The Spectrum for Lambda-fold 2-Perfect 6-Cycle Systems

ELIZABETH J. BILLINGTON AND C. C. LINDNER

The spectrum for the decomposition of $\lambda K_v$ into 2 perfect 6-cycle systems is found for all $\lambda > 1$ and for two outstanding cases when $\lambda = 1$, completing work done by Lindner, Phelps and Rodger in the case $\lambda = 1$.

1. INTRODUCTION

Much work has been done in recent years on the decomposition of the complete graph $K_v$ into edge-disjoint copies of a graph $G$. Recent results are surveyed by Rodger [8]. When $G$ is a cycle of length $m$, such a decomposition is called an $m$-cycle system of $K_v$. Recent results on $m$-cycle systems include [2–7]. We denote an $m$-cycle by $(x_0, x_1, \ldots, x_{m-1})$ or sometimes by $x_0x_1 \cdots x_{m-1}$, so that $x_ix_{i+1}$ is an edge for $0 \leq i \leq m-1$ (reducing subscripts modulo $m$). Thus an $m$-cycle system of $K_v$ is essentially an ordered pair $(V, C)$, where $V$ is the vertex set of $K_v$ and $C$ is a set of $m$-cycles which induce a partition of the edge set of $K_v$. So the $m$-cycles in $C$ form an edge-disjoint decomposition of $K_v$.

If $(a, b, c, d, e, f)$ is a 6-cycle or hexagon, we can consider the associated graph obtained from this hexagon by joining all vertices which are distance 2 apart. Thus $(a, c, e)$ and $(b, d, f)$, two triangles, arise from this hexagon in this way. A 2-perfect 6-cycle system of $K_v$ is a 6-cycle system $(V, C)$ of $K_v$ with the additional property that if each hexagon in $C$ is replaced by the two triangles obtained by joining all vertices in the hexagon at distance 2, then the resulting collection of triangles also forms a decomposition of $K_v$, that is, a Steiner triple system (which necessarily contains an even number of blocks). The values of $v$ for which such a decomposition of $K_v$ into 2-perfect 6-cycle systems exists is called the spectrum; this spectrum for 2-perfect 6-cycle systems was shown in [6] to be all positive integers $v \equiv 1$ or 9 (mod 12), $v \neq 9$, except possibly $v \in \{45, 57\}$.

In this paper we shall consider a decomposition of the complete multigraph $\lambda K_v$, into a 2-perfect 6-cycle system, and find the spectrum of such a decomposition. We shall also remove the two possible exceptions in the case $\lambda = 1$ by exhibiting 2-perfect 6-cycle systems of $K_{45}$ and $K_{57}$.

For $\lambda K_v$, to be decomposable into a 2-perfect 6-cycle system, it is clearly necessary that $v \geq 6$ and that the number of edges, $\lambda v(v-1)/2$, should be a multiple of 6. Moreover, the degree of each vertex, $\lambda(v-1)$, must be even. These necessary conditions are summarized in Table 1.

Our main result is as follows.

MAI N T H EORE M. The necessary conditions for the existence of a 2-perfect 6-cycle system of $\lambda K_v$, given in Table 1, are sufficient, with the following exceptions: there is no such system when $(\lambda, v) = (1, 9)$ or when $\lambda = 2$ (mod 4) and $v = 6$.

2. THE CASE $\lambda = 2$

The necessary conditions are that $v \geq 6$ and $v \equiv 0$ or 1 modulo 3. We consider five cases, with $v = 1, 3$ or 4 (mod 6) and $v \equiv 0$ or 6 (mod 12).
E. J. Billington and C. C. Lindner

Necessary conditions for existence of a 2-perfect 6-cycle system of $\lambda K_v$

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2.1. $v = 1 \pmod{6}$

A 2-perfect 6-cycle system of $2K_7$ with vertex set $\mathbb{Z}_7$ is given by the following hexagons:

$$\{(0, 2, 1, 5, 3, 4) + i \mid 0 \leq i \leq 6\}.$$ 

In the case of $2K_{13}$ we may take two copies of a 2-perfect 6-cycle system of $K_{13}$ (when $\lambda = 1$); see [6].

Now let $v = 6n + 1$, $n \geq 3$. The following two constructions define a 2-perfect 6-cycle system of $K_{6n+1}$ on the vertex set $\{\infty\} \cup \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq 6\}$.

**Construction A.** If $n \equiv 0$ or 1 (modulo 3), then it is well known that there exists a 2-fold triple system of order $n$. Take hexagons as follows:

A(i) on $\infty \cup \{(i, j) \mid 1 \leq j \leq 6\}$, for $1 \leq i \leq n$, take a copy of the decomposition given above for $2K_7$;

A(ii) if $\{x, y, z\}$ is a block of the 2-fold triple system of order $n$ that we chose, then on the vertex set $\{(x, j), (y, j), (z, j) \mid 1 \leq j \leq 6\}$ we place one copy of a 2-perfect 6-cycle system of $K_{6,6,6}$ (see [6], Lemma 2.2).

