Journal of Combinatorial Theory, Series B **80**, 172–177 (2000)
doi:10.1006/jctb.2000.1980, available online at http://www.idealibrary.com on IDEAL®

Unordered Canonical Ramsey Numbers

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Received September 20, 1999

We define a weak form of canonical colouring, based on that of P. Erdős and R. Rado (1950, *J. London Math. Soc.* **25**, 249–255). This yields a class of unordered canonical Ramsey numbers $CR(s, t)$, again related to the canonical Ramsey [illegible]
known corresponding bounds for $ER(2; s)$. © 2000 Academic Press

Key Words: Ramsey theory; canonical colouring.

Erdős and Rado extended Ramsey's original theory of monochromatic graphs [6] to allow infinitely many colours [4]. This extension created the theory of canonical colourings. The Erdős–Rado theorem assumes that there is a pre-existing ordering on the vertices of the graph. It states that, for sufficiently large n, an arbitrary colouring of the edges of K_n contains some K_s which is coloured according to one of the following methods (where $c(e)$ is the colour of the edge e and uv is an edge with $u < v$):

- K_s is monochromatic;
- $c(uv) = f(u)$, where f injective, i.e., minimum-coloured;
- $c(uv) = f(v)$, where f injective, i.e., maximum-coloured; or
- $c(uv) = f(uv)$, where f injective, i.e., K_s is *distinctly edge coloured.*

Let $ER(2; s)$ be the smallest n for which the above holds. The best bounds known for $ER(2; s)$ are those given by Lefmann and Rödl in [5], $2^{c_1 s^2} \leqslant ER(2; s) \leqslant 2^{c_2 s^2 \log s}$ for some constants c_1, c_2. When considering k-uniform hypergraphs, the best upper bounds for $ER(k; s)$ for general fixed k are due to Shelah [7]. The upper bounds obtained by Shelah are towers of height k, the same height as the towers of the lower bounds of the corresponding Ramsey numbers, and therefore in a sense these bounds are best possible.

0095-8956/00 $35.00

Here we present a weaker definition of canonical colourings, which does not require a fixed ordering on the vertices of our graph. In our definition, we combine the first three categories into a single type. Take some $G = K_s$ for arbitrary s. Colour the edges of G with arbitrary colours. We call such a colouring *orderable* if we can order $V(G)$ such that the colour of each edge is completely determined by the smaller of its two vertices. That is, if $c(e)$ denotes the colour of the edge e, and uv is an edge with $u < v$, then $c(uv) = f(u)$. Note that this definition does not require that f be injective, and hence monochromatic graphs are orderable. Also note that the minimum- and maximum-colourings specified by Erdős and Rado are orderable, as we can select either the given ordering of the vertices (to satisfy a minimum-colouring) or reverse ordering (to satisfy a maximum-colouring).

Let $CR(s, t)$ be the smallest N such that every colouring of the edges of K_N contains either an orderable K_s or a distinctly edge coloured K_t. Certainly $CR(s, t) \leqslant ER(2; \max(s, t))$, and so in particular $CR(s, t)$ exists. In this paper we prove both an upper bound (in Theorem 0.1) and lower bound (Theorem 0.2) for $CR(s, t)$, having the same order of magnitude.

The notion of orderable colourings was stimulated originally by the proof of Ramsey's theorem given by Erdős and Szekeres [2]. Restricting ourselves to two colours, we have an analogue of the standard Ramsey numbers, rather than the Erdős–Rado numbers. Denote by $\rho(n)$ the smallest k such that any 2-colouring of K_k must contain an orderable K_n. Now a monochromatic K_n is orderable, and conversely any 2-coloured orderable K_{2n-2} must contain a monochromatic K_n. (This is why we do not require f to be injective in our definition of orderable graphs.) Thus $\rho(n) \leqslant r(n) \leqslant \rho(2n-2)$, and so the standard bounds on $r(n)$ [2, 3] show that $\rho(n)$ grows exponentially in n. Moreover, the argument of Erdős and Szekeres [2] shows $\rho(n+1) \leqslant 2\rho(n)$, and since it is easily checked that $\rho(3) = 3$ and $\rho(4) = 6$ we obtain the upper bound $\rho(n) \leqslant 3 \cdot 2^{n-3}$.

We now return to our main purpose, dealing with arbitrarily many colours and thereby obtaining bounds on $CR(s, t)$. We begin with an upper bound.

