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Coherent and squeezed states in black-hole evaporation

A.N.St.J. Farley *, P.D. D'Eath

Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom

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Abstract

In earlier Letters, we adopted a complex approach to quantum processes in the formation and evaporation of black holes. Taking Feynman's $+i\epsilon$ prescription, rather than one of the more usual approaches, we calculated the quantum amplitude (not just the probability density) for final weak-field configurations following gravitational collapse to a black hole with subsequent evaporation. What we have done is to find quantum amplitudes relating to a pure state at late times following black-hole matter collapse. Such pure states are then shown to be susceptible to a description in terms of coherent and squeezed states—in practice, this description is not very different from that for the well-known highly-squeezed final state of the relic radiation background in inflationary cosmology. The simplest such collapse model involves Einstein gravity with a massless scalar field. The Feynman approach involves making the boundary-value problem for gravity and a massless scalar field well-posed. To define this, let T be the proper-time separation, measured at spatial infinity, between two space-like hypersurfaces on which initial (collapse) and final (evaporation) data are posed. Then, in this approach, one rotates $T \to |T| \exp(-i\delta)$ into the lower half-plane. In an adiabatic approximation, the resulting quantum amplitude may be expressed in terms of generalised coherent states of the quantum oscillator, and a physical interpretation is given. A squeezed-state representation, as above, then follows.

1. Introduction

We begin by describing Feynman's $+i\epsilon$ approach [1] in the context of black-hole evaporation. In [2–12], this treatment was described and applied to the calculation of quantum amplitudes (not just probabilities) for particle production, following gravitational collapse to a black hole. Suppose, for definiteness, that one's Lagrangian contains Einstein gravity coupled to a real massless scalar field. Asymptotically-flat initial data are posed on an initial space-like hypersurface Σ_I, and final data on a surface Σ_F, separated from Σ_I by a (large) real Lorentzian time-interval T, as measured at spatial infinity. Suppose further, for simplicity, that the initial data on Σ_I are spherically symmetric, corresponding to a diffuse slowly-moving initial matter distribution. The final data for gravity + scalar are taken to have a 'background' spherically-symmetric part, plus small non-spherical perturbations, which correspond to gravitons and massless-scalar particles.

Following Feynman's $+i\epsilon$ procedure [1], one rotates the time-interval T into the complex: $T \to |T| \exp(-i\delta)$, with $0 < \delta \leqslant \pi/2$. The classical boundary-value problem, for a complex 4-metric $g_{\mu\nu}$ and scalar field ϕ given the above data on Σ_I, Σ_F, is then expected to be well-posed, unlike the ill-posed case $\delta = 0$ (or equivalently T real) [3,13,14]. One can evaluate the second-variation classical action $S^{(2)}_{\text{class}}$ as a functional of the (still real) boundary data and as a function of the complex variable T. One then computes the corresponding semi-classical quantum amplitude, proportional to $\exp(iS^{(2)}_{\text{class}})$, and can also include loop corrections, if appropriate. Finally, the Lorentzian quantum amplitude for black-hole evaporation (again, not just the probability density) is recovered by taking the limit as $\delta \to 0_+$.

In this Letter, we study such black-hole evaporation amplitudes, which were constructed in detail in [5–8,10–12], but now in the context of coherent states [15], which resemble 'classical states', and of squeezed states [16], which are purely quantum-

* Corresponding author.
E-mail address: a.n.s.j.farley@damtp.cam.ac.uk (A.N.St.J. Farley).

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mechanical. Although our motivation originated with the question of black-hole radiation, there are also strong connections between this work and the study of the relic cosmic microwave background radiation (CMBR) induced by inflationary cosmological perturbations.

In inflationary cosmology, the field modes are in their adiabatic ground state, with short wavelengths near the start of inflation. Due to the accelerated expansion of the universe during inflation, quantum fluctuations are amplified into macroscopic or classical perturbations. The early-time fluctuations lead to the formation of large-scale structure in the universe, and also contribute to the anisotropies in the CMBR. The final state for the perturbations is a two-mode highly-squeezed state for modes whose radius is much greater than the Hubble radius [17], pairs of field quanta being produced at late times with opposite momenta. Tensor ($s = 2$) fluctuations in the metric, for example, are predicted to give rise to relic gravitational waves. By comparison, electromagnetic waves ($s = 1$) cannot be squeezed in the same way.

In either case, cosmological or black hole, one works within an adiabatic approximation for the perturbative modes. Writing k for a typical perturbative frequency, one requires $k \gg H$ in the cosmological case, where $H = (\dot{a}/a)$ and $a(t)$ is the scale factor. In the black-hole case, the space–time geometry at late times, in the region containing a stream of outgoing radiation, is given by a Vaidya metric [4,8,18,19] with a slowly-varying 'mass function' $m(t, r)$. The adiabatic condition then reads $k \gg |\dot{m}/m|$.