**Construction B.** If $n \equiv 2$ (modulo 3), then we know that there exists a maximum packing of $2K_n$ with edge-disjoint triangles such that the leave consists of one multiple edge (see [1] or [9]). On the vertex set $\{i \mid 1 \leq i \leq n\}$ let this leave consist of the multiple edge joining the vertices 1 and 2. Then hexagons are taken in $2K_{6n+1}$ as follows:

B(i) on $\infty \cup \{(1, j), (2, j) \mid 1 \leq j \leq 6\}$, take a 2-perfect 6-cycle decomposition of $2K_{13}$;

B(ii) on $\infty \cup \{(i, j) \mid 1 \leq j \leq 6\}$, for $3 \leq i \leq n$, take a 2-perfect 6-cycle decomposition of $2K_7$;

B(iii) if $\{x, y, z\}$ is a block of the maximum packing of $2K_n$ described above, then on the vertex set $\{(x, j), (y, j), (z, j) \mid 1 \leq j \leq 6\}$, place one copy of a 2-perfect 6-cycle system on $K_{6,6,6}$ ([6], Lemma 2.2) (here $\{1, 2\} \notin \{x, y, 2\}$).

2.2. $v = 3 \pmod{6}$

In this case we need the concept of a 2-perfect 6-cycle system with a hole. A 2-perfect 6-cycle decomposition of $K_n$ with a hole of size $u$ is a partial decomposition of $K_n$ into hexagons and corresponding pairs of triangles, in which all pairs of vertices are adjacent both in a 6-cycle and in a 3-cycle, except for pairs occurring in a distinguished set of vertices of size $u$. 

**Table 1**

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For example, a decomposition of $2K_9$ with a hole of size 3 into 2-perfect 6-cycles is as follows, where the vertices of the hole are 1, 2, 3:

$\begin{array}{cccc}
4 & 7 & 5 & 9 \\
4 & 7 & 5 & 9 \\
1 & 5 & 4 & 2 \\
1 & 6 & 7 & 3 \\
2 & 6 & 5 & 3 \\
1 & 6 & 5 & 2 \\
\end{array}$

$\begin{array}{cccc}
1 & 4 & 9 & 3 \\
2 & 4 & 6 & 3 \\
1 & 4 & 6 & 2 \\
1 & 5 & 8 & 3 \\
2 & 7 & 9 & 3 \\
1 & 6 & 5 & 2 \\
\end{array}$

When $v = 3 \pmod{6}$ we also need decompositions of $2K_9$ and $2K_{14}$. A decomposition of $2K_9$ is as follows:

$\begin{array}{cccc}
1 & 5 & 2 & 4 \\
4 & 9 & 5 & 8 \\
1 & 3 & 7 & 6 \\
1 & 2 & 5 & 7 \\
2 & 8 & 6 & 4 \\
1 & 3 & 6 & 5 \\
\end{array}$

$\begin{array}{cccc}
6 & 8 & 2 & 7 \\
1 & 2 & 4 & 8 \\
2 & 6 & 5 & 3 \\
1 & 4 & 5 & 3 \\
1 & 4 & 8 & 2 \\
2 & 3 & 4 & 5 \\
\end{array}$

**CONSTRUCTION FOR $2K_{15}$**. We already have a decomposition of $2K_7$. A decomposition of $2K_7$ with a hole $\{A, B, C\}$ and vertex set $\{A, B, C, 1, 2, 3, 4\}$ is given by:

$\begin{array}{cc}
A & 3 \\
A & 1 \\
B & 2 \\
B & 1 \\
\end{array}$

$\begin{array}{cc}
B & 4 \\
B & 3 \\
C & 2 \\
C & 4 \\
\end{array}$

A 2-perfect 6-cycle system on $2K_{6n+3}$ is given by taking 6-cycles as follows. If $n = 0$ or 1 (mod 3), a construction similar to Construction A in Section 2.1 above may be used. We take a 2-fold triple system on $n$ elements, and 6-cycles as follows:

(i) on $\{A, B, C\} \cup \{(i, j) \mid 1 \leq j \leq 6\}$ place a decomposition of $2K_9$ with hole $\{A, B, C\}$;

(ii) on $\{A, B, C\} \cup \{9, 10, 11, 12\}$ take a decomposition of $2K_7$ (with no hole);

(iii) on $2K_{4,4,4}$ with vertex set $\{1, 2, 3, 4\} \cup \{5, 6, 7, 8\} \cup \{9, 10, 11, 12\}$ take two copies of a 2-perfect 6-cycle system ([6], Lemma 2.2).

