0, \\
x(\theta) = \phi(\theta), \quad \theta \in [-h, 0],
\]

where \( A_0, A_1 \in \mathbb{R}^{n \times n} \) and \( h > 0 \) are given.

It is well known that the system (42) is exponentially stable if and only if

\[
\det F(s) \neq 0 \quad \forall s \in \mathbb{C}, \ \Re s \geq 0,
\]

where \( F(s) := sI_n - A_0 - A_1 e^{-hs} \) is the characteristic matrix of the system (42), see e.g. [12]. Furthermore, the system (42) is said to be delay-independently stable if

\[
\det(sI_n - A_0 - A_1 e^{-hs}) \neq 0 \quad \forall s \in \mathbb{C}, \ \Re s \geq 0, \forall h \geq 0,
\]

see e.g. [3,21]. This is equivalent to

\[
\det(sI_n - A_0 - zA_1) \neq 0 \quad \forall (s,z) \in \mathbb{C}^2, \ \Re s \geq 0, s \neq 0, |z| \leq 1 \text{ or } s = 0, z = 1,
\]

see e.g. [13]. Denote by

\[
H(s,z) = (sI_n - A_0 - zA_1), \quad (s,z) \in \mathbb{C}^2.
\]

**Definition 5.1.** [3] The system (42) is called strongly delay-independently stable if

\[
\det H(s,z) \neq 0, \quad \forall (s,z) \in \mathbb{C}^2, \ \Re s \geq 0, |z| \leq 1.
\]

**Remark 5.2.** It is important to note that the exponential stability and strongly delay-independent stability of linear time-delay system (42) are the properties robust with respect to perturbations of the system matrices \( A_0, A_1 \), whereas the delay-independent stability of the system is not, see e.g. [3,38]. So, we introduce below two types of stability radius of the system.

We now assume that the system (42) is exponentially stable (strongly delay-independently stable) and the system matrices \( A_i, i \in I := \{0, 1\} \), are subjected to multi perturbations of the form

\[
A_i \rightarrow A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E, \quad i \in I,
\]

where \( D_{ij} \in \mathbb{R}^{n \times l_{ij}}, i \in I, j \in N := \{1, 2, \ldots, N\}, E \in \mathbb{R}^{q \times n} \) are given matrices determining the structure of the perturbations and \( \Delta_{ij} \in \mathbb{K}^{l_{ij} \times q}, i \in I, j \in N \) (\( \mathbb{K} = \mathbb{R}, \mathbb{C} \)) are unknown disturbances. Let us define

\[
F_\Delta(s) = sI_n - \left( A_0 + \sum_{j=1}^{N} D_{0j} \Delta_{0j} E \right) - e^{-sh} \left( A_1 + \sum_{j=1}^{N} D_{1j} \Delta_{1j} E \right), \quad s \in \mathbb{C},
\]
and
\[ H_\Delta(s, z) = sI_n - \left( A_0 + \sum_{j=1}^{N} D_{0j} \Delta_{0j} E \right) - z \left( A_1 + \sum_{j=1}^{N} D_{1j} \Delta_{1j} E \right), \quad (s, z) \in \mathbb{C}^2. \] (47)

Denote by
\[ \gamma(\Delta) := \sum_{i=1}^{N} \| \Delta_{0i} \| + \sum_{i=1}^{N} \| \Delta_{1i} \|. \] (48)

**Definition 5.3.** Let the linear time-delay system (42) be exponentially stable. The complex stability radius of the system (42) with respect to multi perturbations of the form (45) is defined by
\[ r_C = \inf \{ \gamma(\Delta): \Delta_{ij} \in \mathbb{C}^{l_{ij} \times q} \forall i \in I, j \in N, \det F_\Delta(s) = 0, \] for some \( s \in \mathbb{C}, \Re s \geq 0 \}. \] (49)

If the disturbance matrices \( \Delta_{ij} \) in (49) are restricted, respectively, to the spaces \( \mathbb{R}^{l_{ij} \times q}, \mathbb{R}_{+}^{l_{ij} \times q} \), \( i \in I, j \in N \), then we obtain the real, positive stability radii \( r_C^R, r_C^+ \), respectively.

