Color-character of uncolorable cubic graphs

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Let \( G = (V, E) \) be a cubic graph with chromatic index 4 and \( c : E \to \{0, 1, 2, 3\} \) a proper
4-edge-coloring of \( G \). Let \( E_i = \{e \in E \mid c(e) = i\} \) and \( \sigma(c) = \min \{ |E_i| \mid i = 0, 1, 2, 3 \} \). If \( \mathcal{C}(G) \) denotes all the proper 4-edge-colorings of \( G \), then \( m(G) = \min_{c \in \mathcal{C}(G)} \sigma(c) \) is defined
to be the color-character of \( G \). In this work, we prove that \( m(G) \) is a constant under some
operations, and give a relation between \( m(G) \) and another parameter of \( G \).

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1. Introduction

In this work, by a graph we mean a finite graph with multiple edges and without loops. The chromatic index of a graph \( G \),
denoted by \( \chi'(G) \), is the smallest integer \( k \) for which a proper \( k \)-edge-coloring exists and a \( k \)-coloring is defined to be a proper
\( k \)-edge-coloring. It is well known that the chromatic index of a cubic graph is either 3 or 4. Graphs with chromatic index 4 are called uncolorable graphs while the others are called colorable. Uncolorable graphs are of great interest, since there are
counterexamples to some hard conjectures, e.g. Tutte’s 5-flow Conjecture, the Cycle Double-Cover Conjecture (CDCC) (see
e.g. [1,2]); then they must be uncolorable. These famous conjectures are obstructed by uncolorable cubic graphs. Moreover,
the Four-Color Theorem is equivalent to the statement that every bridgeless cubic planar graph is colorable (see e.g. [3,4]).
Motivated by these researches we focus our attention on the study of uncolorable cubic graphs.

We consider a parameter: the color-character of uncolorable cubic graphs. Let \( G \) be an uncolorable cubic graph. Suppose \( c \) is a 4-coloring of \( G \) and \( E_i = \{e \in E \mid c(e) = i\} \). If \( \sigma(c) = \min \{ |E_i| \mid i = 0, 1, 2, 3 \} \) and define \( m(G) = \min_{c \in \mathcal{C}(G)} \sigma(c) \) to be the color-character of \( G \), where \( \mathcal{C}(G) \) consists of all the 4-colorings of \( G \). If \( G \) is a cubic graph with \( \chi'(G) = 3 \), we define \( m(G) = 0 \). If \( c \in \mathcal{C}(G) \) is said to be a character-coloring of \( G \) if \( \sigma(c) = m(G) \). Intuitively, \( m(G) \) is the
minimal number of edges which have to be removed from \( G \) to obtain a graph with chromatic index 3.

Steffen proved some properties of the color-character of uncolorable cubic graphs in [5] (in [5], \( m(G) \) is called color number and denoted by \( c(G) \)), and discussed the relationship between the color-character and oddness of uncolorable cubic graphs in [6].

In this work, we prove that \( m(G) \) is an invariant under some operations of \( G \) and give a relation between \( m(G) \) and another parameter of \( G \). Note that color-character provides a way to classify uncolorable cubic graphs. So it enables us to
study those graphs class by class, and especially to verify those famous conjectures.

2. Notation and known results

Let \( G \) be an uncolorable cubic graph. Suppose \( c \) is a character-coloring of \( G \) using colors 0, 1, 2, 3. Without loss of generality, in this work we always assume that \( m(G) = \sigma(c) = |E_0| \). For \( v \in V(G) \), let \( E(v) = \{e \in E(G) \mid v \text{ is an end of } e\} \) and

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c(\(v\)) = \{c(e) \mid e \in E(\(v\))\}. For any e = xy \in E_0, define t(e) = (c(x) \cap c(y)) \setminus \{0\} and \(\overline{t(e)} = c(x) \cup c(y)\), where "\(\cup\)" is the symmetric difference. Since c is a character-coloring, it is easy to see that |\(\overline{t(e)}\)| = 2. Define c(E) = \{c(e) \mid e \in E\}.