Now let $v = 6n + 3$ with $n \geq 3$. Take the vertex set

$$\{A, B, C\} \cup \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq 6\}.$$
$2K_{15}$, and on $\{A, B, C\} \cup \{(i, j) \mid 1 \leq j \leq 6\}$, $3 \leq i \leq n$, we place a decomposition of $2K_9$ with a hole $\{A, B, C\}$ of size 3, while repeating (iii) as in Construction B.

2.3. $v = 4 \pmod{6}$

This case is exactly like that of Section 2.2 when $v = 3 \pmod{6}$. In that construction we replace $2K_9$ by $2K_{10}$, $2K_9$ with a hole of size 3 by $2K_{10}$ with a hole of size 4, and $2K_{15}$ by $2K_{16}$.

A decomposition of $2K_{10}$ with a hole $\{0, 1, 2, 3\}$ of size 4 and vertex set $\mathbb{Z}_{10}$ is as follows:

\[
\begin{array}{cccccccc}
4 & 7 & 5 & 8 & 6 & 9 & 0 & 8 \\
0 & 5 & 9 & 1 & 6 & 8 & 0 & 4 \\
0 & 5 & 4 & 2 & 8 & 7 & 0 & 6 \\
0 & 4 & 7 & 3 & 6 & 9 & 1 & 5 \\
1 & 4 & 8 & 3 & 5 & 7 & 1 & 4 \\
1 & 5 & 9 & 3 & 7 & 6 & 2 & 5 \\
2 & 6 & 4 & 3 & 9 & 8 & & \\
\end{array}
\]

A decomposition of $2K_{10}$ with vertex set $\mathbb{Z}_{10}$ is as follows:

\[
\begin{array}{cccccccc}
4 & 7 & 5 & 8 & 6 & 9 & 0 & 8 \\
0 & 5 & 9 & 1 & 6 & 8 & 0 & 4 \\
0 & 5 & 4 & 2 & 8 & 7 & 0 & 6 \\
0 & 4 & 7 & 3 & 6 & 9 & 1 & 5 \\
1 & 4 & 8 & 3 & 5 & 7 & 1 & 4 \\
1 & 5 & 9 & 3 & 7 & 6 & 2 & 5 \\
2 & 6 & 4 & 3 & 9 & 8 & & \\
\end{array}
\]

A decomposition of $2K_{16}$ with vertex set $\mathbb{Z}_{16}$, we take the following hexagons:

\[
\{(0, 4, 2, 12, 8, 6) + i, (8, 4, 10, 12, 0, 14) + i \mid 0 \leq i \leq 3\}
\]

(with addition modulo 16); then 32 more hexagons as follows, again with the addition modulo 16:

\[
\{(0, 14, 1, 8, 6, 9) + i, (0, 15, 2, 8, 7, 10) + i, \\
(0, 11, 3, 8, 7, 15) + i, (0, 12, 3, 8, 4, 11) + i \mid 0 \leq i \leq 7\}.
\]

2.4. $v = 0 \pmod{12}$

Two small cases are needed here. (a) When $v = 12$, let $K_{12}$ have vertex set $\mathbb{Z}_{11} \cup \{\infty\}$. Then

\[
\{(\infty, 3, 4, 1, 0, 9) + i, (0, 4, 1, 8, 3, 9) + i \mid 0 \leq i \leq 10\}
\]

is a 2-perfect 6-cycle system of $2K_{12}$. (b) When $v = 24$, let $K_{24}$ have vertex set $\mathbb{Z}_{23} \cup \{\infty\}$. Then

\[
\{(\infty, 0, 21, 5, 13, 6) + i, (0, 3, 8, 13, 4, 10) + i, \\
(0, 2, 3, 15, 12, 4) + i, (0, 13, 9, 8, 2, 14) + i \mid 0 \leq i \leq 22\}
\]

is a 2-perfect 6-cycle system of $2K_{24}$. 
Now suppose that \( v = 12n \) with \( n \geq 3 \), and let the vertex set of \( 2K_n \) be \( \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq 12\} \). If \( n \equiv 0 \) or \( 1 \) (mod 3), we mimic Construction A of Section 2.1, using a 2-fold triple system of order \( n \). So the hexagons may be taken as follows:

(i) on the set \( \{(i, j) \mid 1 \leq i \leq 12\} \) (for each \( i \)), place a decomposition of \( 2K_{12} \);
(ii) if \( \{x, y, z\} \) is a triple in the chosen 2-fold triple system of order \( n \), then take a decomposition of \( K_{12,12,12} \) (see [6], Lemma 2.2) on the set \( \{(x, j), (y, j), (z, j) \mid 1 \leq j \leq 12\} \).