THEOREM 0.1. $CR(s, t) \leqslant 7^{3-s} t^{4s-4}$.

Proof. Let $n = CR(s, t) - 1$. Then there exists some colouring G of K_n such that G contains no orderable K_s or distinctly edge coloured K_t. Hence every $K_t \subset G$ contains a pair of like-coloured edges. We say that a pair of edges of the same colour which lie in the same K_t *block* that K_t (as they prevent it from being distinctly edge coloured). As each K_t is blocked by at least two edges, some edge e blocks at least $2\binom{n}{t}/\binom{n}{2}$ distinct K_t.

Two disjoint edges of like colour block $\binom{n-4}{t-4}$ distinct K_t, while two adjacent edges of like colour block $\binom{n-3}{t-3}$ distinct K_t. If we denote by k the number of edges incident with e of the same colour as e, and by l the number of other edges of the same colour as e, then

$$l\binom{n-4}{t-4}+k\binom{n-3}{t-3}\geqslant 2\frac{\binom{n}{t}}{\binom{n}{2}}$$

So there must be a large number of edges of the same colour as e. Note, however, that G is also free of orderable K_s graphs, and this restricts the number of edges of any single colour. Let $c=CR(s-1,t)$. If some vertex $v\in G$ had degree at least c in one colour, then either G contains a K_t distinctly edge coloured, or the neighbourhood of v in that colour contains an orderable K_{s-1}, and hence G contains an orderable K_s. Hence each vertex incident with e has at most $c-2$ edges of the same colour as e besides e itself. Hence $k\leqslant 2c-4$.

Similarly, none of the $n-2$ vertices disjoint from e can have c incident edges of the same colour as e. The maximum number of edges coloured similarly to e but disjoint from e is therefore $\frac{1}{2}(c-1)(n-2)$. Hence

$$\binom{n-4}{t-4}\left(\frac{n-2}{2}\right)(c-1)+(2c-4)\binom{n-3}{t-3}\geqslant 2\frac{\binom{n}{t}}{\binom{n}{2}}$$

So, writing $t_{(4)}$ for $t(t-1)(t-2)(t-3)$, we have

$$\frac{n-2}{2(n-3)}(c-1)+\frac{2}{t-3}(c-2)\geqslant\frac{4(n-2)}{t_{(4)}}.$$

Note that $CR(s,t)=n+1\geqslant c$. The above inequality therefore holds whenever

$$\frac{c}{2}+\frac{3}{2(n-3)}+\frac{2}{t-3}(c-2)\geqslant\frac{4(n-2)}{t_{(4)}}$$

and so

$$CR(s,t)\leqslant 3+\frac{t_{(4)}}{8}\left(c+\frac{3}{CR(s,t)-4}\right)+\frac{t_{(3)}(c-2)}{2}$$
$$\leqslant\frac{t^4}{8}\left(c+\frac{3}{CR(s,t)-4}\right)$$

whenever $n\geqslant 4$, $t\geqslant 4$.

If $CR(s, t) < 10$, then as $t \geqslant 2$ $CR(s, t) \leqslant 3t^4/2$. Otherwise, if $CR(s-1, t) < 6$, then from the equation above we have that

$$CR(s, t) \leqslant \frac{t^4}{8}\left(5 + \frac{3}{CR(s, t) - 4}\right),$$

and as $CR(s, t) \geqslant 10$ this means $CR(s, t) \leqslant 3t^4/2$. If neither of these bounds holds, then we instead bound $CR(s, t)$ above by

$$\frac{t^4}{8}\left(CR(s-1, t) + \frac{1}{2}\right),$$

so

$$CR(s, t) \leqslant \frac{t^4}{7}\, CR(s-1, t).$$

Note also that $CR(2, t) = 2$ for every $t \in \mathbf{N}$ (as any single edge is an orderable K_2). Hence we can obtain an upper bound for any $CR(s, t)$ by using the third bound,

$$CR(s, t) \leqslant \frac{t^4}{7}\, CR(s-1, t),$$

at most $s-2$ times in succession, until either $CR(s, t) < 10$ or $CR(s-1, t) < 6$, both of which then yield a fixed bound of at most $3t^4/2$.