**Definition 5.4.** Let the linear time-delay system (42) be strongly delay-independently stable. The complex strong delay-independent stability radius (or shortly, complex strong stability radius) of the system (42) with respect to multi perturbations of the form (45) is defined by
\[ r_C = \inf \{ \gamma(\Delta): \Delta_{ij} \in \mathbb{C}^{l_{ij} \times q} \forall i \in I, j \in N, \det H_\Delta(s, z) = 0, \] for some \( (s, z) \in \mathbb{C}^2, \Re s \geq 0, |z| \leq 1 \}. \] (50)

If the disturbance matrices \( \Delta_{ij} \) in (50) are restricted, respectively, to the spaces \( \mathbb{R}^{l_{ij} \times q}, \mathbb{R}_{+}^{l_{ij} \times q} \), \( i \in I, j \in N \), then we obtain the real, positive strong stability radii \( r_C^R, r_C^+ \), respectively.

The following theorem is a particular case of Theorem 3.5 when \( m = 1, \alpha = 1 \).

**Theorem 5.5.** Let the linear time-delay system (42) be strongly delay-independently stable. Then
\[ r_C = \frac{1}{\max(\max_{j \in N} \{ \max_{\Re s \geq 0, |z| \leq 1} \| G_{0j}(s, z) \| \}, \max_{j \in N} \{ \max_{\Re s \geq 0, |z| \leq 1} \| G_{1j}(s, z) \| \| z \| \})}. \] (51)

Furthermore, if the system (42) is positive (i.e., \( A_0 \) is a Metzler matrix and \( A_1 \) is a nonnegative matrix, see e.g. [27,38], for further details) then from Theorem 3.10, we have the following.

**Theorem 5.6.** Let the linear system (42) be positive, strongly delay-independently stable. Assume that the system matrices \( A_i, i \in I \), are subjected to the multi perturbations of the form (45) where \( D_{ij} \in \mathbb{R}^{n \times l_{ij}}_+ \) for all \( i \in I, j \in N \) and \( E \in \mathbb{R}^{q \times n}_+ \). Then
\[ r_C = r_R = r_+ = \frac{1}{\max(\max_{j \in N} \| G_{0j}(0, 1) \|, \max_{j \in N} \| G_{1j}(0, 1) \|)}. \] (52)
Then, by setting \(m = 1, \alpha = 1\) in Theorem 4.1, we get the formula for the strong stability radii of linear time-delay system (42) under affine perturbations.

We now prove that under the assumptions of positivity of the system (42) and of the matrices defining the structure of perturbation \(D_{ij}, i \in I, j \in \mathbb{N}, E \in \mathbb{R}^{q \times n}\), the stability radii and strong stability radii under multi-perturbations (45) coincide. In fact, it is important to note that if the system (42) is positive then the exponential stability, delay-independent stability and strong delay-independent stability of the system are equivalent, see e.g. [30, 31]. By Definitions 5.3, 5.4, we have the following inequalities:

\[
 r_C \leq r^e_C, \quad r_R \leq r^e_R, \quad r_+ \leq r^e_+.
\]

Moreover, it follows from Lemma 3.4(iii) and Theorem 5.6 that

\[
 r_+ = r^e_+.
\]

On the other hand, we have

\[
 r^e_C = \frac{1}{\max_{i,j} \{ \max_{s \in C, \Re s \geq 0} \| E(sI_n - A_0 - e^{-sh}A_1)^{-1}D_{ij} \| \}},
\]

see, e.g. [26]. By Theorem 3.7, we get

\[
 \max_{s \in C, \Re s \geq 0} \| E(sI_n - A_0 - e^{-sh}A_1)^{-1}D_{ij} \| = \| E(-A_0 - A_1)^{-1}D_{ij} \|,
\]

for every \(i \in I, j \in \mathbb{N}\). Combining (53)–(56), we obtain

\[
 r^e_C = r^e_R = r^e_+ = r_C = r_R = r_+ = \frac{1}{\max_{i,j} \| E(-A_0 - A_1)^{-1}D_{ij} \|}.
\]

Finally, it is worth noticing that by the same way, under the assumptions of positivity of the system (42) and of matrices defining the structure of affine perturbation (3), we also show that the stability radii and the strong stability radii of the system (42) under affine perturbations coincide.