We modify the definition of an edge to create a semiedge which is like an edge but has just one end vertex; pairs of semiedges will combined to form edges. A multipole M = (V, E, S) consists of a set of vertices \(V = V(M)\), a set of edges \(E = E(M)\) and a set of semiedges \(S = S(M)\). If \(|S(M)| = k\), then M is also called a k-pole. An n-coloring of M = (V, E, S) is a mapping c from E \(\cup S\) into \{1, 2, \ldots, n\} such that c(\(e_1\)) \(\neq c(\(e_2\)) for all adjacent (semi)edges \(e_1, e_2\). Let \(f_1\) and \(f_2\) be two semiedges of M, and incident with \(v_1\) and \(v_2\) respectively. We say that M' is obtained from M by identifying semiedges \(f_1\) and \(f_2\) if these two semiedges are replaced by an edge \(v_1v_2\).

For a colored graph G and colors x, y we define an \((x, y)\)-path to be a path in G whose edges are colored with x and y alternately.

A triangle T in cubic graph G along with the pendant edges \(e_7, e_8\) and \(e_9\) is called the T-star, which is denoted by S(T) (see Fig. 1(a)). Contracting T into a new vertex \(v_T\), we get a new cubic graph G/T from G (see Fig. 1(b)), where the T-star S(T) becomes the \(x_T\)-star at vertex \(v_T\) which is incident with three new edges \(e_7', e_8'\) and \(e_9'\) and the corresponding operation is called a \(\Delta\)-reduction on T.

Suppose \(P = v_0 \ldots v_k\) is a path in G, where the degree of \(v_1\) and \(v_k\) is 3 and the degree of \(v_1, \ldots, v_{k-1}\) is 2. We delete \(v_1, v_{k-1}\) and add a new edge \(v_0v_k\) to G. Such an operation is called suppressing \(v_1, \ldots, v_{k-1}\). Let G be a cubic graph and suppose Q = \(v_1e_1v_2e_2v_3e_3v_4e_4v_5\) is a square (4-cycle) of G. We construct a new cubic graph by deleting the edges \(e_1, e_3\) and suppressing the two resulting paths of two vertices of degree 2. Such a graph is denoted by G \(\odot\{e_1, e_3\}\), and we call the corresponding operation a \(\odot\)-reduction. We prove that either G \(\odot\{e_1, e_3\}\) or G \(\odot\{e_2, e_4\}\) has the same color-character as G.

Let \(G_n\) be the collection of cubic graphs of order n. Then \(G_n\) can be partitioned into the cubic graphs with chromatic index 3, denoted by \(G_n^{(3)}\), and the cubic graphs with chromatic index 4, denoted by \(G_n^{(4)}\).

Suppose G is a cubic graph in \(G_n\), and \(e_1\) and \(e_2\) are two edges of G. Subdividing \(e_1\) in G by a new vertex \(u_i\) (i = 1, 2) and adding a new edge \(u_1u_2\), we get a new cubic graph G(\(e_1, e_2\)) from G (note that \(e_1 = e_2\) is permitted; in this case \(u_1\) and \(u_2\) will be two subdividing vertices on the same edge \(e_1\)). Since this process is reversible, \(G_{n+2} = \{G(\(e_1, e_2\) \mid e_1, e_2 \in G \in G_n\}\).

It is convenient to regard G(\(e_1, e_2\)) as an operation on G and \(\{e_1, e_2\}\), which is denoted by G(\(e_1, e_2\)) = G | \(\{e_1, e_2\}\).

Let S be a set of some pairs of edges of G, and \(E_S = \bigcup_{\{e_1, e_2\} \in S} \{e_1, e_2\}\). For each edge \(e \in E_S\), let \(n_e\) be the number of pairs of S containing e. Suppose S satisfies the following two conditions:

(i) \(e_1 \neq e_2\) for all \(\{e_1, e_2\} \in S\);
(ii) \(n_e \leq 2\) for all \(e \in E_S\), and there is at most one edge, say \(e_0\), in E_S such that \(n_{e_0} = 2\).