So, for all $s \geqslant 2$, $t \geqslant 4$,

$$CR(s, t) \leqslant \frac{3t^4}{2}\frac{t^{4s-8}}{7^{s-2}} = \frac{3}{2}\, 7^{2-s} t^{4s-4} \leqslant 7^{3-s} t^{4s-4}. \quad \blacksquare$$

We now consider lower bounds for $CR(s, t)$. Like the original Ramsey numbers we can obtain a lower bound for $CR(s, t)$ probabilistically. Random colourings using $\binom{t}{2} - 1$ colours give $CR(s, t) \geqslant (\binom{t}{2} - 1)^{(s-1)(s-2)/2s}$. But, unlike the original Ramsey numbers, for the unordered canonical Ramsey numbers $CR(s, t)$ we can give a lower bound by a simple construction, which is better than the probabilistic bound.

Theorem 0.2. $CR(s, t) \geqslant (\binom{t}{2} - 1)(CR(s-1, t) - 1) + 1$.

Proof. Let G be a 2-coloured graph of order $CR(s-1, t) - 1$. Take $\binom{t}{2} - 1$ copies of G, labelled

$$G_0, ..., G_{\binom{t}{2}-2}.$$

Let c_k, $0 \leqslant k \leqslant \binom{t}{2} - 2$ be distinct colours not used in G. Colour all edges between any pair of copies G_i and G_j with colour c_l, where $l \equiv i + j \bmod (\binom{t}{2} - 1)$.

As a result, the colours connecting each G_j to the $\binom{t}{2}-2$ other copies of G are distinct for every j. Call the resultant complete graph H. We claim that H contains no distinctly edge coloured K_t, and no orderable K_s.

Suppose that there exists $H' \subset H$, distinctly edge coloured and of order t. We know that $H' \not\subset G_k$ for every $1 \leqslant k \leqslant \binom{t}{2}-1$, as each G_k is free of distinctly edge coloured K_ts. Alternatively, if no two vertices of H' lie in the same $G_k \subset H$, then all edges in H' have colours c_k for some $0 \leqslant k \leqslant \binom{t}{2}-2$, and hence at least two edges in H' must have the same colour. So there must be some $G_k \subset H$ such that $H' \not\subset G_k$, but $|H' \cap G_k| \geqslant 2$. Then there exists G_l, $l \neq k$, such that at least one vertex of H' lies in G_l. But then H' contains two edges between G_k and G_l, which must be of the same colour. We therefore know that there is no distinctly edge coloured K_t inside H.

We now show that H contains no orderable K_s. For suppose such a K_s exists. G contains no orderable K_{s-1}, hence any orderable $K_s \subset H$ must contain either (i) vertices from three different copies of G, or (ii) at least two vertices from each of two distinct copies of G. In case (i) the three vertices from distinct copies will form a triangle in which each edge is a distinct colour, which prevents the K_s from being orderable. In case (ii), regardless of how we order the vertices, the first vertex v_1 has at least one neighbour in the same copy of G, and one neighbour in a different copy of G. These two vertices must be connected to v_1, by edges of different colours, ensuring that G is not orderable. Hence $CR(s, t) \geqslant |H| + 1 = (\binom{t}{2}-1)(CR(s-1, t)-1)+1$. ∎

As $CR(2, t) = 2$ for every $t \geqslant 2$, we can use Theorem 0.2 recursively to obtain the following general lower bound.

Corollary 0.1. $CR(s, t) \geqslant (\binom{t}{2}-1)^{s-2}+1$.

Further, a theorem of Babai [1] gives us a way to improve the exponent in this lower bound, but in a non-constructive way. Babai proved that there exists an edge-coloured complete graph of order $\Theta(t^3/\log t)$ in which no two edges of the same colour are adjacent, but which does not contain any distinctly edge-coloured K_t. So by using the nesting argument of Theorem 0.2 with graphs of order $\Theta(t^3/\log t)$, rather than order $\binom{t}{2}-1$ as used in Theorem 0.2, we obtain a graph of order $\Theta(t^{3s}/(\log t)^s)$. This graph is free of distinctly edge coloured K_t by [1], and is a so free of orderable graphs of order s, hence $CR(s, t) = \Omega(t^{3s}/(\log t)^s)$.

ACKNOWLEDGMENTS

I thank my Ph.D. supervisor, Dr. Andrew Thomason, for his invaluable guidance during the development of this paper. I also thank the referees for their useful comments and suggestions.

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