5.2. Stability radii of positive linear functional differential equations

Consider a linear retarded system described by the following general functional differential equation

\[
 \begin{align*}
 \dot{x}(t) &= A_0x(t) + Lx_t, \quad t \geq 0, \ x(t) \in \mathbb{R}^n, \\
 x(\theta) &= \phi(\theta), \quad \theta \in [-h, 0],
\end{align*}
\]

where, for each \(t \geq 0, x_t \in C([-h, 0], \mathbb{R}^n)\) is defined by \(x_t(\theta) = x(t + \theta), \theta \in [-h, 0]\), \(A_0 \in \mathbb{R}^{n \times n}\) is a given matrix and \(L : C([-h, 0], \mathbb{R}^n) \to \mathbb{R}^n\) is a linear bounded operator defined by

\[
 L\phi = \int_{-h}^{0} d[\eta(\theta)]\phi(\theta), \quad t \geq 0, \ \phi \in C([-h, 0], \mathbb{R}^n),
\]

where \(\eta(\cdot) \in NBV([-h, 0], \mathbb{R}^{n \times n})\) is given (real \(n \times n\)-matrix function of bounded variation on \([-h, 0]\) such that \(\eta\) vanishes at \(-h\) and is continuous from the left on \([-h, 0]\)).

It is well known that, for any given \(\phi \in C := C([-h, 0], \mathbb{R}^n)\), the system (58)–(59) has a unique solution \(x(\phi, \cdot)\) defined and continuous on \([-h, \infty)\), see, e.g. [12].
The system (58)–(59) is said to be *exponentially stable* if there are constants $M > 0, \alpha > 0$ such that for all $\phi \in \mathcal{C}$, the solution $x(\phi, \cdot)$ of (58)–(59) satisfies

$$
\|x(\phi, t)\| \leq Me^{-\alpha t}\|\phi\|, \quad t \geq 0.
$$

Denote by $H(s)$ the characteristic quasi-polynomial of the system (58)–(59), that is,

$$
H(s) = sI - A_0 - \int_{-h}^{0} e^{\theta d}[\eta(\theta)].
$$

Then the necessary and sufficient condition for the system (58)–(59) to be exponentially stable is

$$
\sigma(A_0, \eta) \subseteq \mathbb{C}^{-} := \{s \in \mathbb{C}: \text{Re } s < 0\},
$$

where $\sigma(A_0, \eta)$ denotes the set of all roots of the characteristic equation of the system (58)–(59):

$$
\sigma(A_0, \eta) := \{s \in \mathbb{C}: \det H(s) = 0\},
$$

see, e.g. [12].

Assume that the retarded system (58)–(59) is exponentially stable and subjected to multi perturbations of the type

$$
A_0 \rightarrow A_{0\Delta} = A_0 + \sum_{j=1}^{N} D_{0j} \Delta_j E, \quad \Delta_j \in \mathbb{C}^{(l_j) \times q}, \quad j \in N,
$$

$$
\eta \rightarrow \eta_\delta = \eta + \sum_{j=1}^{N} D_{1j} \delta_j E, \quad \delta_j \in \text{NBV}([-h,0], \mathbb{C}^{(l_j) \times q}), \quad j \in N,
$$

and thus the perturbed system is described by

$$
\dot{x}(t) = \left(A_0 + \sum_{j=1}^{N} D_{0j} \Delta_j E\right)x(t)
$$

$$
+ \int_{-h}^{0} d \left[\eta(\theta) + \sum_{j=1}^{N} D_{1j} \delta_j(\theta) E\right]x(t + \theta), \quad t \geq 0.
$$

Here $D_{ij} \in \mathbb{C}^{n \times l_{ij}}, E \in \mathbb{C}^{q \times n}, i \in I := \{0,1\}, j \in N,$ are given matrices determining the *structure* of perturbations, $\Delta_j$ and $\delta_j(\cdot), j \in N,$ are unknown disturbances. We shall measure the size of each perturbation $\tilde{\Delta} := [\Delta, \delta]$ where

$$
\Delta := (\Delta_1, \ldots, \Delta_N), \quad \delta := (\delta_1, \ldots, \delta_N),
$$

$$
\Delta_j \in \mathbb{C}^{(l_j) \times q}, \quad \delta_j \in \text{NBV}([-h,0], \mathbb{C}^{(l_j) \times q}), \quad j \in N,
$$

by the norm

$$
\|\tilde{\Delta}\| := \sum_{j=1}^{N} \|\Delta_j\| + \sum_{j=1}^{N} \|\delta_j\|. \quad \|\delta_j\| := \text{Var}(\delta_j; -h, 0), \quad j \in N.
$$