If \( n \equiv 2 \) (mod 3), we mimic Construction B of Section 2.1, using a maximum packing of \( 2K_n \) by triangles on the vertex set \( \{i \mid 1 \leq i \leq n\} \), with leave \( \{1, 2\} \): place a decomposition of \( 2K_{24} \) on \( \{(1, j), (2, j) \mid 1 \leq j \leq 12\} \); a decomposition of \( 2K_{12} \) on \( \{(i, j) \mid 1 \leq j \leq 12\} \) for \( 3 \leq j \leq n \); and a decomposition of \( K_{12,12,12} \) on \( \{(x, j), (y, j), (z, j) \mid 1 \leq j \leq 12\} \) whenever \( \{x, y, z\} \) is a triple of the chosen maximum packing of \( 2K_n \) with triangles and with leave \( \{1, 2\} \).

2.5. \( v \equiv 6 \) (mod 12)

First we consider the case \( v = 6 \), and general \( \lambda \), not merely \( \lambda = 2 \). The obvious necessary condition for the existence of a 2-perfect 6-cycle system of \( \lambda K_n \) is that \( \lambda \) must be even. However, the next result shows that in fact we need \( \lambda \) to be a multiple of 4 in order to have a 2-perfect 6-cycle decomposition of \( \lambda K_n \).

**Lemma.** There exists a 2-perfect 6-cycle decomposition of \( \lambda K_n \) if and only if \( \lambda \equiv 0 \) (modulo 4).

**Proof.** A 2-perfect 6-cycle decomposition of \( 4K_n \) is given by:

\[
\begin{align*}
1 & \ 5 \ 3 \ 4 \ 2 \ 6 \ 1 \ 3 \ 2 \ 5 \ 4 \ 6 \\
1 & \ 4 \ 5 \ 3 \ 2 \ 6 \ 1 \ 3 \ 6 \ 5 \ 2 \ 4 \\
1 & \ 5 \ 3 \ 6 \ 4 \ 2 \ 1 \ 2 \ 3 \ 6 \ 5 \ 4 \\
1 & \ 2 \ 3 \ 4 \ 6 \ 5 \ 1 \ 6 \ 5 \ 2 \ 4 \ 3 \\
1 & \ 2 \ 6 \ 3 \ 4 \ 5 \ 1 \ 3 \ 5 \ 2 \ 6 \ 4 
\end{align*}
\]

Appropriate multiples of this decomposition will deal with all cases in which \( \lambda \equiv 0 \) (modulo 4).

We now show that \( \lambda \equiv 0 \) (mod 4) is necessary. So, let \( (\lambda K_n, H) \) be a \( \lambda \)-fold 2-perfect 6-cycle system of order 6 based on \( \{1, 2, 3, 4, 5, 6\} \) and let \( (1, a, 2, b, 3, c) \in H \). Let \( H_1 \) be the set of hexagons containing the triple \( \{1, 2, 3\} \), \( H_2 \) the set of hexagons containing triples of the form \( \{1, 2, x\}, x \neq 3 \), \( H_3 \) the set of hexagons of the form \( \{1, 3, y\}, y \neq 2 \), and \( H_4 \) the set of hexagons containing triples of the form \( \{2, 3, z\}, z \neq 1 \). Then \( H = H_1 \cup H_2 \cup H_3 \cup H_4 \) and the \( H \)'s partition \( H \). Hence

\[
(5\lambda)/2 = |H| = |H_1| + |H_2| + |H_3| + |H_4|
\]

\[
= |H_1| + 3(\lambda - |H_1|) = 3\lambda - 2|H_1|,
\]

which gives

\[
4 |H_1| = \lambda.
\]

This completes the proof. \( \square \)

Now let \( v = 12n + 6 \), with \( n \geq 3 \), and take the vertex set

\[
\{\infty_i \mid 1 \leq i \leq 6\} \cup \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq 12\}.
\]

The construction when \( n \equiv 0 \) or \( 1 \) (modulo 3), \( n \geq 3 \), follows Construction A of Section 2.1 above. It uses a decomposition of \( 2K_{18} \) and a decomposition of \( 2K_{18} \) with a hole of size 6. We also have a decomposition of \( 2K_{30} \). These decompositions are given in the
Appendix. Likewise, when \( n = 2 \mod 3 \), \( n \geq 3 \), a maximum packing is used, as in Construction B of Section 2.1.

This completes the decomposition of \( 2K_n \) into 2-perfect 6-cycles in all admissible cases.

3. Two Remaining Cases When \( \lambda = 1 \)

In [6] the spectrum for 2-perfect 6-cycle systems of \( K_v \) was determined with the two possible exceptions \( v = 45, 57 \). We deal with these two cases now by exhibiting a 2-perfect 6-cycle system in each case.

In order to do this we will use a 2-perfect 6-cycle system of \( K_{21} \) with a hole of size 9. One of these is given in the Appendix.