In applying the squeezed-state formalism, one finds, in the case of cosmological perturbations, that these evolve essentially according to a set of Schrödinger equations [20]. Such perturbations, whether of density, rotational or gravitational type, starting in an initial vacuum state, are transformed into a highly-squeezed vacuum state, with many particles, having a large variance in their amplitude (particle number), but small (squeezed) phase variations. The squeezing of cosmological perturbations may be suppressed at small wavelengths, but it should be present at long wavelengths, especially for gravitational waves [21]. These perturbations also induce the anisotropies at large angular scales, as observed in the CMBR. Their wavelengths today are comparable with or greater than the Hubble radius. The above amplification of the initial zero-point fluctuations gives rise to standing waves with a fixed phase, rather than traveling waves. The relic perturbations in the high-squeezing or WKB limit can be described as a stochastic collection of standing waves. Although this paragraph has reviewed the application to cosmology, a similar picture emerges in the application to black-hole evaporation.

Section 2 outlines the main features of the above complex approach to the calculation of quantum amplitudes (not just probabilities) for data (spins $s = 0, 1, 2$) prescribed on a late-time final hypersurface Σ_F. This requires a rotation: $T \to |T| \exp(-i\delta)$ into the lower half-plane. The resulting amplitudes are then related to coherent and squeezed states. Sections 3–5 describe coherent states, generalised coherent states and squeezed states, respectively. In Section 6, the small angle δ (above), through which the time T at infinity is rotated into the complex, is related to the large amount of squeezing which has been applied to give the final state. Section 7 contains a brief conclusion.

2. The quantum amplitude for late-time data

Consider first the case of a rotation into the complex of the time-interval T, measured at spatial infinity, by a moderately small angle δ, as above. One expects that the resulting classical solution $(g_{\mu\nu}, \phi)$ of the coupled Einstein/massless-scalar field equations is slightly complexified, by comparison with a Lorentzian-signature solution. By suitable choice of coordinates (t, r, θ, φ), the spherically-symmetric 'background' part of the metric may be written in the form [2,6]

$$ds^2 = -e^b\,dt^2 + e^a\,dr^2 + r^2\big(d\theta^2 + \sin^2\theta\,d\varphi^2\big), \tag{2.1}$$

where $b = b(t, r)$, $a = a(t, r)$, and the spherically-symmetric 'background' part Φ of the scalar field has the form $\Phi = \Phi(t, r)$. The coupled Lorentzian-signature Einstein/scalar field equations for this spherically-symmetric configuration are given by the analytic continuation of the Riemannian field equations (3.7)–(3.11) of [5], on making the replacement

$$t = \tau\exp(-i\vartheta), \tag{2.2}$$

where τ is the 'Riemannian time-coordinate' of [5], and where the real number ϑ should be rotated from 0 to $\pi/2$.

Small non-spherical perturbations in the boundary data given on the final late-time hypersurface Σ_F consist of the perturbed part of the intrinsic 3-dimensional spatial metric h_{ijF} on Σ_F, together with the perturbations in the scalar field ϕ_F on Σ_F. As above, these correspond to gravitons and to massless-scalar particles, propagating on the spherically-symmetric classical background $(g_{\mu\nu}, \Phi)$. For example, the linearised scalar perturbations $\phi^{(1)}$, given [2] by $\phi = \Phi + \phi^{(1)}$, may be first decomposed as in Eq. (6) of [2], namely as:

$$\phi^{(1)}(t, r, \theta, \varphi) = \frac{1}{r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{m=\ell} Y_{\ell m}(\Omega)R_{\ell m}(t, r). \tag{2.3}$$

Here, $Y_{\ell m}(\Omega)$ denotes the (ℓ, m) spherical harmonic of [22]. The scalar field equation decouples for each (ℓ, m), leading to the mode equation

$$\begin{aligned}&\big(e^{(b-a)/2}\partial_r\big)^2 R_{\ell m} - (\partial_t)^2 R_{\ell m}\\ &\quad - \frac{1}{2}\big(\partial_t(a-b)\big)(\partial_t R_{\ell m}) - V_\ell(t, r)R_{\ell m} = 0,\end{aligned} \tag{2.4}$$

where

$$V_\ell(t, r) = \frac{e^{b(t,r)}}{r^2}\left(\ell(\ell+1) + \frac{2m(t, r)}{r}\right) \tag{2.5}$$

is real and positive in the Lorentzian-signature case. The 'mass function' $m(t, r)$, which would equal the constant mass M for an exact Schwarzschild geometry [23], is defined by

$$e^{-a(t,r)} = 1 - \frac{2m(t, r)}{r}. \tag{2.6}$$

An analogous harmonic decomposition can be given for weak gravitational-wave perturbations about the spherical background [10].

