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Multivalued Essential Maps of Approximable and Acyclic Type

R. P. AGARWAL
Department of Mathematics
National University of Singapore
10 Kent Ridge Crescent, Singapore 119260

D. O'REGAN
Department of Mathematics
National University of Ireland, Galway, Ireland

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Abstract—This paper discusses multivalued approximable and acyclic closed maps. We show if F is essential and $F \cong G$, then G has a fixed point.

Keywords—Essential maps, Approximable closed maps, Acyclic closed maps.

1. INTRODUCTION

The notion of an essential map introduced by Granas in [1], is more general than the notion of degree. In [1], he showed that if F is essential and $F \cong G$, then G is essential. However, to be essential is quite general and as a result Granas was only able to show this homotopy property for particular classes of maps (usually condensing). The notion was extended by many authors (see, for example, [2]) to other classes of maps. However, from an application viewpoint the authors in [1,2] were asking too much (and therefore, they could only establish their continuation theory for particular classes of maps). What one needs usually in applications is the following question to be answered: if F is essential and $F \cong G$, does G have a fixed point? In this paper, we discuss this question in detail. To illustrate the ideas involved, we discuss in particular approximable and acyclic closed maps and we show in Section 2 and in Section 3 that the above property holds for these classes of maps. (Indeed this property also holds for many other classes of maps in the literature, see for example those maps in [2,3].) In [3,4], nonlinear alternatives of Leray-Schauder type were presented for general classes of maps (of course the results in this paper automatically include those in [3,4]). This paper should be viewed as a stepping stone towards obtaining a general continuation theory (i.e., if F is essential and $F \cong G$, then G is essential) for general classes of maps. These continuation type results are currently under investigation by the authors.

To conclude the introduction, we present some concepts which will be needed in Section 2 and in Section 3. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2,

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respectively, and $F : X \to 2^Y$ (here 2^Y denotes the family of all nonempty subsets of Y) is a multifunction. Given two open neighborhoods U and V of the origins in E_1 and E_2, respectively, a (U,V)-approximate continuous selection of F is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (F[(x+U)\cap X] + V)\cap Y, \qquad \text{for every } x \in X.$$

F is said to be *approximable* if its restriction $F|_K$ to any compact subset K of X admits a (U,V)-approximate continuous selection for any open neighborhoods U and V of the origins in E_1 and E_2, respectively.

Let (Z,d) be a metric space and let Ω_Z be the family of all bounded subsets of Z. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_Z \to [0,\infty]$ defined by (here $B \in \Omega_Z$),

$$\alpha(B) = \inf\left\{r > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } \operatorname{diam}(B_i) \le r\right\}.$$

Let S be a nonempty subset of Z and suppose $G : S \to 2^X$. Then (i). $G : S \to 2^X$ is k-set contractive (here $k \ge 0$) if $\alpha(G(A)) \le k\alpha(A)$ for all nonempty, bounded sets A of S, and (ii). $G : S \to 2^X$ is condensing if G is 1-set contractive and $\alpha(G(A)) < \alpha(A)$ for all bounded sets A of S with $\alpha(A) \ne 0$.

2. APPROXIMABLE CLOSED MAPS

Let E be a Fréchet space and U an open subset of E with $0 \in U$.

DEFINITION 2.1. *We say $F \in APCG(\overline{U},E)$ if $F : \overline{U} \to Cc(E)$ is a closed (i.e., has closed graph), approximable, condensing, bounded (i.e., $F(\overline{U})$ is bounded) map; here $Cc(E)$ denotes the family of nonempty, closed subsets of E and $\overline{U}$ denotes the closure of U in E.*

DEFINITION 2.2. *We let $APM_{\partial U}(\overline{U},E)$ denote the set of all maps $F \in APCG(\overline{U},E)$ with $0 \notin (I-F)(x)$ for $x \in \partial U$; here I is the identity map and ∂U denotes the boundary of U in E.*

DEFINITION 2.3. *A map $F \in APM_{\partial U}(\overline{U},E)$ is essential if for every $G \in APM_{\partial U}(\overline{U},E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $0 \in (I-G)(x)$.*