**Construction 3.1.** Define a 2-perfect 6-cycle system of \( K_{45} \) on the vertex set 
\[
\{i \mid 1 \leq i \leq 9\} \cup \{(i, j) \mid 1 \leq i \leq 3, 1 \leq j \leq 12\}
\]
as follows:

(i) on \( \{i \mid 1 \leq i \leq 9\} \cup \{(1, j), (2, j) \mid 1 \leq j \leq 12\} \), place a 2-perfect 6-cycle system of \( K_{21} \), which was given in Example 2.1 of [6];

(ii) on \( \{i \mid 1 \leq i \leq 9\} \cup \{(i, j) \mid 1 \leq j \leq 12\} \), for \( i = 2 \) and then for \( i = 3 \), place a 2-perfect 6-cycle system of \( K_{21} \) with hole \( \{i \mid 1 \leq i \leq 9\} \);

(iii) on \( \{(i, j) \mid 1 \leq i \leq 3, 1 \leq j \leq 12\} \), place a 2-perfect 6-cycle system of \( K_{12,12,12} \) [6, Lemma 2.2].

**Construction 3.2.** Let the vertices of \( K_{57} \) be
\[
\{i \mid 1 \leq i \leq 9\} \cup \{(i, j) \mid 1 \leq i \leq 8, 1 \leq j \leq 6\}.
\]

Take a maximum packing of \( K_8 \) by triangles; this has leave consisting of a 1-factor of \( K_8 \). Let \( \{i \mid 1 \leq i \leq 8\} \) be the vertices of such a \( K_8 \), and without loss of generality let the leave of a maximum packing be \( \{1, 2\} \cup \{3, 4\} \cup \{5, 6\} \cup \{7, 8\} \).

Then the following defines a 2-perfect 6-cycle system of \( K_{57} \):

(i) on \( \{i \mid 1 \leq i \leq 9\} \cup \{(x, j), (y, j) \mid 1 \leq j \leq 6\} \), each time \( \{x, y\} \) is in the leave of the chosen packing of \( K_8 \), place a decomposition of \( K_{21} \) with a hole of size 9, except on one occasion use \( K_{21} \) and omit the hole;

(ii) whenever \( \{x, y, z\} \) is a triple in the maximum packing, take a 2-perfect 6-cycle system of \( K_{6,6,6} \) on the vertex set \( \{(x, j), (y, j), (z, j) \mid 1 \leq j \leq 6\} \).

4. The Case \( \lambda = 3 \)

As detailed in Table 1, for \( 3K_n \), to have any hope of possessing a decomposition into a 2-perfect 6-cycle system, it is necessary that \( v = 1 \mod 4 \). This may be re-expressed as \( v = 1, 5 \) or \( 9 \mod 12 \), and by using results in the case \( \lambda = 1 \) (see [6] and also Section 3 above), we need only consider \( v = 9 \) and \( v = 5 \mod 12 \).

A 2-perfect 6-cycle system of \( 3K_9 \), on the vertex set \( \mathbb{Z}_9 \), is given by the 18 hexagons
\[
\{(0, 4, 3, 6, 1, 7) + i, (0, 3, 4, 2, 6, 7) + i \mid 0 \leq i \leq 8\}.
\]

Now suppose that \( v = 12n + 5 \). It is known (see, for example, [6]) that there exists a group-divisible design on \( 2n \) elements with group sizes \( 2 \) and block size \( 3 \) when \( 2n = 0 \) or \( 2 \mod 6 \), and with one group of size \( 4 \) and the rest all of size \( 2 \), and block size \( 3 \),
when $2n = 4 \pmod{6}$. So take the following vertex set for $K_n$:

$$\{x_i \mid 1 \leq i \leq 5\} \cup \{(i, j) \mid 1 \leq i \leq 2n, 1 \leq j \leq 6\}.$$ 

Then a 2-perfect 6-cycle system of $3K_{12n+5}$ is given by the following cycles

(i) If $2n = 0$ or $2 \pmod{6}$, suppose that the groups of the group-divisible design are $\{2\alpha - 1, 2\alpha\}$, for $\alpha = 1, 2, \ldots, n$. Then place a 2-perfect 6-cycle system of $3K_{17}$ on $\{x_i \mid 1 \leq i \leq 5\} \cup \{(i, j) \mid i = 1, 2, \ldots, j \leq 6\}$; place a 2-perfect 6-cycle system of $3K_{17}$ with a hole of size 5 on $\{x_i \mid 1 \leq i \leq 5\} \cup \{(i, j) \mid i = 2\alpha - 1, 2\alpha; 2 \leq \alpha \leq n\}$. If $2n = 4 \pmod{6}$, suppose that the groups of the group-divisible design are $\{1, 2, 3, 4\} \cup \{2\alpha - 1, 2\alpha\}$ for $\alpha = 3, 4, \ldots, n$. Then place a 2-perfect 6-cycle system of $3K_{29}$ on $\{x_i \mid 1 \leq i \leq 5\} \cup \{(i, j) \mid i = 1, 2, 3, 4, 1 \leq j \leq 6\}$, and a 2-perfect 6-cycle system of $3K_{17}$ with a hole of size 5 on $\{x_i \mid 1 \leq i \leq 5\} \cup \{(i, j) \mid i = 2\alpha - 1, 2\alpha; 3 \leq \alpha \leq n\}$.