Then we construct G(S) as follows. Subdivide \(e \in E_S \setminus \{e_0\}\) by a new vertex \(u_e\), and subdivide \(e_0\) (if it exists) by two new vertices \(u_{e_0}\) and \(u'_{e_0}\). For each pair \(\{e_1, e_2\} \in S, e_1 \neq e_2\), add a new edge \(u_eu_{e_0}\) to G, and for the two pairs \(\{e_1, e_0\}\) and \(\{e_0, e_1\}\), add two new edges \(u_{e_0}u'_{e_0}\) and \(u_{e_0}u'_{e_0}\) or \(u_{e_0}u'_{e_0}\) and \(u_{e_0}u_{e_0}\) to G. Note that if such \(e_0\) exists, then G(S) are two graphs. Clearly, G(S) \(\in G_{n+2k}\) if G \(\in G_n\) and \(|S| = k\). For G \(\in G_n\), let S be a minimum set satisfying (i), (ii) and G(S) \(\in G_{n+2k}^{(1)}\), for some k. We define \(s(G) = |S|\) (if \(\chi'(G) = 3\), we define \(s(G) = 0\). In Section 4, we prove that \(s(G) = \lceil m(G) \rangle - 2\rangle\). Namely, for any G \(\in G_n^{(2)}\), there exists a set of \(\lceil m(G) \rangle\) pairs of edges of G, say S, that satisfies (i), (ii) and G(S) \(\in G_{n+2x}[m(G)]\). And \(\chi'(G(S))\) will never be 3 if \(|S| < \lceil m(G) \rangle\).

Here are some known results about cubic graphs. These results are famous or will be used in this work. We start with the Parity Lemma of [7], which is a very useful tool for proofs in the field of uncolorable cubic graphs.

Lemma 2.1 ([7]). Let G be a colorable cubic graph that has been 3-colored with colors 1, 2 and 3. If a cutset consisting of n edges contains \(n_i\) edges of color i, for \(i = 1, 2, 3\), then

\(n_1 \equiv n_2 \equiv n_3 \equiv n \mod 2\).
Lemma 2.1. Without loss of generality, assume that \( H_i \neq \emptyset \) if \( i \neq j \), and \( E_0 = H_1 \cup H_2 \cup H_3 \). Lemma 2.2 in [5] is obtained from Lemma 2.1.

Lemma 2.2 ([5]). Let \( G \) be an uncolorable cubic graph, and \( c \) be a character-coloring of \( G \) with \( E_0 = H_1 \cup H_2 \cup H_3 \). Then \( |H_1| = |H_2| = |H_3| = m(G) \) (mod 2).

In [5], Steffen also proved the following two lemmas, which will be used in this work.

Lemma 2.3 ([5]). Let \( e = vw \in E_0 \) and \( t(e) = \{y, z\} \). Then there is a \((y, z)\)-path from \( v \) to \( w \).

Let \( e = vw \) be the edge of Lemma 2.3; then \( e \) together with the \((y, z)\)-path from \( v \) to \( w \) forms a cycle \( C_e \) of odd length in \( G \). We will say that \( e \) admits cycle \( C_e \).

Lemma 2.4 ([5]). Let \( e_1 \) and \( e_2 \) be two different edges of \( E_0 \). Then \( c_{e_1} \) and \( c_{e_2} \) are disjoint.

Lemma 2.5 in [8] shows that the chromatic index of a cubic graph is a constant under \( \sqcup \)-reduction, while Lemma 2.6 in [3] is a result about \( \square \)-reduction.

Lemma 2.6 ([3]). If \( G \) is a cubic graph and \( Q = v_1e_1v_2e_2v_3e_3v_4e_4v_1 \) is a square of \( G \), then \( G \) is 3-colorable if and only if one of \( G \oplus c(e_1, e_3) \) and \( G \oplus c(e_2, e_4) \) is.

In Section 3, we consider the color-character of cubic graphs under \( \sqcup \)-reductions and \( \square \)-reductions. Then in Section 4, we give a relation between \( s(G) \) and \( m(G) \).

3. \( m(G) \) is an invariant under some operations

Theorem 3.1. Let \( G \) be an uncolorable cubic graph, and let \( \{u_1v_1, u_2v_2, u_3v_3\} \) be a 3-edge-cut of \( G \). For \( i = 1, 2, 3 \), replace \( u_iv_i \) by two semiedges \( f_i \) and \( g_i \), which are incident to \( u_i \) and \( v_i \) respectively, and we get two 3-poles \( M_1 \) and \( M_2 \). See Fig. 2. If \( \chi'(M_1) = 3 \), and \( G_2 \) is the cubic graph obtained from \( M_2 \) by joining the three semiedges of \( M_2 \) to a new vertex \( x \), then \( m(G) = m(G_2) \).