THEOREM 2.1. *Let E be a Fréchet space and U an open subset of E and $0 \in U$. Suppose $F \in APM_{\partial U}(\overline{U},E)$ is essential. Let $H : \overline{U} \times [0,1] \to Cc(E)$ be a closed map with the following properties:*

$$H(x,0) = F(x), \qquad \text{for } x \in \overline{U}, \tag{2.1}$$

$$0 \notin (I-H_t)(x) \text{ for any } x \in \partial U \text{ and } t \in (0,1] \text{ (here } H_t(x) = H(x,t)) \tag{2.2}$$

and

$$\begin{gathered}\text{for any continuous } \mu : \overline{U} \to [0,1] \text{ with } \mu(\partial U) = 0, \text{ the map}\\ R_\mu : \overline{U} \to Cc(E) \text{ defined by } R_\mu(x) = H(x,\mu(x)) \text{ is in } APCG\left(\overline{U},E\right).\end{gathered} \tag{2.3}$$

Then H_1 has a fixed point in U.

PROOF. Let

$$B = \left\{x \in \overline{U} : 0 \in (I-H_t)(x) \text{ for some } t \in [0,1]\right\}.$$

When $t = 0$, we have $I - H_0 = I - F$, and since $F \in APM_{\partial U}(\overline{U},E)$ is essential, there exists $x \in U$ with $0 \in (I-F)(x)$. Thus, $B \ne \emptyset$. In addition, B is closed since $H : \overline{U} \times [0,1] \to Cc(E)$ is a closed map. Also (2.2) (together with $F \in APM_{\partial U}(\overline{U},E)$) implies $B \cap \partial U = \emptyset$. Thus, there exists a continuous $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map $R : \overline{U} \to Cc(E)$ by

$$R(x) = H(x,\mu(x)).$$

From (2.3), we have $R \in APCG(\overline{U}, E)$. Moreover, for $x \in \partial U$, $(I - R)(x) = (I - H_0)(x) = (I - F)(x)$ and so $R \in APM_{\partial U}(\overline{U}, E)$. Also notice

$$R|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$$

and since $F \in APM_{\partial U}(\overline{U}, E)$ is essential, there exists $x \in U$ with $0 \in (I - R)(x)$ (i.e., $0 \in (I - H_{\mu(x)})(x)$). Thus, $x \in B$ and so $\mu(x) = 1$. Consequently, $0 \in (I - H_1)(x)$. ∎

We now use Theorem 2.1 to obtain a nonlinear alternative of Leray-Schauder type for approximable maps. To prove our result we need the following well-known result from the literature [5, pp. 192–193].

THEOREM 2.2. *Let E be a Fréchet space, Q a nonempty, closed, convex subset of E and $J \in APCG(Q, Q)$. Then J has a fixed point in Q.*

THEOREM 2.3. *Let E be a Fréchet space, U an open subset of E and $0 \in U$. Suppose $G \in APCG(\overline{U}, E)$ with*

$$x \notin tG(x), \qquad \text{for } x \in \partial U \quad \text{and} \quad t \in (0,1). \tag{2.4}$$

Then G has a fixed point in $\overline{U}$.

PROOF. We assume $x \notin G(x)$ for $x \in \partial U$ (otherwise, we are finished). Then

$$x \notin tG(x), \qquad \text{for } x \in \partial U \quad \text{and} \quad t \in [0,1]. \tag{2.5}$$

Let $H(x,t) = tG(x)$ for $(x,t) \in \overline{U} \times [0,1]$ and $F(x) = \{0\}$ for $x \in \overline{U}$. First, we show $H : \overline{U} \times [0,1] \to Cc(E)$ is a closed map. To see this, let $(x_\alpha, t_\alpha, y_\alpha)$ be a net in $\overline{U} \times [0,1] \times E$ with $y_\alpha \in H(x_\alpha, t_\alpha) = t_\alpha G(x_\alpha)$ and $(x_\alpha, t_\alpha, y_\alpha) \to (x, t, y)$. We must show $y \in H(x,t)$. Without loss of generality assume $t \in (0,1]$. Since $y_\alpha \in t_\alpha G(x_\alpha)$, there exists $z_\alpha \in G(x_\alpha)$ with $y_\alpha = t_\alpha z_\alpha$. Now $y_\alpha \to y$, $z_\alpha \to (1/t)y$ together with the closedness of G implies $(1/t)y \in G(x)$, i.e., $y \in tG(x)$. Thus, $y \in H(x,t)$ and so $H : \overline{U} \times [0,1] \to Cc(E)$ is a closed map. In addition, (2.1) and (2.2) hold. Also, notice $R_\mu(x) = \mu(x)G(x)$ and we will now show that $R \in APCG(\overline{U}, E)$. Notice that an argument similar to the one above shows R_μ is a closed map. In addition, for any bounded set $A \subseteq \overline{U}$ we have