Set
\[ \Delta_C := \{ \tilde{\Delta} = [\Delta, \delta] : \Delta_j \in \mathbb{C}^{(l_j) \times q}, \delta_j \in NBV([-h, 0], \mathbb{C}^{(l_j) \times q}), j \in \mathbb{N} \}, \]
\[ \Delta_R := \{ \tilde{\Delta} = [\Delta, \delta] : \Delta_j \in \mathbb{R}^{(l_j) \times q}, \delta_j \in NBV([-h, 0], \mathbb{R}^{(l_j) \times q}), j \in \mathbb{N} \}\]
and
\[ \Delta_+ := \{ \tilde{\Delta} = [\Delta, \delta] : \Delta_j \in \mathbb{R}_+^{(l_j) \times q}, \delta_j \in NBV([-h, 0], \mathbb{R}_+^{(l_j) \times q}) \text{ is increasing}, j \in \mathbb{N} \}. \]

**Definition 5.7.** Let the linear retarded system (58)–(59) be exponentially stable. The complex, real and positive stability radii of the system with respect to perturbations of the form (63), measured by the norm (65), are defined, respectively, by
\[ r_C(A_0, \eta) := \inf \{ \| \tilde{\Delta} \| : \tilde{\Delta} \in \Delta_C, \sigma(A_0, \eta) \nsubseteq \mathbb{C}^- \}, \]
\[ r_R(A_0, \eta) := \inf \{ \| \tilde{\Delta} \| : \tilde{\Delta} \in \Delta_R, \sigma(A_0, \eta) \nsubseteq \mathbb{C}^- \}, \]
and
\[ r_+(A_0, \eta) := \inf \{ \| \tilde{\Delta} \| : \tilde{\Delta} \in \Delta_+, \sigma(A_0, \eta) \nsubseteq \mathbb{C}^- \}. \]

**Definition 5.8.** The system (58)–(59) is called positive if for every nonnegative initial function \( \phi_0 \in C([-h, 0], \mathbb{R}^n_+) \), the corresponding solution \( x(\phi_0, \cdot) \) satisfies \( x(\phi_0, t) \in \mathbb{R}^n_+ \) for every \( t \geq 0 \).

**Definition 5.9.** The function \( \eta \in NBV([-h, 0], \mathbb{R}^{n \times n}) \) is said to be an increasing matrix function if for \( \theta_1, \theta_2 \in [-h, 0], \theta_2 > \theta_1 \) then \( \eta(\theta_2) \supseteq \eta(\theta_1) \).

**Theorem 5.10.** [31,39] The system (58)–(59) is positive if and only if \( A_0 \) is a Metzler matrix and \( \eta \in NBV([-h, 0], \mathbb{R}^{n \times n}) \) is an increasing matrix function.

We need the following result on the exponential stability of the positive systems of the form (58)–(59).

**Theorem 5.11.** [31] Suppose that the system (58)–(59) is positive. Then the system (58)–(59) is exponentially stable if and only if \( \mu(A_0 + \eta(0)) < 0 \).

We now use our results to derive the explicit formulae for stability radii of the positive systems of the form (58)–(59) under multi perturbations. To do this, we assume that the system (58)–(59) is exponentially stable, positive and \( D_{ij} \in \mathbb{R}_+^{n \times l_{ij}}, \quad E \in \mathbb{R}_+^{q \times n}, \quad i \in I, \quad j \in \mathbb{N} \). We associate the positive linear functional differential system (58)–(59) with the following time-invariant parameter-varying differential system:
\[ \dot{x}(t) = A(z)x(t), \quad t \geq 0; \quad A(z) := A_0 + z\eta(0), \quad z \in \mathbb{C}, \quad |z| \leq 1. \]  
(69)

Since \( A_0 \) is a Metzler matrix and \( \eta(0) \geq 0 \), by Theorem 2.1(iv) and Theorem 5.11, we get \( \mu(A_0 + z\eta(0)) \leq \mu(A_0 + \eta(0)) < 0 \), for every \( z \in \mathbb{C}, |z| \leq 1 \). Therefore, the positive system of the form (58)–(59) is exponentially stable if and only if the associated linear parameter-varying differential system (69) is stable. Let \( r_C, r_R, r_+ \) be the complex, real and positive stability radii of the system (69) with respect to multi perturbations of the form
\[ A_0 \to A_0 + \sum_{j=1}^{N} D_{0j} \Delta_j E, \quad \Delta_j \in \mathbb{C}^{l_{ij} \times q}, \quad j \in \mathbb{N}, \]
\[ \eta(0) \to \eta(0) + \sum_{j=1}^{N} D_{1j} \Delta_j' E, \quad \Delta_j' \in \mathbb{C}^{l_{1j} \times q}, \ j \in \mathbb{N}. \] (70)