Proof. Without loss of generality, assume that \( \{u_1, u_2, u_3\} \subseteq V(M_1) \) (Fig. 2). Let \( c_2 \) be a character-coloring of \( G_2 \) and \( m(G_2) = c(c_2) = |E_0| \). If \( v_i \in c_2(x) \), say \( c_2(xv_1) = 0 \), since \( c_2 \) is a character-coloring, there exists a color \( i \in c_2(v_1) \) and \( i \notin c_2(x) \), say \( c_2(v_1y) = i \). Then let \( P \) be the longest \((0, i)\)-path that contains \( xv_1 \) and \( v_1y \) in \( M_2 \). Interchanging colors 0 and 1 along \( P \) we get a character-coloring of \( G_2 \) such that no edge incident with \( x \) is colored with 0. So we may assume that there is no edge incident with \( x \) that is colored with 0 under \( c_2 \). Lemma 2.1 implies that for any 3-coloring of \( M_1, f_1, f_2 \) and \( f_3 \) must be colored with three distinct colors. Moreover, there is a 3-coloring \( c_1 \) of \( M_1 \) where \( c_1(f_i) = c_2(v_ix), i = 1, 2, 3 \). So we can get a 4-coloring \( c \) of \( G \) from \( c_1 \) and \( c_2 \) such that \( o(c) = o(c_2) \). So \( m(G) \leq m(G_2) \).

Suppose \( c \) is a character-coloring of \( G \) and \( m(G) = c(c) = |E_0| \). Let \( A = (c(u_1v_1), c(u_2v_2), c(u_3v_3)) \). Let \( c_1 \) be a coloring of \( M_1 \) derived from \( c \), where \( c_1(e) = c(e) \) for \( e \in E(M_1) \) and \( c_1(f_i) = c(u_iv_i) \). If no edge and no semiedge of \( M_1 \) are colored with 0 under \( c_1 \), then by Lemma 2.1, \( A = (c_1(f_1), c_1(f_2), c_1(f_3)) = \{1, 2, 3\} \). Then we can naturally derive a 4-coloring \( c_2 \) of \( G_2 \) from \( c_1 \), where \( c_2(e) = c(e) \) for \( e \in E(M_2) \), and \( c_2(v_ix) = c(u_iv_i) \). So \( o(c_2) \leq o(c) \), and thus \( m(G_2) \leq m(G) \). Now suppose that there is at least one edge or one semiedge of \( M_1 \) which is colored with 0 under \( c \). Let \( c_{v_i} = \{c(e) \mid e \text{ is an edge incident with } v_i \text{ in } M_2\} \).

Case 1: \( 0 \notin c_{v_i} \) for \( i = 1, 2, 3 \).
Suppose that $c_{v_1}, c_{v_2}$ and $c_{v_3}$ are distinct sets. Then we can derive a 4-coloring $c_2$ of $G_2$ from $c$, where $c_2(e) = c(e)$ for $e \in E(M_2)$, and $c_2(v_1x) = \{1, 2, 3\} \setminus c_{v_1}$. So $\o(c_2) \leq \o(c)$, and thus $m(G_2) \leq m(G)$. If exactly two of them are alike, then there is a 4-coloring $c_2$ of $G_2$ where $c_2(e) = c(e)$ for $e \in E(M_2)$ and only one edge incident with $x$ is colored with 0. Since there is at least one edge or one semiedge of $M_1$, which is colored with 0 under $c$, $\o(c_2) \leq \o(c)$, and thus $m(G_2) \leq m(G)$. If $c_{v_1} = c_{v_2} = c_{v_3}$, say $c_{v_1} = c_{v_2} = c_{v_3} = \{1, 2\}$, then in $M_2$, there exists a $(1, 3)$-path starting from one vertex of $v_1, v_2$ and $v_3$, but that does not end in $v_1, v_2$ or $v_3$. Interchanging colors 1 and 3 along this path yields the above case. So $m(G_2) \leq m(G)$.