In most regions of the classical space–time, except for the central region where the black hole is formed, the metric functions $a(t,r)$ and $b(t,r)$ vary 'slowly' or 'adiabatically'. In this case, one can consider a radial mode solution for (say) a perturbed scalar field, of the form [6]

$$R_{\ell m}(t,r) \sim \exp(ikt)\xi_{k\ell m}(t,r), \tag{2.7}$$

where $\xi_{k\ell m}(t,r)$ varies 'slowly' with respect to t. This will occur near spatial infinity, and it will also occur, provided that the time-interval T is sufficiently large, in a neighbourhood of the final hypersurface Σ_F. The mode equation (2.4), (2.5) then reduces [6] to

$$e^{(b-a)/2}\frac{\partial}{\partial r}\left(e^{(b-a)/2}\frac{\partial \xi_{k\ell m}}{\partial r}\right) + \left(k^2 - V_\ell\right)\xi_{k\ell m} = 0. \tag{2.8}$$

The spherically-symmetric background metric in this region can be represented to high accuracy by a Vaidya metric [8,18,19], which describes the (on average) spherically-symmetric outflow of massless matter. The principal condition for the validity of the adiabatic expansion is [6] that

$$|k| \gg |\dot{m}/m|. \tag{2.9}$$

In analysing the behaviour of the radial mode equation (2.8), it is natural to define a generalisation r^* of the standard Regge–Wheeler or 'tortoise' coordinate r^*_S for the Schwarzschild geometry [23], according to

$$\frac{\partial}{\partial r^*} = e^{(b-a)/2}\frac{\partial}{\partial r}. \tag{2.10}$$

The approximate (adiabatic) mode equation (2.8) then reads

$$\frac{\partial^2 \xi_{k\ell m}}{\partial r^{*2}} + \left(k^2 - V_\ell\right)\xi_{k\ell m} = 0. \tag{2.11}$$

We consider here, for definiteness, a set of suitable radial functions $\{\xi_{k\ell m}(r)\}$ on the final surface Σ_F, since it is here that the non-trivial boundary data are posed. Since the mode equation (2.11) does not depend on the quantum number m, we may choose $\xi_{k\ell m}(r) = \xi_{k\ell}(r)$, independently of m. The boundary condition of regularity at the spatial origin $\{r = 0\}$ [6] implies that

$$\xi_{k\ell}(r) = \text{const} \times (kr)^{\ell+1} + O\left((kr)^{\ell+3}\right) \tag{2.12}$$

as $r \to 0_+$. For the boundary condition on the $\xi_{k\ell}(r)$ as $r \to \infty$, note that the potential $V_\ell(r)$ decreases sufficiently rapidly, as $r \to \infty$, that a real solution to Eq. (2.11) behaves near $\{r = \infty\}$ according to

$$\xi_{k\ell}(r) \sim \left(z_{k\ell}\exp\left(ikr^*_S\right) + z^*_{k\ell}\exp\left(-ikr^*_S\right)\right). \tag{2.13}$$

Here, the $z_{k\ell}$ are certain dimensionless complex coefficients, which must be determined by using the differential equation (2.11) together with the regularity conditions. Further [6], there is a natural normalisation of the basis $\{\xi_{k\ell}(r)\}$ of radial wavefunctions.

We continue, for purposes of exposition, to study the case of scalar perturbations, with a slightly complexified time-interval at infinity, $T = |T|\exp(-i\delta)$, for $0 < \delta \leqslant \pi/2$. The relevant boundary data for anisotropic perturbations $\phi^{(1)}$ of the scalar field ϕ_F on Σ_F can be described [6] by expanding out the interior classical boundary-value solution near Σ_F in the form

$$\phi^{(1)} = \frac{1}{r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty} dk\, a_{k\ell m}\xi_{k\ell}(t,r)\frac{\sin(kt)}{\sin(kT)}Y_{\ell m}(\Omega). \tag{2.14}$$

Here, the real quantities $\{a_{k\ell m}\}$ characterise the final data.

More generally, for perturbative boundary data for a field of any spin, posed on Σ_F in describing a final state resulting from black-hole evaporation, we denote by $\{a_{sk\ell mP}\}$ a set of analogous 'Fourier-like' coefficients, where s gives the particle spin, k the frequency, (ℓ, m) the angular quantum numbers, and $P = \pm 1$ the parity (for $s \neq 0$). For massless perturbations of spins $s = 0, 1, 2$ [2,3,5–7,10,11], we found that the quantum amplitude or wave functional is of semi-classical form, being given by

$$\Psi\left[\{a_{sk\ell mP}\}; T\right] = N\exp\left(iS_{\text{class}}\left[\{a_{sk\ell mP}\}; T\right]\right), \tag{2.15}$$

where the pre-factor N depends only on T. Here, S_{class} denotes the (second-variation) action of the classical infilling solution, as a functional of the boundary data. For simplicity, we denote the collection $a_{sk\ell mP}$ of indices by j. Further, we write M_I for the total (time-independent) ADM (Arnowitt–Deser–Misner) mass of the 'space–time', as measured at spatial infinity [23]. The ADM mass M_I, which is the limit at large radius of the variable mass $m(t,r)$ of the Vaidya metric, is a functional of the final field configurations $\{a_j\}$ on Σ_F, since it depends on the full gravitational field which results from classical solution of the complexified boundary-value problem.

