Monomials of Entire Algebroid Functions

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Hayman has shown that if \( f \) is a transcendental entire function and \( n \geq 2 \), then \( f^n f^9 \) assumes all values except possibly zero infinitely often. We extend his result in three directions by considering an entire algebroid function \( w \), its monomial
\[
\sum_{k=0}^{n_0} \cdots \cdots \sum_{k=n_k} (w^k)^{n_k},
\]
and by estimating the growth of the number of \( \alpha \)-points of the monomial.

1. INTRODUCTION AND RESULTS

As Hayman [3, p. 34] noted, the problem of possible Picard values of derivatives of a meromorphic function having no zeros reduces to the problem of whether certain differential polynomials of an entire function necessarily have zeros. In this connection he proved the following theorem:

**Theorem A.** If \( f(z) \) is a transcendental entire function and \( n \geq 2 \), then \( f^n f^9(z) \) assumes all values except possibly zero infinitely often.

Clunie [1] showed that Theorem A is also true for \( n = 1 \).

It is assumed that the reader is familiar with the notation of standard Nevanlinna theory [4] and its algebroid counterpart [9, 11, 12]. A characterization of algebroid functions and the basic results of their value distribution can be found in [6, Sects. 1 and 2].

Sons [10] extended Theorem A to more general differential monomials:

**Theorem B.** If \( f(z) \) is a transcendental entire function and
\[
\psi(z) := f(z)^{n_0} f'(z)^{n_1} \cdots f^{(k)}(z)^{n_k},
\]

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where $n_0 \geq 2$, $n_k \geq 1$, and $n_i \geq 0$ for $i \neq 0, k$, then, for $a \neq 0$,
\[
\delta(a, \psi) := 1 - \limsup_{r \to \infty} \frac{N(r, 1/(\psi - a))}{T(r, \psi)} < 1.
\]

In [8], Theorem A was generalized as follows:

**Theorem C.** Let $w(z)$ be a $\nu$-valued transcendental entire algebroid function and set
\[
\psi(z) := w(z)^{n_0}w'(z)^{n_1}, \quad n_0, n_1 \in \mathbb{N}.
\]
Then if $n_0 \geq 4\nu - 2 + 2(\nu - 1)(n_1 - 1)$, we have for each $a \in \mathbb{C} \setminus \{0\}$,
\[
\overline{N}\left(r, \frac{1}{\psi - a}\right) \geq pT(r, w) - S(r, w),
\]
where $p := n_0 - 4\nu + 3 - 2(\nu - 1)(n_1 - 1) \geq 1$.

In this paper we will consider monomials as in Theorem B and obtain a result analogous to Theorem C, where the transformation $u(z) := 1/w(z)$ was applied.

Referring to (1), denote
\[
\gamma := n_0 + n_1 + \cdots + n_k,
\]
\[
\bar{\mu} := n_0 + 2n_2 + \cdots + kn_k,
\]
\[
\sigma := n_0 + 3n_2 + \cdots + (2k - 1)n_k.
\]

We can now state the result:

**Theorem.** Let $w(z)$ be a $\nu$-valued transcendental entire algebroid function and set
\[
\psi(z) := w(z)^{n_0}w'(z)^{n_1} \cdots w^{(k)}(z)^{n_k},
\]
where $k \geq 2$, $n_k \geq 1$, and $n_i \geq 0$ for $i \neq 0, k$. Then if
\[
n_0 \geq k + \sum_{i=2}^{k} (i - 1)n_i + 2(\nu - 1)\left(k^2 + 2 + \sum_{i=1}^{k} i^2n_i\right) := m,
\]
we have for each $a \in \mathbb{C} \setminus \{0\}$,
\[
\overline{N}\left(r, \frac{1}{\psi - a}\right) \geq pT(r, w) - S(r, w),
\]
where $p := (\alpha/(\alpha + 2))(n_0 - m + 1 - 8(\nu - 1)/\alpha) \geq \frac{1}{2}$ with $\alpha := \gamma - \nu\bar{\mu} - 1 \geq 18(\nu - 1) + 1$. 

Corollary. With the same hypotheses, we have for each \( a \in \mathbb{C} \setminus \{0\} \),
\[
\Theta(a, \psi) := 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, 1/(\psi - a))}{T(r, \psi)} \leq 1 - \frac{p}{\gamma + 2(\nu - 1)\sigma}.
\]

2. Lemmas

Let \( w \) be a \( \nu \)-valued algebroid function. It has expansions of the general form
\[
w(z) = w(z_0) + c_r (z - z_0)^{r/\lambda} + \cdots \quad \text{or} \quad w(z) = c_{-r} (z - z_0)^{-r/\lambda} + \cdots,
\]
where \( \tau, \lambda \in \mathbb{N} \) and \( \lambda \leq \nu \). We define the counting functions of branch points and multiple points of \( w \) by
\[
n_\delta(r, w) := \sum_{|z| \leq r, w \in \mathbb{C}} (\lambda - 1) \quad \text{and} \quad n_{\Xi}(r, w) := \sum_{|z| \leq r, w \in \mathbb{C}} (\tau - 1),
\]
respectively, and denote by \( N_\delta(r, w) \) and \( N_{\Xi}(r, w) \) the corresponding integrated counting functions. We regard \( \psi, w'/w, \) etc., as functions on the Riemann surface of \( w \); see [7, Sect. 2].