**Case 2:** $0 \in c_{v_3}$, for some $i = 1, 2$ or 3.

Without loss of generality, we assume that $0 \in c_{v_3}$. If there exists a set, say $c_{v_1}$, such that $0 \not\in c_{v_1}$, then we can define a 4-coloring $c_2$ of $G_2$ where $c_2(e) = c(e)$ for $e \in E(M_2)$, $c_2(v_1x) = 0$, $c_2(v_2x) \in \{1, 2, 3\} \setminus c_{v_2}$ and $c_2(v_3x) \in \{0, 1, 2, 3\} \setminus (c_{v_1} \cup c_{v_2}(x))$. Then $\o(c_2) \leq \o(c)$, and thus $m(G_2) \leq m(G)$. Suppose $0 \in c_{v_1}, c_{v_2}$ and $c_{v_3}$. Let $y, z$ be the two neighbors of $v_1$ in $M_2$, and assume $c(v_1y) = 0$. Since $c$ is a character-coloring of $G$, there is an edge, say $yu$ ($u \not\in v_1$), incident with $y$ such that $c(yu) \neq c(v_1z)$ and $c(v_1u)$. Suppose that $c(yu) = 1$. Then let $P$ be the longest $(0, 1)$-path that contains $v_1y$ and $yu$ in $M_2$. Interchanging colors 0 and 1 along this path yields the above case. So $m(G_2) \leq m(G)$. □

**Corollary 3.1.** If $G$ is a cubic graph, $T$ is a triangle in $G$ and $G/T$ is defined above, then $m(G) = m(G/T)$.

**Proof.** If $m(G) = 0$, the corollary holds by Lemma 2.5. If $m(G) > 0$, let $M_1$ be the $T$-star of $G$. Then the proposition follows from Theorem 3.1 directly. □

From Corollary 3.1, we immediately have:

**Corollary 3.2.** Let $G$ be a cubic graph. If $G'$ is obtained from $G$ via a sequence of $\Delta$-reductions, then $m(G') = m(G)$.

Corollary 3.2 implies that the color-character of a cubic graph is a constant under the $\Delta$-reduction. But it is not always true for $\square$-reduction since a square inserted along any two non-adjacent edges of the Petersen graph yields a graph with chromatic index 3. But we have the following weaker theorem.

**Theorem 3.2.** If $G$ is a cubic graph and $Q = v_1e_1v_2e_2v_3e_3v_4e_4v_1$ is a square of $G$, then either $m(G \oplus \{e_1, e_3\}) = m(G)$ or $m(G \oplus \{e_2, e_4\}) = m(G)$.

**Proof.** If $m(G) = 0$, the theorem holds by Lemma 2.6. So assume that $\chi'(G) = 4$. Let $G \oplus \{e_1, e_3\} = G_1$ and $G \oplus \{e_2, e_4\} = G_2$. Note that $v_1$ has only one neighbor which is not in $V(Q)$; denote this neighbor by $u_i$.

We can reconstruct $G$ from $G_1$, and it is easy to see that any proper coloring of $c_1$ of $G_1$ can be extended to a proper coloring of $c$ of $G$ such that $\o(c) = \o(c_1)$. So $m(G) \leq m(G_1)$. Similarly, $m(G) \leq m(G_2)$. Now, suppose $c$ is a character-coloring of $G$. In the following, we prove that there is a 4-coloring of $c'$ of $G_1$ or $G_2$ such that $\o(c') \leq \o(c)$, and thus $m(G_1) \leq m(G)$ or $m(G_2) \leq m(G)$. Let $e'_i = u_iv_i, i = 1, 2, 3, 4$, and $E' = \{e'_1, e'_2, e'_3, e'_4\}$. By Lemma 2.4 we can always assume that $0 \not\in c(E')$. We consider the following three cases.

**Case 1:** $c(e'_1) = c(e'_2) = c(e'_3) = c(e'_4)$.

In this case, the result is trivial.

Let $u'_i$ and $u'_i$ be the two neighbors of $u_i$ other than $v_i$, $E' = \{u'_iu'_i, u'_iu'_i\}$, and $D = \bigcup_{i=1}^{4} E'_i$.