As was found (for example) in the scalar case $s = 0$ in [2,3,6], the classical action is dominated by contributions from frequencies k with the values

$$k = k_n = \frac{n\pi}{T}, \quad n = 1, 2, 3, \ldots. \tag{2.16}$$

We also define Δk_j to be the spacing between neighbouring k_j-values:

$$\Delta k_j = \frac{\pi}{T}. \tag{2.17}$$

Following [2,3,5–7,10,11], the classical action functional S_{class} is found to be a sum over individual 'harmonics' labelled by j, which depend on the corresponding indices $\{sk_j\ell mP\}$ through the quantity $|A_j|^2$, defined by

$$\begin{aligned} |A_j|^2 = 2(-1)^s c_s \frac{(\ell-s)!}{(\ell+s)!}|z_j|^2 \\ \times \left|a_j + (-1)^s P a_{s,-k_j\ell mP}\right|^2. \end{aligned} \tag{2.18}$$

Here, the coefficients c_s for bosonic spins s are given by $c_0 = 2\pi$, $c_1 = 1/4$, $c_2 = 1/8$. The quantities z_j are the complex numbers appearing in Eq. (2.13), which arise in solving the adiabatic radial mode equation (2.11). This leads to the form of the quantum amplitude:

$$\Psi\big[\{A_j\}; T\big] = \hat{N} e^{-\frac{1}{2} i M_I T} \prod_j \Psi(A_j; T), \tag{2.19}$$

where $\hat{N}$ also depends only on T.

Taking the classical action S_{class} in the form found in [6] for the scalar $s = 0$ case (for example), one deduces that the wave functional for given boundary data can be written:

$$\Psi\big[\{A_j\}; T\big] = \hat{N} e^{-\frac{1}{2} i M_I T} \prod_j \frac{1}{2i \sin(k_j T)} \times \exp\left[\frac{i}{2}(\Delta k_j) k_j |A_j|^2 \cot(k_j T)\right]. \tag{2.20}$$

This will be related to the coherent-state description in the following Section 3.

3. Coherent states

It is possible to rewrite the quantum amplitude (2.20) with the help of the Laguerre polynomials [24]. First, we introduce the associated Laguerre polynomials $L_n^{(m-n)}(x)$, defined by

$$L_n^{(m-n)}(x) = \sum_{p=0}^{n} \binom{m}{n-p} \frac{(-x)^p}{p!} \tag{3.1}$$

for $m \geqslant n \geqslant 0$. The Laguerre polynomials $L_n(x)$ [24] are given by

$$L_n(x) = L_n^{(0)}(x). \tag{3.2}$$

The set $\{L_n(x)\}$ obeys the completeness relation

$$\sum_{n=0}^{\infty} e^{-(x/2)} L_n(x) e^{-(y/2)} L_n(y) = \delta(x, y). \tag{3.3}$$

Writing $z = x + iy$, consider now the function $L_n(|z|^2)$, which appears in Eq. (3.5) below. For $n > 0$, this cannot be written as a product of two (decoupled) wave functions of x and y in an excited state, due to pair correlations [25]. But, in terms of Hermite polynomials $H_p(x)$ [24], one can write

$$L_n\big(x^2 + y^2\big) = \frac{(-1)^n}{2^{2n} n!} \sum_{p=0}^{n} \binom{n}{p} H_{2p}(x) H_{2n-2p}(y). \tag{3.4}$$

From this, one can further decompose the quantum amplitude (2.20) as

$$\Psi\big[\{A_j\}; T\big] = \hat{N} e^{-\frac{1}{2} i M_I T} e^{-\Sigma_j (\Delta k_j) k_j |A_j|^2/2} \times \prod_j \sum_{n=0}^{\infty} e^{-2i E_n T} L_n\big[k_j(\Delta k_j)|A_j|^2\big], \tag{3.5}$$

where $E_n = (n + \frac{1}{2}) k_j$ is the quantum energy of the linear harmonic oscillator. Note also the dependence of the quantum amplitude on $|A_j|$—it is spherically symmetric.

The Schrödinger-picture wave functions

$$\Psi_{nj}(x_j, T) = \frac{N}{\pi} e^{-(x_j/2)} e^{-2i E_n T} L_n(x_j) \tag{3.6}$$

appear in the wave-function (3.5), with $x_j = k_j(\Delta k_j)|A_j|^2$. The wave functions (3.6) have a strong connection with the exact solution of the forced-harmonic-oscillator problem [26], with Hamiltonian

$$H = \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 q^2 + q F(t), \tag{3.7}$$

where $F(t)$ denotes an external force, μ the oscillator mass and ω the oscillator frequency. Assume that $F(t) = 0$ for $t < t_0$ and for $t > T$, so that the asymptotic states, at early and late times t, are free-oscillator states. One can calculate the amplitude A_{km} to make a transition from the free-oscillator state $|m\rangle$ (with m particles) at early times $t < t_0$, to the free-oscillator state $|k\rangle$ at late times $t > T$. Define the 'Fourier transform' of the force:

$$\beta = \int_{t_0}^{T} dt\, F(t) e^{-i\omega t}, \tag{3.8}$$

and set

$$z = \frac{|\beta|^2}{2\mu\omega}. \tag{3.9}$$

It has been shown [27–29], in the case $m \geqslant k$, that

$$A_{km} = e^{i\lambda} e^{-(z/2)} \left(\frac{k!}{m!}\right)^{\frac{1}{2}} \left(\frac{i\beta}{\sqrt{2\mu\omega}}\right)^{m-k} L_k^{(m-k)}(z), \tag{3.10}$$

where λ is a real phase. This expression also gives A_{km} for $m \leqslant k$, since $A_{km} = A_{mk}$ is symmetric.