By Theorem 3.10, we have
\[ r = r_\mathbb{R} = r_+ = \frac{1}{\max_{i \in I, j \in \mathbb{N}} \| E(-A_0 - \eta(0))^{-1} D_{ij} \|}. \] (71)

We now show that
\[ r_+ = r_+(A_0, \eta) = \frac{1}{\max_{i \in I, j \in \mathbb{N}} \| E(-A_0 - \eta(0))^{-1} D_{ij} \|}. \] (72)

In fact, let \( \tilde{\Delta} = \left[ \Delta, \delta \right] \in \Delta_+ \) satisfy
\[ \| \tilde{\Delta} \| := \sum_{j=1}^{N} \| \Delta_j \| + \sum_{j=1}^{N} \| \delta_j \| < \frac{1}{\max_{i \in I, j \in \mathbb{N}} \| E(-A_0 - \eta(0))^{-1} D_{ij} \|}. \]

This implies that
\[ \sum_{j=1}^{N} \| \Delta_j \| + \sum_{j=1}^{N} \| \delta_j(0) \| < \frac{1}{\max_{i \in I, j \in \mathbb{N}} \| E(-A_0 - \eta(0))^{-1} D_{ij} \|}. \]

By the definition of \( r_+ \) and Theorem 5.11, we derive that the perturbed systems (64) are exponentially stable. Therefore,
\[ r_+(A_0, \eta) \geq \frac{1}{\max_{i \in I, j \in \mathbb{N}} \| E(-A_0 - \eta(0))^{-1} D_{ij} \|}. \]

On the other hand, by Lemma 3.4(iii) and Theorem 5.11, we have
\[ r_+(A_0, \eta) \leq \frac{1}{\max_{i \in I, j \in \mathbb{N}} \| E(-A_0 - \eta(0))^{-1} D_{ij} \|}. \]

We thus get (72). Next, it is not difficult to show that
\[ r_\mathbb{C}(A_0, \eta) = \frac{1}{\max_{i \in I, j \in \mathbb{N}} \{ \max_{\Re s \geq 0} \| E(s I_n - A_0 - \int_{-h}^{0} e^{\Re s \theta} d[\eta(\theta))]^{-1} D_{ij} \| \}}. \] (73)

Since \( A_0 \) is a Metzler matrix and \( \eta \) is increasing, it follows from Theorem 2.1(iv) that \( \mu(A_0 + \int_{-h}^{0} e^{\Re s \theta} d[\eta(\theta)]) < \mu(A_0 + \eta(0)) < 0, \forall s \in \mathbb{C}, \Re s \geq 0. \) Then by a similar argument as in the proof of Theorem 3.7, we get
\[ \max_{\Re s \geq 0} \| E \left( s I_n - A_0 - \int_{-h}^{0} e^{\Re s \theta} d[\eta(\theta)] \right)^{-1} D_{ij} \| = \| E(-A_0 - \eta(0))^{-1} D_{ij} \|. \] (74)

Therefore,
\[ r_\mathbb{C}(A_0, \eta) = \frac{1}{\max_{i \in I, j \in \mathbb{N}} \| E(-A_0 - \eta(0))^{-1} D_{ij} \|}. \] (75)

Combining (71), (72), (75) and the inequalities \( r_\mathbb{C}(A_0, \eta) \leq r_\mathbb{R}(A_0, \eta) \leq r_+(A_0, \eta) \), we conclude that
\[ r_C(A_0, \eta) = r_R(A_0, \eta) = r_+(A_0, \eta) = r_C = r_R = r_+ = \frac{1}{\max_{i,j \in \mathbb{N}} \|E(-A_0 - \eta(0))^{-1}D_{ij}\|}. \]  

(76)

In particular, we get back the main result of [29].

By the same way, we can get the formula for the stability radii of the linear functional differential equation (58)–(59) under affine parameter perturbations which were given in [29].

6. Examples

Example 6.1. Consider the positive linear delay system

\[ \dot{x}(t) = A_0x(t) + A_1x(t - 1), \quad t \geq 0, \]

where

\[ A_0 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \]