(ii) If $\{x, y, z\}$ is a block in the group-divisible design, place three copies of a 2-perfect 6-cycle system of $K_{6,6,6}$ on the vertices $\{(i, j) \mid i = x, y, z; 1 \leq j \leq 6\}$.

Decompositions of $3K_{17}, 3K_{29}$ and $3K_{17}$ with a hole of size 5, into 2-perfect 6-cycles, are given in the Appendix.

This completes the case $\lambda = 3$.

5. The case $\lambda = 6$

Apart from $v \geq 6$, there is no restriction on $6K_v$ for a 2-perfect 6-cycle system to exist. We have already shown, in the lemma of Section 2.5 above, that $6K_v$ cannot be decomposed in this way.

It is straightforward to see that the only values of $v$ here which have not already been dealt with by combining decompositions of $\lambda K_v$ for smaller values of $\lambda$ are $v = 2$, 8 and 11 (mod 12). So we shall give constructions for decomposing $6K_v$ into a 2-perfect 6-cycle system whenever $v = 2$ or 5 (mod 6).

First, as usual, we need some small examples. In the Appendix are listed 2-perfect 6-cycles systems of $6K_8$, $6K_8$ with a hole of size 2, $6K_{14}$, $6K_{11}$, and $6K_{11}$ with a hole of size 5. We also need a decomposition of $6K_{17}$, for which two copies of a decomposition of $3K_{17}$ may be taken.

Let $K_{6n+2}$ have vertex set $\{x_1, x_2\} \cup \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq 6\}$, where $n \geq 3$.

Take a 6-fold triple system of order $n$. On $\{x_1, x_2\} \cup \{(i, j) \mid 1 \leq j \leq 6\}$ place a 2-perfect 6-cycle system of $6K_8$. On $\{x_1, x_2\} \cup \{(i, j) \mid 1 \leq j \leq 6\}$, for each $i$ such that $2 < i < n$, place a 2-perfect 6-cycle system of $6K_8$ with hole $\{x_1, x_2\}$. Finally, whenever $\{x, y, z\}$ is a block of the 6-fold triple system, place a 2-perfect 6-cycle decomposition of $K_{6,6,6}$ on the vertex set $\{(x, j), (y, j), (z, j) \mid 1 \leq j \leq 6\}$. We also have a 2-perfect 6-cycle system on $6K_{14}$.

In the case $K_{6n+5}$ with $n \geq 3$, the above construction is mimicked, with $6K_8$ replaced by $6K_{11}$, and the hole of size 2 is replaced by one of size 5. Thus in this case we use decompositions of $6K_{11}$, and $6K_{11}$ with a hole of size 5; we also have a decomposition of $6K_{17}$.

This completes the case $\lambda = 6$.

6. The General Case, for any $\lambda$

A quick check of the necessary conditions when $\lambda = 4$ or $\lambda = 5$ shows that (apart from $\lambda K_6$, which was dealt with in Section 2.5 above) these are catered for by $\lambda = 2$ and $\lambda = 1$ respectively.
Finally, for any value of $\lambda$, by combining appropriate decompositions of $6K_v$ and $\lambda_0K_v$, where $1 \leq \lambda_0 \leq 5$, we obtain a decomposition of $\lambda K_v$ into 2-perfect 6-cycles. Thus we have proved the following.

**Main Theorem.** A 2-perfect 6-cycle system of $\lambda K_v$ exists if and only if $v \geq 6$ and:

1. $\lambda \equiv 2$ or $4 \pmod{6}$ and $v \equiv 0$ or $1 \pmod{3}$, or
2. $\lambda \equiv 3 \pmod{6}$ and $v \equiv 1 \pmod{4}$, or
3. $\lambda \equiv 0 \pmod{6}$ and $v \geq 6$, or
4. $\lambda \equiv 1$ or $5 \pmod{6}$ and $v \equiv 1$ or $9 \pmod{12}$.

except that there is no such system when $(\lambda, v) = (1, 9)$ or when $\lambda \equiv 2 \pmod{4}$ and $v = 6$.

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**References**


**Appendix**

A 2-Perfect 6-Cycle System of $2K_{18}$. Take the vertex set $\{\infty\} \cup \mathbb{Z}_{17}$, and fifty-one 6-cycles as follows, where $\infty$ is kept fixed (addition is modulo 17):

$\{(\infty, 7, 6, 0, 8, 4) + i, (0, 3, 6, 4, 2, 13) + i, (0, 12, 3, 13, 8, 7) + i \mid 0 \leq i \leq 16\}$.