In the adiabatic limit, in which the force $F(t)$ changes extremely slowly, one has $z \ll 1$, and from general considerations a state which begins as $|k\rangle$ must end up in the same state $|k\rangle$ after the time-dependent force has been removed. From Eq. (3.10), one has

$$A_{kk} = e^{i\lambda} e^{-(z/2)} L_k(z). \tag{3.11}$$

The corresponding probability that there should be no change in the number of particles is $|A_{kk}|^2 = e^{-z}[L_k(z)]^2$. Apart from the introduction of mode labels j denoting the 'quantum numbers' $\{sk\ell mP\}$, together with a necessary re-interpretation for z, these amplitudes are effectively the wave functions (3.5) derived from our boundary-value problem.

One further viewpoint can be brought to bear on Eq. (3.10), arising from the coherent-state representation. Coherent states $|\alpha\rangle$ can be regarded as displaced vacuum states; that is [15]

$$|\alpha\rangle = D(\alpha)|0\rangle, \tag{3.12}$$

where

$$D(\alpha) = \exp\big(\alpha a^\dagger - \alpha^* a\big) \tag{3.13}$$

is a unitary displacement operator, obeying

$$D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha), \tag{3.14}$$

and where the states $|\alpha\rangle$ are eigenstates of the annihilation operator a with complex eigenvalue α. Among quantum states for the harmonic oscillator, they are the closest to classical states, in that they attain the minimum demanded by the uncertainty principle. Coherent states form an over-complete set, and are not orthogonal. In terms of the Fock-number eigenstates

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \tag{3.15}$$

one has [29]

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{3.16}$$

The coherent state labelled by $\alpha = 0$ is the ground state of the oscillator. If, for example, the system started in a vacuum state, the amplitude to find it subsequently in a coherent state $|\alpha\rangle$ is

$$\langle 0|\alpha\rangle = \langle 0|D(\alpha)|0\rangle = e^{-|\alpha|^2/2}, \tag{3.17}$$

up to a phase.

To make complete contact with the amplitude (3.10), using coherent-state methods, we note that, in terms of the displacement operators $D(\xi)$:

$$\langle m|D(\xi)|\alpha\rangle = \frac{1}{\sqrt{m!}}(\xi+\alpha)^m \exp\left[-\frac{1}{2}\left(|\alpha|^2+|\xi|^2+2\xi^*\alpha\right)\right], \tag{3.18}$$

and

$$\langle m|D(\xi)|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle m|D(\xi)|n\rangle. \tag{3.19}$$

On equating these, one finds that

$$(1+y)^m e^{-y|\xi|^2} = e^{|\xi|^2/2}\sum_{n=0}^{\infty}\sqrt{\frac{m!}{n!}}\xi^{n-m}y^n\langle m|D(\xi)|n\rangle. \tag{3.20}$$

But, from the generating function for the associated Laguerre polynomials [25],

$$(1+y)^m e^{-yx} = \sum_{n=0}^{\infty}L_n^{(m-n)}(x)y^n, \quad |y|<1, \tag{3.21}$$

one deduces that the matrix element between initial and final states is

$$\langle m|D(\xi)|n\rangle = \left(\frac{n!}{m!}\right)^{\frac{1}{2}}\xi^{m-n}e^{-|\xi|^2/2}L_n^{(m-n)}\left(|\xi|^2\right), \tag{3.22}$$

which agrees with Eq. (3.10), up to an unimportant phase factor.

4. Generalised coherent states

These amplitudes can also be interpreted in terms of generalised coherent states of the harmonic oscillator [28]. Define:

$$|n,\alpha\rangle = e^{-iE_n t}D\left(\alpha(t)\right)|n\rangle. \tag{4.1}$$

Then, in the Fock representation,

$$|n,\alpha\rangle = \sum_{m=0}^{\infty}\langle m|D\left(\alpha(0)\right)|n\rangle|m\rangle e^{-iE_m t}. \tag{4.2}$$

For generalised coherent states, the ground state ($n = 0$) is a coherent state and not a vacuum state. Generalised coherent states are to the coherent states what the Fock states $|n\rangle$ are to the vacuum state, that is, excited coherent states. In addition, denoting by I the identity operator, one finds that [27]:

$$I = \frac{1}{\pi}\int d^2\alpha\, |n,\alpha\rangle\langle n,\alpha|, \tag{4.3}$$

$$\langle n,\beta|n,\alpha\rangle = L_n\left(|\alpha-\beta|^2\right)e^{\beta^*\alpha-\frac{1}{2}(|\alpha|^2+|\beta|^2)}, \tag{4.4}$$

$$\langle n,\beta|\psi\rangle = \frac{e^{-|\beta|^2/2}}{\pi}\int d^2\alpha\, L_n\left(|\alpha-\beta|^2\right)e^{\beta^*\alpha}e^{-|\alpha|^2/2}\langle n,\alpha|\psi\rangle, \tag{4.5}$$

for an arbitrary state $|\psi\rangle$, with the definition:

$$\int d^2\alpha = \int d\left[\mathrm{Re}(\alpha)\right]d\left[\mathrm{Im}(\alpha)\right]. \tag{4.6}$$