Since \( A_0 \) is a Metzler matrix and \( A_1 \) is a nonnegative matrix, it follows from Theorem 2.1(iv) that \( \mu(A_0 + zA_1) \leq \mu(A_0 + A_1) < 0 \) for every \( z \in \mathbb{C}, \|z\| \leq 1 \). Hence \( \det(sI_2 - A_0 - zA_1) \neq 0 \) for every \( (s, z) \in \mathbb{C}, \Re s \geq 0, \|z\| \leq 1 \). Therefore the above delay system is strongly delay-independently stable. Suppose the system matrices \( A_0, A_1 \) are subjected to parameter perturbations of the form

\[ A_0 \rightarrow A_0 + D_{01}\Delta_{01}E + D_{02}\Delta_{02}E, \]

\[ A_1 \rightarrow A_1 + D_{11}\Delta_{11}E + D_{12}\Delta_{12}E, \]

where

\[ D_{01} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D_{02} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad E = I_2, \]

\[ \Delta_{01}, \Delta_{11} \in \mathbb{K}^{1 \times 2}, \quad \Delta_{02}, \Delta_{12} \in \mathbb{K}^{2 \times 2}, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}. \]

Then we get

\[ G_{01}(0, 1) = (-A_0 - A_1)^{-1}D_{01} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]

\[ G_{02}(0, 1) = (-A_0 - A_1)^{-1}D_{02} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}. \]

\[ G_{11}(0, 1) = (-A_0 - A_1)^{-1}D_{11} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

\[ G_{12}(0, 1) = (-A_0 - A_1)^{-1}D_{12} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}. \]

Therefore, if \( \mathbb{R}^2 \) is endowed, respectively, with the 1-norm, 2-norm, \( \infty \)-norm, then by (57) \( r_+ = r_C = r_R = r_+ = \frac{1}{\sqrt{9 + \sqrt{80}}} \frac{1}{3} \), respectively.
Example 6.2. Consider a positive linear time-delay system described by the following scalar equation:

\[
\dot{x}(t) = -x(t) + \int_{-1}^{0} e^\theta x(t + \theta) \, d\theta, \quad t \geq 0, \quad x(t) \in \mathbb{R}.
\]  

(77)

The system (77) can be represented by the form (58)–(59), with \( \eta = e^\theta - e^{-1} \). Clearly, \( \eta \in NBV([-1, 0], \mathbb{R}) \) and it is increasing on \([-1, 0]\). Using Theorem 5.11, it is easy to see that the system (77) is exponentially stable. Assume the system (77) is perturbed as follows

\[
\dot{x}(t) = (-1 + \delta_0)x(t) + \int_{-1}^{0} (e^\theta + 2005\Delta_1(\theta) + 2006\Delta_2(\theta))x(t + \theta) \, d\theta,
\]  

(78)

where \( \delta_0 \in \mathbb{R} \) is an unknown parameter scalar and \( \Delta_1(\cdot), \Delta_2(\cdot) \) are unknown integrable functions on \([-1, 0]\). This perturbed system can be rewritten in the form

\[
\dot{x}(t) = (-1 + \delta_0)x(t) + \int_{-1}^{0} d[\eta(\theta) + 2005\delta_1(\theta) + 2006\delta_2(\theta)]x(t + \theta),
\]  

(79)

where

\[
\delta_1(\theta) = \int_{-1}^{\theta} \Delta_1(\tau) \, d\tau, \quad \delta_2(\theta) = \int_{-1}^{\theta} \Delta_2(\tau) \, d\tau, \quad \theta \in [-1, 0].
\]

By (76), we conclude that the perturbed system (78) is exponentially stable for all \( \delta_0 \in \mathbb{R} \), \( \Delta_1(\cdot), \Delta_2(\cdot) \in L_1([-1, 0], \mathbb{R}) \) satisfying

\[
|\delta_0| + V_0(-1)(\delta_1) + V_0(-1)(\delta_2) = |\delta_0| + \int_{-1}^{0} |\Delta_1(\theta)| \, d\theta + \int_{-1}^{0} |\Delta_2(\theta)| \, d\theta < \frac{1}{2006e}.
\]

On the other hand, if we take the perturbation \( \delta_0 = 0, \Delta_1(\theta) = 0, \Delta_2(\theta) = \frac{1-e^\theta}{2006}, \theta \in [-1, 0] \), then the perturbed system becomes

\[
\dot{x}(t) = -x(t) + \int_{-1}^{0} x(t + \theta) \, d\theta, \quad t \geq 0, \quad x(t) \in \mathbb{R},
\]

for which the characteristic quasi-polynomial is

\[
H(s) = s + 1 - \int_{-1}^{0} e^{s\theta} \, d\theta.
\]

Since, clearly, \( H(0) = 0 \) it follows that the perturbed system is not exponentially stable. Note that

\[
|\delta_0| + V_0(-1)(\delta_1) + V_0(-1)(\delta_2) = \int_{-1}^{0} |\Delta_1(\theta)| \, d\theta = \frac{1}{2006e}.
\]
References