**Case 2:** Exactly three edges of $E'$ have the same color.

Assume, without loss of generality, that $c(e'_1) = 1$ and $c(e'_2) = c(e'_3) = c(e'_4) = 2$. Clearly, in this case, $0 \in c(E(Q))$. We claim that $0 \not\in c(D)$. In fact, if $0 \in E^{1}$ and $c(u_iu'_i) = 0$, we can find a character-coloring $c_0$ of $G$ corresponding to $c$, where $c_0(v_1v_2) = 0, c_0(v_2v_3) = c_0(v_3v_4) = 1, c_0(v_1v_4) = 1, c_0(e) = c(e)$ for other edges of $G$. Under character-coloring $c_0$, cycle $c_{v_1v_2}$ contains vertex $u_1$, which must be contained in cycle $c_{u_1u'_1}$. This is a contradiction to Lemma 2.4. So $0 \not\in c(E')$. Similarly, $0 \not\in c(E^{2})$. And since we can also find a character-coloring of $G$ under which $v_1v_2$ is colored with 0 and $v_1v_2, v_3v_4$ are both colored with 3, $0 \not\in c(E^{3})$ and $0 \not\in c(E^{4})$. So $0 \not\in c(D)$. Then we can get a 4-coloring of $c'$ of $G_1$ where $c(u_1u_2) = 0, c'(u_3u_4) = 2$, and $c'(e) = c(e)$ for each $e \in E(G) \cap E(G_1)$; since $0 \in c(E(Q))$, $\o(c') \leq \o(c)$. (Similarly, such a $c'$ exists for $G_2$.)

**Case 3:** No three edges of $E'$ have the same color.

**Subcase 3.1.** Exactly one pair of edges of $E'$ have the same color.

First, suppose that $c(e'_i) = c(e'_{i+1})$ for some $i \in \{1, 2, 3, 4\}$ (when $i = 4$, let $e_{i+1} = e'_1$). Without loss of generality, assume that $c(e'_1) = c(e'_2) = 1, c(e'_3) = 2$ and $c(e'_4) = 3$. It is easy to see that $0 \in c(E(Q))$. We claim that $0 \not\in c(E^{2}), c(E^{3})$. Otherwise, suppose $0 \in c(E^{3})$ and $c(u'_1u'_2) = 0$. Then we can find a character-coloring $c_0$ of $G$ corresponding to $c$, where $c_0(v_1v_2) = 2, c_0(v_2v_3) = 0, c_0(v_3v_4) = 1, c_0(u_1u_2) = 2$, and $c_0(e) = c(e)$ for other edges of $G$. Under character-coloring $c_0$, cycle $c_{v_1v_2}$ contains vertex $u_1$, which must be contained in cycle $c_{u_1u'_1}$. This is a contradiction to Lemma 2.4. So $0 \not\in c(E^{3})$. Similarly, we can also find a character-coloring of $G$ such that $c_0(v_1v_2) = 2, c_0(v_2v_3) = 3, c_0(v_3v_4) = 1, c_0(v_1v_4) = 0$, and $c_0(e) = c(e)$ for other edges of $G$. So $0 \not\in c(E^{3})$. Then we can derive a 4-coloring of $c'$ of $G_1$, where $c'(u_1u_2) = 1, c'(u_3u_4) = 0, and c'(e) = c(e)$ for each $e \in E(G) \cap E(G_1)$. So $\o(c') \leq \o(c)$. Thus $m(G_1) \leq m(G)$.
Now suppose that $c(e'_i) = c(e'_{i+2})$, $i = 1$ or 2; assume $c(e'_1) = c(e'_3) = 1$, $c(e'_2) = 2$ and $c(e'_4) = 3$. Then it is easy to see that $c_0(v_1v_2) = 0$, $c_0(v_2v_3) = c_0(v_3v_4)$ = 0, $c_0(v_4v_1) = 2$, and $c_0(e) = c(e)$ for other edges in $G$ is a character-coloring of $G$. By Lemma 2.4, this case will not happen.

Subcase 3.2. There are two pairs of edges of $E'$ such that the members of each pair have the same color.