$$R_\mu(A) \subseteq \operatorname{co}(G(A) \cup \{0\}),$$

so it is immediate that R_μ is a bounded, condensing map. It remains to show R_μ is approximable. Let K be a compact subset of $\overline{U}$. Let U_1 and V_1 be two neighborhoods of the origin. We may assume without loss of generality that U_1 is symmetric. Let $V_2 \subseteq V_1$ be a balanced open neighborhood of the origin with $V_2 + V_2 \subseteq V_1$. Now K is compact, G is a closed map, $G(K)$ is bounded, and μ is continuous, so for any $x \in K$ there exists a neighborhood $W_x \subseteq U_1$ of the origin with

$$\mu\left([x + W_x] \cap K\right) G\left([x + W_x] \cap K\right) \subseteq \mu(x)G(x) + V_2 = R_\mu(x) + V_2.$$

Let $Z_x \subseteq W_x$ be a neighborhood of the origin with $Z_x + Z_x \subseteq W_x$. Now let $\{x_i + Z_{x_i}\}_1^n$ (here $x_i \in K$) be an open covering of K and let $U_2 = \cap_1^n Z_{x_i}$. Also let $s : K \to E$ be the (U_2, V_2)-approximate continuous selection of $G|_K$. Let $s_1 : K \to E$ be defined by $s_1(x) = \mu(x)s(x)$. We now check that $s_1 : K \to E$ is a (U_1, V_1)-approximate continuous selection of $R_\mu|_K$. Fix $x \in K$. Then $x \in x_i + Z_{x_i}$ for some $i \in \{1, 2, \dots\}$. Now since

$$s(x) \in G(x + U_2) + V_2,$$

we have

$$\mu(x)s(x) \in \mu(x_i + W_{x_i})G(x + U_2) + V_2$$

and so

$$\mu(x)s(x) \in \mu(x_i + W_{x_i})G(x_i + W_{x_i}) + V_2.$$

Thus,

$$\mu(x)s(x) \in R_\mu(x_i) + V_1$$

and so

$$s_1(x) \in R_\mu([x + U_1] \cap K) + V_1.$$

Consequently, $s_1 : K \to E$ is a (U_1, V_1)-approximate continuous selection of $R_\mu|_K$. Thus, (2.3) is satisfied. We can apply Theorem 2.1 if we show F is essential. To see this let $\theta \in APM_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U} = \{0\}$. We must show that there exists $x \in U$ with $x \in \theta(x)$. Let $Q = \overline{\text{co}}(\theta(\overline{U}))$ and let $J : Q \to Q$ be defined by

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U}, \\ \{0\}, & x \notin \overline{U}. \end{cases}$$

We now show $J \in APCG(Q, Q)$. It is clear that J is a closed map. In addition, for any bounded set $\Omega \subseteq Q$ we have

$$J(\Omega) \subseteq \text{co}\left(\theta\left(\Omega \cap \overline{U}\right) \cup \{0\}\right),$$

so it is immediate that J is a bounded, condensing map. It remains to show J is approximable. Let K be a compact subset of Q. Let U_1 and V_1 be two neighborhoods of the origin and let $r : K \cap \overline{U} \to E$ be the (U_1, V_1)-approximate continuous selection of $\theta|_{K \cap \overline{U}}$. Let $r_1 : K \to E$ be defined by

$$r_1(x) = \begin{cases} r(x), & x \in K \cap \overline{U}, \\ 0, & \text{otherwise}. \end{cases}$$

It is immediate that $r_1 : K \to E$ is a (U_1, V_1)-approximate continuous selection of J. Thus, $J \in APCG(Q, Q)$. Theorem 2.2 implies that there exists $x \in Q$ with $x \in J(x)$. Now if $x \notin U$, we have $x \in J(x) = \{0\}$, which is a contradiction since $0 \in U$. Thus, $x \in U$ so $x \in J(x) = \theta(x)$. Hence, F is essential and we may apply Theorem 2.1 to deduce the result. ∎

3. ACYCLIC CLOSED MAPS

Let E be a Fréchet space and U an open subset of E with $0 \in U$.