In particular, from Eq. (4.4) with $\beta = 0$, one has

$$\langle n,0|n,\alpha\rangle \equiv \langle n|n,\alpha\rangle = e^{-|\alpha|^2/2}L_n\left(|\alpha|^2\right), \tag{4.7}$$

again giving Eq. (3.6) up to a phase. The initial state should be seen not as a vacuum state, but as a Fock state, while the final state should be seen as a generalised coherent state.

As shown by Hollenhorst [30], the amplitudes of Eq. (3.22) have yet a further interpretation: they are the matrix elements for a transition from state $|k\rangle$ to state $|m\rangle$ under the influence of a linearised gravitational wave, with the force $F(t)$ proportional to the Riemann curvature-tensor component $R_{txtx}(t)$:

$$F(t) = \mu\ell R_{xtxt}(t) = -\frac{1}{2}\mu\ell(\partial_t)^2 h_{xx}^{TT}, \tag{4.8}$$

where ℓ is the distance between two particles along the x-axis, each being of mass $(\mu/2)$, while h_{xx}^{TT} is the transverse-traceless gravitational-wave component of the metric [23], and x is the change in the separation of the masses.

In the context of black-hole evaporation, one expects that the role of the force is played by the time-dependent background space–time—which approximates a Vaidya space–time in the high-frequency limit at late times [4,8,18,19].

An important point which we should mention is that, under the influence of a time-dependent force, an initial vacuum state transforms into a coherent state. Below, we discuss how, by changing a phase parameter of the perturbations appearing in their frequencies (parametric amplification), an initial vacuum state transforms into a squeezed vacuum state. This phase is not an oscillator phase, but a small angle, δ, through which the time T at infinity is rotated into the lower complex plane.

5. Squeezed-state formalism

In this section and in the following Section 6, we shall see how, by rotating the asymptotic Lorentzian time T into the complex plane, and in the case of spherically-symmetric initial matter and gravitational fields, one obtains a quantum-mechanical highly-squeezed-state interpretation for the final state in black-hole evaporation, in the limit of an infinitesimal rotation angle.

Grishchuk and Sidorov [17] were the first to formulate particle creation in strong gravitational fields explicitly in terms of squeezed states, although the formalism does appear in Parker's original paper on cosmological particle production [31]. In [17], it was shown that relic gravitons (as well as other perturbations), created from zero-point quantum fluctuations as the universe evolves, should now be in a strongly squeezed state. Squeezing is just the quantum process corresponding to parametric amplification.

Black-hole radiation in the squeezed-state representation was first discussed in [17]. The 'squeeze parameter' r_j (see below) was there related to the frequency ω_j and the black-hole mass M through

$$\tanh(r_j) = \exp(-4\pi M\omega_j). \tag{5.1}$$

In this language, the vacuum quantum state in a black-hole space–time for each mode is a two-mode squeezed vacuum. However, our approach to squeezed states in black-hole evaporation is new; arising from a two-surface boundary-value problem and Feynman's $+i\epsilon$ prescription [1]. We now give a brief account of quantum-mechanical squeezed states.

A general one-mode squeezed state (or squeezed coherent state) is defined [16] as

$$|\alpha, z\rangle = D(\gamma)S(r,\phi)|0\rangle = D(\gamma)S(z)|0\rangle. \tag{5.2}$$

Here, $D(\gamma)$ is the single-mode displacement operator, and

$$S(r,\phi) \equiv S(z) = \exp\left(\frac{1}{2}\left(za^2 - z^* a^{\dagger 2}\right)\right) \tag{5.3}$$

in terms of annihilation and creation operators a and $a^\dagger$, respectively, together with the relation

$$z = re^{-2i\phi}, \tag{5.4}$$

gives the unitary squeezing operator for $|\alpha, z\rangle$, obeying

$$S^\dagger(z)S(z) = S(z)S^\dagger(z) = 1, \tag{5.5}$$

with γ given by

$$\gamma = \alpha\cosh r + \alpha^* e^{-2i\phi}\sinh r. \tag{5.6}$$

The state Eq. (5.2) is a Gaussian wave-packet, displaced from the origin in position and momentum space. While the (real) squeezing parameter r ($0 \leqslant r < \infty$) determines the magnitude of the squeezing, the squeezing angle ϕ ($|\phi| < \pi/2$) gives the distribution of the squeezing between conjugate variables. The squeezed vacuum state occurs when $\alpha = 0$:

$$|z\rangle \equiv |0, z\rangle = S(z)|0\rangle. \tag{5.7}$$

The limit of high squeezing occurs when $r \gg 1$, where the state $|z\rangle$ is highly localised in momentum space.