If the two pairs are $(e'_1, e'_2)$ and $(e'_3, e'_4)$ or $(e'_1, e'_3)$ and $(e'_2, e'_4)$, then the result holds trivially. So assume that the two pairs are $(e'_1, e'_3)$ and $(e'_2, e'_4)$. Suppose that $c(e'_1) = c(e'_3) = 1$ and $c(e'_2) = c(e'_4) = 2$. It is easy to see that $c_0(v_1v_2) = c_0(v_3v_4) = 0$, $c_0(v_2v_3) = c_0(v_4v_1) = 3$, and $c_0(e) = c(e)$ for other edges in G is a character-coloring of $G$. As above, we can observe that $4 \not\in c(D)$. So let $c'$ be a coloring of $G$ such that $c'(u_1u_2) = c'(u_3u_4) = 0$, and $c'(e) = c(e)$ for each $e \in E(G) \cap E(G_i)$. Since there are two edges of $E(Q)$ colored 0 under $c$, $o(c') \leq o(c)$. (Similarly, such a $c'$ exists for $G_2$.) This completes the proof of the theorem. □

4. Relation between $s(G)$ and $m(G)$

Let $G$ be an uncolorable cubic graph. We can see from the definition of $m(G)$ that it is hard to ascertain its color-character. So we wish to establish relations between $m(G)$ and other parameters of $G$. In this section, we prove a relation between $s(G)$ and $m(G)$ which is given by Corollary 4.1.

As we introduced in Section 2, let $S$ be a set of some pairs of edges of $G$, and $E_S = \bigcup_{\{e, e'\} \in S} \{e_1, e_2\}$. For each edge $e \in E_S$, let $n_e$ be the number of pairs of $S$ containing $e$. Then we state the following two conditions:

(i) $e_1 \neq e_2$, for all $\{e_1, e_2\} \in S$;
(ii) $n_e \leq 2$ for all $e \in E_S$, and there is at most one edge, say $e_0$, in $E_S$ such that $n_{e_0} = 2$.

Theorem 4.1. Let $G \in \mathcal{G}_n^{(2)}$ and $S$ be a set satisfying conditions (i) and (ii). If $\chi'(G(S)) = 3$ and $|S| = k$, then $m(G) \leq 2k$.

Proof. First, we construct a new cubic graph $G'$ from $G(S)$. Let $e_1 = xy$ and $e_2 = xz$ be two adjacent edges in $E_S = \bigcup_{\{e, e'\} \in S} \{e_1, e_2\}$. Let $u_1, u_2 \in V(G(S))$ be two vertices subdividing $e_1$ and $e_2$, and $xu_1, xu_2 \in E(G(S))$. Again subdivide $xu_1$ and $xu_2$ by $v_1$ and $v_2$ respectively, and add a new edge $v_1v_2$ to $G(S)$. Treat any two adjacent edges in $E_S$ like this, with the resulting graph denoted by $G'$. Then $\chi'(G'(S)) = 3$ by Lemma 2.5. Now, let $\mathcal{P} = \{x = uv \mid u, v \in V(G(S)) \setminus V(G)\}$ and $G_0 = G' - \mathcal{P}$. Let $P = \{P \mid P$ is a path of $G_0$, the degree of its two ends is 3 and the degree of its internal vertices is exactly 2$\}$. By replacing each $P$ in $\mathcal{P}$ with an edge we get a new cubic graph $G''$. Then $G'$ is a graph obtained from $G''$ via a sequence of $\Delta$-reductions. Denote the edge set corresponding to $\mathcal{P}$ by $E_P$. Then $E_P$ is an independent edge set of $G''$ and $|E_P| = |E_P'| \leq 2k$. Let $c'$ be a proper 3-edge-coloring of $G'$ using color 1, 2, and 3. Then we can obtain a proper 4-edge-coloring $c = \{E_0, E_1, E_2, E_3\}$ of $G''$, which derives from $c'$ as follows:

\[ c(e) = \begin{cases} c'(e) & \text{if } e \not\in E_P; \\ 0 & \text{if } e \in E_P. \end{cases} \]

Then $|E_0| = |E_P| \leq 2k$, so $m(G'') \leq 2k$. By Corollary 3.2, $m(G) = m(G'')$. Thus $m(G) \leq 2k$. □

Theorem 4.2. If $G \in \mathcal{G}_n^{(2)}$, then there exists a set $S$ satisfying conditions (i), (ii) and $|S| = \frac{m(G)}{2}$ such that $\chi'(G(S)) = 3$.