DEFINITION 3.1. *We say $F \in ACG(\overline{U}, E)$, if $F : \overline{U} \to CD(E)$ is closed, condensing, and bounded; here $CD(E)$ denotes the family of nonempty, closed, acyclic subsets of E.*

DEFINITION 3.2. *We let $AM_{\partial U}(\overline{U}, E)$ denote the set of all maps $F \in ACG(\overline{U}, E)$ with $0 \notin (I - F)(x)$ for $x \in \partial U$.*

DEFINITION 3.3. *A map $F \in AM_{\partial U}(\overline{U}, E)$ is essential if for every $G \in AM_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $0 \in (I - G)(x)$.*

THEOREM 3.1. *Let E be a Fréchet space and U an open subset of E and $0 \in U$. Suppose $F \in AM_{\partial U}(\overline{U}, E)$ is essential. Let $H : \overline{U} \times [0, 1] \to Cc(E)$ be a closed map with the following properties:*

$$H(x, 0) = F(x), \qquad \text{for } x \in \overline{U}, \tag{3.1}$$

$$0 \notin (I - H_t)(x), \qquad \text{for any } x \in \partial U \quad \text{and} \quad t \in (0, 1] \quad (\text{here } H_t(x) = H(x, t)) \tag{3.2}$$

and

$$\begin{gathered} \text{for any continuous } \mu : \overline{U} \to [0, 1] \text{ with } \mu(\partial U) = 0, \text{ the map} \\ R_\mu : \overline{U} \to CD(E) \text{ defined by } R_\mu(x) = H(x, \mu(x)) \text{ is in } ACG\left(\overline{U}, E\right). \end{gathered} \tag{3.3}$$

Then H_1 has a fixed point in U.

PROOF. Essentially, the same reasoning as in Theorem 2.1 establishes the result. ∎

Next, we recall the following well-known result from the literature [5, pp. 193].

THEOREM 3.2. *Let E be a Fréchet space, Q a nonempty, closed, convex subset of E and $J \in ACG(Q,Q)$. Then J has a fixed point in Q.*

THEOREM 3.3. *Let E be a Fréchet space, U an open subset of E and $0 \in U$. Suppose $G \in ACG(\overline{U}, E)$ with*

$$x \notin tG(x), \qquad \text{for } x \in \partial U \quad \text{and} \quad t \in (0,1). \tag{3.4}$$

Then G has a fixed point in $\overline{U}$.

PROOF. We assume $x \notin G(x)$ for $x \in \partial U$ and so

$$x \notin tG(x) \text{ for } x \in \partial U \quad \text{and} \quad t \in [0,1]. \tag{3.5}$$

Let $H(x,t) = tG(x)$ for $(x,t) \in \overline{U} \times [0,1]$ and $F(x) = \{0\}$ for $x \in \overline{U}$. As in Theorem 2.3 it is easy to see that $H : \overline{U} \times [0,1] \to Cc(E)$ is a closed map. It is also immediate that (3.1), (3.2), and (3.3) hold. We can apply Theorem 3.1 if we show F is essential. To see this let $\theta \in AM_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U} = \{0\}$. Let $Q = \overline{\text{co}}(\theta(\overline{U}))$ and let $J : Q \to Q$ be defined by

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U}, \\ \{0\}, & x \notin \overline{U}. \end{cases}$$

Now it is immediate that $J \in ACG(Q,Q)$, so Theorem 3.2 implies that there exists $x \in Q$ with $x \in J(x)$. Also, if $x \notin U$ we have $x \in J(x) = \{0\}$, which is a contradiction since $0 \in U$. Thus, $x \in U$, so $x \in J(x) = \theta(x)$. Hence, F is essential and we may apply Theorem 3.1 to deduce the result. ∎

REMARK. The ideas in this paper could be extended to many other classes of maps (for example, the ideas extend to the weakly inward approximable maps in [3]).

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