A 2-Perfect 6-Cycle System of $2K_{18}$ with a Hole of Size 6. Take the vertex set $\{A, B, C, D, E, F\} \cup \mathbb{Z}_{12}$. The forty-six 6-cycles we take as follows; here the hole consists of $\{A, B, C, D, E, F\}$ of course, and these elements remain fixed. The addition is performed modulo 12:

$\{(A, 5, 1, C, 2, 8) + i, (B, 2, 1, F, 3, 8) + i, (D, 10, 1, E, 6, 5) + i \mid 1 \leq i \leq 12\}$

$\cup \{(1, 8, 4, 7, 2, 10) + i \mid i = 0, 1, 2, 3, 4, 5\} \cup \{(1, 3, 5, 7, 9, 11) + i \mid i = 0, 1, 2, 3\}$.

A 2-Perfect 6-Cycle System of $2K_{39}$. Take the vertex set $\{\infty\} \cup \mathbb{Z}_{29}$, and 145 6-cycles obtained from the following (addition is modulo 29 with $\infty$ kept fixed):

$\{(\infty, 11, 0, 3, 12, 2) + i, (0, 14, 7, 12, 1, 25) + i, (2, 4, 14, 8, 0, 17) + i,$

$0, 6, 10, 2, 3, 16) + i, (0, 1, 8, 6, 3, 12) + i \mid 0 \leq i \leq 28\}$. 
A 2-PERFECT 6-CYCLE SYSTEM OF $K_{21}$ WITH A HOLE OF SIZE 9. Let the vertex set of $K_{21}$ be
\[ \{A, B, C, D, E, F, G, H, I\} \cup \{(i, j) \mid 1 \leq i \leq 3, 1 \leq j \leq 4\}. \]

Then the following twenty-nine 6-cycles form a 2-perfect 6-cycle system of $K_{21}$, with a hole $\{A, B, C, D, E, F, G, H, I\}$ of size 9:

\[
\begin{align*}
&((0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2)), \\
&((0, 1), (2, 3), A, (1, 4), (0, 2), B) + (i, 0) \quad \text{for } i = 0, 1, 2; \\
&((0, 1), (2, 2), A, (0, 1), (0, 4), C) + (i, 0) \quad \text{for } i = 0, 1, 2; \\
&((0, 1), (2, 4), B, (1, 3), (1, 2), C) + (i, 0) \quad \text{for } i = 0, 1, 2; \\
&((0, 1), (2, 1), D, (1, 3), (0, 3), E) + (i, 0) \quad \text{for } i = 0, 1, 2; \\
&((0, 2), (1, 2), D, (2, 4), (0, 4), E) + (i, 0) \quad \text{for } i = 0, 1, 2; \\
&((0, 1), (1, 4), F, (1, 1), (1, 3), G) + (i, 0) \quad \text{for } i = 0, 1, 2; \\
&((0, 2), (2, 3), F, (2, 2), (2, 4), G) + (i, 0) \quad \text{for } i = 0, 1, 2; \\
&((0, 1), (1, 3), H, (2, 2), (1, 4), I) + (i, 0) \quad \text{for } i = 0, 1, 2; \\
&((0, 2), (2, 1), H, (1, 4), (1, 3), I) + (i, 0) \quad \text{for } i = 0, 1, 2.
\end{align*}
\]

A 2-PERFECT 6-CYCLE SYSTEM OF $3K_{17}$. Take the vertex set $\mathbb{Z}_{17}$, and the following sixty-eight 6-cycles; addition is modulo 17:

\[
\begin{align*}
&((0, 12, 1, 9, 5, 15) + i, (0, 6, 7, 10, 1, 15) + i, (0, 5, 1, 7, 8, 15) + i, \\
&(0, 3, 2, 6, 13, 8) + i, \quad 0 \leq i \leq 16).
\end{align*}
\]

A 2-PERFECT 6-CYCLE SYSTEM OF $3K_{17}$ WITH A HOLE OF SIZE 5. Let the vertex set of $3K_{17}$ be
\[ \{A, B, C, D\} \cup \{0, 1, 2, \ldots, 11\}. \]

Then the following is a 2-perfect 6-cycle system of $3K_{17}$, with a hole $\{A, B, C, D\}$ of size 5. Note that the elements in the hole are not cycled but remain fixed. Twenty-seven 6-cycles are obtained from the following; addition is modulo 12:

\[
\begin{align*}
&((0, 6, C, 9, 3, D) + i, (3, 9, D, 0, 6, E) + i, (3, 6, B, 0, 9, D) + i, \\
&(0, 6, D, 9, 3, B) + i, (0, 3, B, 6, 9, E) + i, (0, 3, A, 9, 6, D) + i, \\
&(3, 6, B, 9, 0, C) + i, (0, 3, D, 6, 9, B) + i, (3, 6, A, 0, 9, B) + i, \quad 0 \leq i \leq 2.
\end{align*}
\]