Proof. Suppose $c$ is a character-coloring of $G$ and $m(G) = o(c) = |E_0|$. If $H_i$ ($i = 1, 2, 3$) is as defined in Section 2, then $|H_1| = |H_2| = |H_3| = m(G)$ (mod 2) by Lemma 2.2. If $m(G)$ is even, then $|H_1|, |H_2|$ and $|H_3|$ are all even. Let $m(G) = 2n$ and $E_0 = \{e_1, e_2, \ldots, e_{2n}\}$ where $e_i = v_iw_i$ for $i = 1, 2, \ldots, 2n$. For each $i$, subdivide $e_i$ by a new vertex $v_i$ and add a semiedge $f_i$ on $u_i$ if $t(e_i) = x$ and $f_i$ on $u_i$ if $t(e_i) = y$. Color $u_i$ with color 1, and $v_i$ with color 2. Then we get a 3-colored $2n$-pole $M$, and denote this coloring of $M$ by $c'$. Since each of $|H_1|, |H_2|$ and $|H_3|$ is even, we can partition $f_1, \ldots, f_{2n}$ into 3 pairs in an appropriate way such that for each pair $\{f_i, f_j\}$, $c'(f_i) = c'(f_j)$. Then by identifying these pairs, we get a 3-colorable cubic graph. And $S = \{(e_1, e_2) \mid f_i$ and $f_j$ are in some pair$\}$ is the set desired.

If $m(G)$ is odd, then $H_1, H_2$ and $H_3$ are all odd. Let $m(G) = 2n + 1$ and $E_0 = \{e_1, e_2, \ldots, e_{2n+1}\}$ where $e_i = v_iw_i$ for $i = 1, 2, \ldots, 2n + 1$. Assume, without loss of generality, that $t(e_1) = 1, t(e_2) = 2$ and $t(e_3) = 3$. For each $i \geq 2$, subdivide $e_i$ by a new vertex $v_i$ and add a semiedge $f_i$ on $u_i$. Subdivide $e_1$ by two new vertices $u_1$ and $v_1$, and add two semiedges $f_1, f'_1$ on $u_1$, $u'_1$ respectively. Then we get a $(2n + 2)$-pole $M$. For each $i > 2$, if $t(e_i) = x$ and $f_i$ on $u_i$ if $t(e_i) = y$, color $u_i$ with color 1, $v_i$ with color 2. Then we get a 3-colored $2n + 3$-pole $M$. For each $i > 2$, if $t(e_i) = x$ and $f_i$ on $u_i$ if $t(e_i) = y$, color $u_i$ with color 1, $v_i$ with color 2. Then we get a 3-colored cubic graph. By Lemma 2.3, there is a $(2, 3)$-path $P$ from $f_1$ to $f'_2$ in $M$. So we can interchange 2 and 3 on $P$, and get a new 3-coloring $c''$ of $M$; note that $c''(f_1) = 3$ and $c''(f'_2) = 2$. Now by identifying $f_1, f_2$ and $f'_1, f'_2$ we also get a 3-colorable cubic graph. Thus let $S = S' \cup \{(e_1, e_2), (e_1, e_3)\}$, and then $S$ is the set required, which completes the proof. □

Corollary 4.1. If $G$ is an uncolorable cubic graph, then $2s(G) - 1 \leq m(G) \leq 2s(G)$.}
Proof. Let $S$ be a set satisfying conditions (i), (ii), $\chi'(G(S)) = 3$ and $|S| = s(G)$. By Theorem 4.1, $|S| > \lceil \frac{m(G)}{2} \rceil$. If $|S| > \lceil \frac{m(G)}{2} \rceil$, by Theorem 4.2, one can find a set $S'$ such that $\chi'(G(S')) = 3$ and $|S'| = \lceil \frac{m(G)}{2} \rceil$. This is contrary to the definition of $s(G)$. So $|S| = \lceil \frac{m(G)}{2} \rceil$, and thus either $m(G) = 2|S|$ or $m(G) = 2|S| - 1$. □

Remark 1. Let $G$ be a cubic graph and $\chi'(G) = 4$. Then $\chi'(G(e_1, e_2)) = 3$ for some $e_1, e_2 \in E(G)$ if and only if $m(G) = 2$.

References