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NOTE

A Note on "Oscillation and Nonoscillation for Second-Order Linear Differential Equations"

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In a recent paper Chunchao Huang [1] studied the oscillatory behavior of the second-order linear differential equation

$$u'' = -p(t)u, \tag{1}$$

where $p(t) \in C[0, \infty)$ and $p(t) \geq 0$.

Let $\alpha_0 = 3 - 2\sqrt{2}$, and let N be the set of natural numbers; then Huang obtained the following theorems:

THEOREM 1. *If there exists $t_0 > 0$ such that for every $n \in N$,*

$$\int_{2^n t_0}^{2^{n+1} t_0} p(t)\, dt \leq \frac{\alpha_0}{2^{n+1} t_0}, \tag{2}$$

then Eq. (1) *is nonoscillatory.*

THEOREM 2. *If there exists $t_0 > 0$ and $\alpha > \alpha_0$ such that for every $n \in N$,*

$$\int_{2^n t_0}^{2^{n+1} t_0} p(t)\, dt \geq \frac{\alpha}{2^n t_0}, \tag{3}$$

then Eq. (1) *is oscillatory.*

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Huang also stated a typical result about the oscillatory properties of Eq. (1) as follows.

THEOREM A [2]. *If*

$$\limsup_{t\to\infty} t\int_t^{\infty} p(s)\,ds < \tfrac{1}{4}, \tag{4}$$

then Eq. (1) *is nonoscillatory, but if*

$$\liminf_{t\to\infty} t\int_t^{\infty} p(s)\,ds > \tfrac{1}{4}, \tag{5}$$

then Eq. (1) *is oscillatory.*

Huang claimed, "It is obvious that (4) and (5) are conditions on the integrals of $p(s)$ in $[t,\infty)$ for arbitrarily large values of t, while (2) and (3) are conditions only concerning the integrals of $p(s)$ in $[2^n t_0, 2^{n+1} t_0]$ for every $n \in N$, therefore they are different kinds of conditions."

However, after careful consideration of Huang's Theorems 1 and 2 and the well-known Theorem A, we find Theorem 1 and Theorem 2 are only special cases of the following Theorem 3, and Theorem 3 is also a special case of Theorem A.

THEOREM 3. *If there exists $t_0 > 0$ such that for every $n \in N$,*

$$\int_{t_0/\varepsilon}^{t_0/\varepsilon^{n+1}} p(t)\,dt \le \frac{\alpha(\varepsilon)}{t_0/\varepsilon^{n+1} - t_0/\varepsilon} = \frac{\alpha(\varepsilon)\varepsilon^{n+1}}{(1-\varepsilon)t_0}, \tag{6}$$

where $0 < \varepsilon < 1$, $\alpha(\varepsilon) = (1-\sqrt{\varepsilon})^2$, then Eq. (1) *is nonoscillatory. If there exists $t_0 > 0$ and $\alpha > \alpha(\varepsilon)$ such that for every $n \in N$,*

$$\int_{t_0/\varepsilon^{n}}^{t_0/\varepsilon^{n+1}} p(t)\,dt \ge \frac{\alpha(\varepsilon)}{t_0/\varepsilon^{n+1} - t_0/\varepsilon} = \frac{\alpha(\varepsilon)\varepsilon^{n+1}}{(1-\varepsilon)t_0}, \tag{7}$$

then Eq. (1) *is oscillatory.*

If we take $\varepsilon = \frac{1}{2}$, $\alpha = \alpha_0/2 = \frac{1}{2}(3-2\sqrt{2})$ in Theorem 3, then we get Theorems 1 and 2.

The proof of Theorem 3 can be performed by a similar method used in [1]. We prove only that Theorem 3 is a special case of Theorem A.

In fact, let $n = n_0, n_0 + 1, \ldots,$ and add both sides of (6) and (7):

$$\int_{t_0/\varepsilon^{n_0}}^{\infty} p(t)\,dt = \sum_{n=n_0}^{\infty} \int_{t_0/\varepsilon^{n}}^{t_0/\varepsilon^{n+1}} p(t)\,dt \leq \frac{\left(1-\sqrt{\varepsilon}\right)^2}{(1-\varepsilon)t_0} \sum_{n=n_0}^{\infty} \varepsilon^{n+1} = \frac{\left(1-\sqrt{\varepsilon}\right)^2 \varepsilon^{n_0+1}}{(1-\varepsilon)^2 t_0} \tag{8}$$

$$\int_{t_0/\varepsilon^{n_0}}^{\infty} p(t)\,dt = \sum_{n=n_0}^{\infty} \int_{t_0/\varepsilon^{n}}^{t_0/\varepsilon^{n+1}} p(t)\,dt \geq \frac{\alpha}{(1-\varepsilon)t_0} \sum_{n=n_0}^{\infty} \varepsilon^{n+1} = \frac{\alpha \varepsilon^{n_0}}{(1-\varepsilon)^2 t_0}. \tag{9}$$

Then (8) and (9) can be written as

$$\frac{t_0}{\varepsilon^{n_0}} \int_{t_0/\varepsilon^{n_0}}^{\infty} p(t)\,dt \leq \left(\frac{1-\sqrt{\varepsilon}}{1-\varepsilon}\right) \varepsilon, \tag{8'}$$

$$\frac{t_0}{\varepsilon^{n_0}} \int_{t_0/\varepsilon^{n_0}}^{\infty} p(t)\,dt \geq \frac{\alpha}{(1-\varepsilon)^2}. \tag{9'}$$

Let

$$f(x) = \left(\frac{1-x}{1-x^2}\right)^2 x^2, \qquad 0 < x < 1,$$

$$g(x) = \frac{\alpha}{(1-x)^2}, \qquad 0 < x < 1.$$

Then

$$f(x) = \left(\frac{x}{1+x}\right)^2; \text{ hence } 0 < f(x) < \frac{1}{4} \quad \text{for } 0 < x < 1,$$

and

$$0 < \left(\frac{1-\sqrt{\varepsilon}}{1-\varepsilon}\right)^2 \varepsilon = f\left(\sqrt{\varepsilon}\right) < \frac{1}{4} \quad \text{for } 0 < \varepsilon < 1, \tag{10}$$

and

$$g(x) > \frac{\alpha(x)}{(1-x^2)^2} = \frac{(1-x)^2}{(1-x)^2} = \frac{1}{(1+x)^2} > \frac{1}{4},$$
$$\frac{\alpha}{(1-\varepsilon)^2} > \frac{\alpha(\varepsilon)}{(1-\varepsilon)^2} = \frac{(1-\sqrt{\varepsilon})^2}{(1-\varepsilon)^2} > \frac{1}{4}. \tag{11}$$

From (8)–(11) and letting $n_0 \to \infty$, $T_0 = t_0/\varepsilon^{n_0}$, we obtain

$$\overline{\lim_{t\to\infty}}\, T_0 \int_{T_0}^{\infty} p(t) < \tfrac{1}{4} \tag{8''}$$

or

$$\overline{\lim_{t\to\infty}}\, T_0 \int_{T_0}^{\infty} p(t) > \tfrac{1}{4} \tag{9''}$$

From Theorem A, we obtain the results of Theorem 3. Hence, Theorems 1 and 2 are also special cases of Theorem A.

REFERENCES

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2. C. A. Swanson, ''Comparison and Oscillatory Theory of Linear Differential Equations,'' Academic Press, New York, 1968.