Single-mode squeezed operators do not conserve momentum, since they describe the creation of particle pairs with momentum k. Two-mode squeezed operators, however, describe the creation and annihilation of two particles (waves) with equal and opposite momenta. A two-mode squeeze operator has the form [32]

$$S(r,\phi) = \exp\left[r\left(e^{-2i\phi}a_+a_- - e^{2i\phi}a_+^\dagger a_-^\dagger\right)\right], \tag{5.8}$$

where $a_\pm$ and $a_\pm^\dagger$ are annihilation and creation operators for the two modes, respectively.

Consider two conjugate operators $\hat{p}$ and $\hat{q}$, with variances $\Delta\hat{p}$ and $\Delta\hat{q}$. In the squeezed-state formalism, one may construct states such that $\Delta\hat{p}$ and $\Delta\hat{q}$ are equal, taking the minimum value possible. The name 'squeezed' refers to the fact that the variance of one variable in a conjugate pair can go below the minimum allowed by the uncertainty principle (the squeezed variable), while the variance of the conjugate variable can exceed the minimum value allowed (the superfluctuant variable) [25,33,34]. The superfluctuant variable is amplified by the squeezing process, and so becomes possible to observe macroscopically, while the subfluctuant variable is squeezed and becomes unobservable. In particle production, whether by black holes or in cosmology, the number operator is a superfluctuant variable, while the phase is squeezed.

6. Analytic continuation and the large-squeezing limit

We shall see here for the black-hole evaporation problem that, when one rotates the time-separation T at infinity: $T \to |T|\exp(-i\delta)$ into the complex by a very small angle $\delta > 0$, one arrives at a very highly-squeezed quantum state. There is no information-loss paradox associated with the relic Hawking radiation, as such a state is a pure state. It is also important to state that we do not take the $|T| \to \infty$ limit. However, one must understand that the observation time at infinity by far exceeds the dynamical collapse time-scale, which is of order πM_I [23]. We now repeat Eq. (2.19):

$$\Psi\left[\{A_j\}; T\right] = \hat{N} e^{-\frac{1}{2}iM_I T}\prod_j \Psi(A_j; T), \tag{6.1}$$

and then define

$$\begin{aligned}\Phi&\left[\{A_j\}; T\right]\\ &= N e^{-\frac{1}{2}iM_I T}\prod_j 2i\sin(k_j T)\Psi(A_j; T)\\ &\equiv N e^{-\frac{1}{2}iM_I T}\prod_j \exp\left[\frac{i}{2}(\Delta k_j)k_j|A_j|^2\cot(k_j T)\right]\\ &= N\exp\left(iS^{(2)}_{\text{class}}\left[\{A_j\}; T\right]\right).\end{aligned} \tag{6.2}$$

We further define the functions $\phi_j(|T|,\delta)$ and $r_j(|T|,\delta)$ by

$$\phi_j\left(|T|,\delta\right) = -k_j|T|\cos\delta, \tag{6.3}$$

$$\tanh r_j\left(|T|,\delta\right) = \exp\left(-2k_j|T|\sin\delta\right), \tag{6.4}$$

whence

$$\exp(-2r_j) = \tanh\left(k_j|T|\sin\delta\right). \tag{6.5}$$

From Eqs. (6.3)–(6.5), one can rewrite Eq. (6.2) in the form

$$\begin{aligned}\Phi&\left[\{A_j\}; |T|,\delta\right]\\ &= \hat{N} e^{-\frac{1}{2}iM_I|T|\cos\delta}e^{-\frac{1}{2}M_I|T|\sin\delta}\\ &\quad\times\prod_j \exp\left[-\frac{1}{2}(\Delta k_j)k_j\left(\frac{1+e^{2i\phi_j}\tanh r_j}{1-e^{2i\phi_j}\tanh r_j}\right)|A_j|^2\right].\end{aligned} \tag{6.6}$$

On comparing with Section 5, we recognise Eq. (6.6) as the coordinate-space representation of a quantum-mechanical squeezed state [35,36], with $r_j(|T|,\delta)$ the squeeze parameter and $\phi_j(|T|,\delta)$ the squeeze angle. The evolution of the squeezed state is taken into account by the $|T|$-dependence in r_j and in ϕ_j, which are in general both complicated functions of time.

We now define

$$\epsilon_j = k_j|T|\sin\delta,$$
$$f\left(k_j,\epsilon_j,|T|\right) = 1 + \frac{\sin^2(k_j|T|)}{\sinh^2\epsilon_j}. \tag{6.7}$$

Then

$$\left|\Phi\left[\{A_j\};|T|,\delta\right]\right|^2 = |N|^2 e^{-M_I|T|\sin\delta} \times \prod_j \exp\left[\frac{-\coth\epsilon_j}{f(k_j,\epsilon_j,|T|)}(\Delta k_j)k_j|A_j|^2\right], \tag{6.8}$$

and, from Eqs. (6.5), (6.7):

$$\epsilon_j \simeq e^{-2r_j}, \qquad \epsilon_j \ll 1, \tag{6.9}$$

corresponding to $r_j \gg 1$, which is the limit of high squeezing. We discuss the form of the normalisation in another paper [9].