Then 36 more 6-cycles are obtained from the following (addition is modulo 12); again the elements from the hole are kept fixed:

\[
\begin{align*}
&((0, 8, 1, A, 5, 10) + i, (0, 4, 5, 3, 1, 8) + i, (0, 1, C, 3, 2, E) + i, \quad 0 \leq i \leq 11.
\end{align*}
\]

A 2-PERFECT 6-CYCLE SYSTEM OF $3K_{29}$. Take the vertex set $\mathbb{Z}_{29}$, and 203 6-cycles as follows; addition is modulo 29:

\[
\begin{align*}
&((0, 3, 5, 24, 10, 26) + i, (0, 17, 9, 18, 2, 25) + i, (0, 8, 9, 27, 3, 23) + i, \\
&(0, 27, 15, 11, 4, 28) + i, (0, 11, 1, 27, 12, 24) + i, \\
&(0, 14, 4, 12, 18, 22) + i, (0, 7, 5, 4, 22, 13) + i, \quad 0 \leq i \leq 28.
\end{align*}
\]

A 2-PERFECT 6-CYCLE SYSTEM OF $6K_{8}$. On the vertex set $\{\infty\} \cup \mathbb{Z}_{7}$, take the following 6-cycles (keep $\infty$ fixed; the addition is modulo 7):

\[
\begin{align*}
&((0, 3, 1, 6, 5, 4) + i, (0, 5, 3, 4, 1, \infty) + i, \\
&(0, 4, 3, 6, 1, \infty) + i, (0, 5, 1, 2, 3, \infty) + i, \quad 0 \leq i \leq 6.
\end{align*}
\]
A 2-PERFECT 6-CYCLE SYSTEM OF $6K_8$ WITH A HOLE OF SIZE 2. Let the vertex set be \{\(A, B\)\} \(\cup\) \(\mathbb{Z}_6\). Take the following 6-cycles: \((0, 1, 2, 3, 4, 5)\) three times, and then 24 more cycles as follows, where the addition is modulo 6 and the elements \(A\) and \(B\) are kept fixed:

\[
\{(0, 3, 2, 4, 1, A) + i, (0, 4, 1, B, 3, 5) + i, \\
(0, 2, A, 5, 3, B) + i, (0, 4, A, 2, 1, B) + i \mid 0 \leq i \leq 5\}.
\]

A 2-PERFECT 6-CYCLE SYSTEM OF $6K_{14}$. Take the vertex set \(\{\infty\} \cup \mathbb{Z}_{13}\); cycles are as follows, where the addition is worked modulo 13, keeping \(\infty\) fixed:

\[
\{(0, 5, \infty, 7, 6, 10) + i, (0, 8, 4, 6, \infty, 12) + i, (0, 7, \infty, 6, 3, 10) + i, (0, 6, 1, 5, 3, 11) + i, \\
(0, 7, 4, 8, 1, 12) + i, (0, 9, 6, 7, 1, 12) + i, (0, 3, 2, 10, 6, 8) + i \mid 0 \leq i \leq 12\}.
\]

A 2-PERFECT 6-CYCLE SYSTEM OF $6K_{11}$. Take the vertex set \(\mathbb{Z}_{11}\); 55 cycles are as follows, where addition is modulo 11:

\[
\{(0, 1, 3, 2, 5, 7) + i, (0, 2, 1, 7, 4, 6) + i, (0, 4, 1, 7, 3, 8) + i, \\
(0, 1, 2, 6, 4, 8) + i, (0, 4, 5, 7, 2, 8) + i \mid 0 \leq i \leq 10\}.
\]

A 2-PERFECT 6-CYCLE SYSTEM OF $6K_{11}$ WITH A HOLE OF SIZE 5. Let the vertex set be \(\{A, B, C, D, E\} \cup \mathbb{Z}_6\); the hole is \(\{A, B, C, D, E\}\). Take the following nine 6-cycles:

\[
(D, 2, 1, B, 4, 5), \quad (E, 2, 1, D, 4, 5), \quad (E, 2, 1, B, 4, 5), \\
(D, 3, 2, B, 5, 0), \quad (E, 3, 2, D, 5, 0), \quad (E, 3, 2, B, 5, 0), \\
(D, 4, 3, B, 0, 1), \quad (E, 4, 3, D, 0, 1), \quad (E, 4, 3, B, 0, 1),
\]

and then 36 more cycles as follows, where the addition is modulo 6, and the hole elements are kept fixed:

\[
\{(A, 3, 2, B, 1, 5) + i, (A, 4, 1, C, 3, 5) + i, (B, 3, 2, D, 1, 5) + i, \\
(C, 4, 3, E, 1, 5) + i, (D, 3, 1, E, 2, 5) + i, (A, 3, 1, C, 2, 5) + i \mid 0 \leq i \leq 5\}.
\]

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