Eq. (6.8) describes a Gaussian non-stationary process in which the variance is an oscillatory function of time. Rather than dealing with travelling waves, one now has standing bosonic waves, where the amplitudes for left- and right-moving waves are large and almost equal—this is similar to the inflationary-cosmology scenario [17]. One consequence of the high-squeezing behaviour is that the variance for the amplitudes $\{x_j\}$ is large, so that there are large statistical deviations of the observable power spectrum from its expected value. This is just a manifestation of the uncertainty principle.

In the squeezed-state formalism, the high-squeezing limit $r_j \gg 1$ may be regarded as the classical limit. For example, in this sense, in the case of black-hole evaporation, the final state of the remnant particle flux becomes more classical (more WKB) in the limit $\delta \to 0$. In this limit, one can effectively consider the final perturbations as being represented by a classical probability distribution [17,33,37]. As in the inflationary scenario in cosmology, the perturbations on the spherically-symmetric black-hole background space–time, of quantum-mechanical origin, cannot be distinguished from classical stochastic perturbations, without the need for an environment for decoherence. There is also a correspondence between the initial conditions for the perturbations in the black hole and in the cosmological cases. In cosmology, the assumption is that, at some early 'time' just prior to inflation, the modes are in their adiabatic ground state. A similar qualitative statement can be made in the black-hole example, provided that the pre-collapse initial data were diffuse, slowly-moving and spherically symmetric.

One further consequence follows, provided that ϵ_j is small (as above). Then, one finds for the probability distribution Eq. (6.8) that, as $\delta \to 0_+$,

$$\left|\Phi\left[\{A_j\};|T|,\delta\right]\right|^2 \sim |N|^2 \prod_{s\ell m P}\prod_{n=1}^{\infty}\exp\left[-(\Delta\omega_n)\omega_n|A_{sn\ell mP}|^2\right], \tag{6.10}$$

where we have used the approximation $\sinh\epsilon_j \sim \epsilon_j$ for small ϵ_j, and the identities

$$\delta(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{(\epsilon^2+x^2)},$$

and

$$\delta\left[f(x)\right] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|},$$

where x_i are zeros of $f(x)$ and $\omega_n = n\pi/|T|$, $\Delta\omega_n = (\omega_{n+1} - \omega_n)$. We have also used the fact that $k_j \to 0$ and that $k_j|A_j|^2 \to 0$ as $k_j \to 0$. In practice, the product over n should be cut off at some large $n_{\max}$, such that $\omega_{n_{\max}} = M_I$.

Further investigation of the derivation of Eq. (6.10) indicates that, in the limit of high squeezing, the random variable ϕ_j associated with the final state is squeezed to discrete values, independently of the quantum numbers $\{s\ell mP\}$ [9]. Note that it is only the squeeze phases $\{\phi_j\}$ of the (standing-wave) perturbations which are fixed and correlated in the high-squeezing limit.

For comparison, in inflationary cosmology, the oscillation phases of standing waves have fixed values, giving rise to zeros in the power spectrum, which are characteristic of the CMBR. The power spectrum of cosmological perturbations in the present universe is not a smooth function of frequency. The standing-wave pattern, due to squeezing, induces oscillations in the power spectrum. This in turn produces Sakharov oscillations [37,38], due to metric and scalar perturbations in the distribution of higher-order multipoles of the angular correlation function for the temperature anisotropies [21,39] in the CMBR, for all perturbations at a given time whose wavelength is comparable with or greater than the Hubble radius defined for that time. That is, the peaks and troughs of the angular power spectrum have a close relationship with the maxima and minima of the metric power spectrum. For long wavelengths, the power spectrum does become smoother.

7. Conclusion

In this Letter, we have illustrated many aspects of the quantum boundary-value formulation, for linearised bosonic fields (spins $s = 0, 1, 2$) propagating in the space–time of an evaporating black hole. When the Lorentzian proper-time separation T between the initial and final space-like hypersurfaces, as measured at spatial infinity, is deformed into the lower complex T-plane, and when the perturbations are initially weak, one obtains a quantum-mechanical squeezed-state formalism. The large-squeezing limit is equivalent to the WKB limit, corresponding to an infinitesimal angle $\delta \ll 1$ of rotation of T into the lower-half complex plane.

Since the final squeezed state is a pure state, there is no information-loss paradox as a result of the Feynman $+i\epsilon$ prescription we have adopted. Our complex approach is new and differs from Grishchuk's original application of squeezed states to black holes. However, as in the cosmological scenario, so the bosonic perturbations on the black-hole background can be regarded as a stochastic collection of standing waves, rather than as traveling waves, in the high-squeezing limit. This leads to the prediction of peaks in the power spectrum of the relic black-hole radiation, analogous to the Sakharov oscillations in the CMBR.

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