By the proof of [13, Theorem 1], we have the following lemma:

Lemma 1. Let \( w \) be a nonconstant algebroid function. Then for \( i \in \mathbb{N} \),
\[
T(r, w^{(i)}) \leq T(r, w) + i\overline{N}(r, w) + (2i - 1)N_\delta(r, w) + S(r, w).
\]

We also have the following estimate; see [5, Lemma 3].

Lemma 2. Let \( w \) be a nonconstant algebroid function and \( \psi \) be of the form (2). Then
\[
T(r, \psi) \leq \gamma T(r, w) + \overline{\rho} \overline{N}(r, w) + \sigma N_\delta(r, w) + S(r, w).
\]

The next lemma was proved in [7].

Lemma 3. Let \( w \) be an algebroid function and \( \psi \) be a rational function of \( w, w', \ldots, w^{(k)} \), \( k \geq 0 \), with meromorphic coefficients. Then \( N_\delta(r, \psi) \leq N_\delta(r, w) \).

By [2, pp. 213, 215], we have the following results:

Lemma 4. Let \( w \) be a nonconstant algebroid function. Then
\[
N_{\Xi}(r, w) = N\left(r, \frac{1}{w}\right) + N(r, w) + N_\delta(r, w) - m\left(r, \frac{w}{w'}\right) + S(r, w)
\]
(3)
and

\[ N \left( r, \frac{w'}{w} \right) \leq \overline{N} \left( r, \frac{1}{w} \right) + \overline{N}(r,w) + N_{\beta}(r,w). \] (4)

We can now state and prove the final two lemmas.

**Lemma 5.** *With the hypotheses of the theorem, we have*

\[ \left( \gamma - \mu - 2(\nu - 1) \sum_{i=1}^{k} i^2 n_i \right) T(r,w) \leq T(r,\psi) + S(r,w). \] (5)

**Remark.** The left-hand side of (5) is \( \geq kT(r,w) \) by the hypothesis on \( n_0 \).

**Proof of Lemma 5.** Set \( q_i := \sum_{j=1}^{k} n_j \) for \( i = 0, 1, \ldots, k \), and write

\[ \psi = w^{q_0} \left( \frac{w'}{w} \right)^{q_1} \left( \frac{w''}{w'} \right)^{q_2} \cdots \left( \frac{w^{(k)}}{w^{(k-1)}} \right)^{q_k}. \]

Using the lemma on the logarithmic derivative with Lemma 1, we get

\[ q_0 T(r,w) \leq T(r,\psi) + \sum_{i=1}^{k} q_i N \left( r, \frac{w^{(i)}}{w^{(i-1)}} \right) + S(r,w). \]

By (4) and the inequality \( N_{\beta}(r,w) \leq 2(\nu - 1)T(r,w) + O(1) \) (see [11, Satz 11, p. 210]), we have

\[ N \left( r, \frac{w'}{w} \right) \leq \overline{N} \left( r, \frac{1}{w} \right) + N_{\beta}(r,w) \leq (1 + 2(\nu - 1)) T(r,w) + O(1). \]

Again by (4), Lemma 1, and the fact that possible poles of the derivatives of \( w \) arise only from branch points of \( w \), we obtain

\[ N \left( r, \frac{w^{(i)}}{w^{(i-1)}} \right) \]

\[ \leq \overline{N} \left( r, \frac{1}{w^{(i-1)}} \right) + \overline{N}(r,w) + N_{\beta}(r,w) \]

\[ \leq T(r,w) + (2i - 1) N_{\beta}(r,w) + 2 N_{\beta}(r,w) + S(r,w) \]

\[ = T(r,w) + (2i - 1) N_{\beta}(r,w) + S(r,w) \]

\[ \leq (1 + 2(\nu - 1)(2i - 1)) T(r,w) + S(r,w) \]
for $i = 2, 3, \ldots, k$. Thus

$$q_0 T(r, w) \leq T(r, \psi) + \sum_{i=1}^{k} q_i (1 + 2(\nu - 1)(2i - 1)) T(r, w) + S(r, w)$$

and, since $q_0 = \gamma$ and $\sum_{i=1}^{k} q_i = \bar{\mu}$,

$$\left( \gamma - \bar{\mu} - 2(\nu - 1) \sum_{i=1}^{k} q_i (2i - 1) \right) T(r, w) \leq T(r, \psi) + S(r, w).$$

But

$$\sum_{i=1}^{k} q_i (2i - 1) = \sum_{i=1}^{k} n_i \sum_{j=1}^{i} (2j - 1) = \sum_{i=1}^{k} i^2 n_i,$$

so that (5) holds.

**Lemma 6.** With the hypotheses of the theorem, we have $\alpha \geq 18(\nu - 1) + 1$.

**Proof.** Since $k \geq 2$, we get

$$\alpha = n_0 + q_1 - \nu \bar{\mu} - 1$$

$$\geq k + \sum_{i=1}^{k} in_i - \sum_{i=1}^{k} n_i + 2(\nu - 1) \left( k^2 + 2 + \sum_{i=1}^{k} i^2 n_i \right)$$

$$+ \sum_{i=1}^{k} n_i - \nu \sum_{i=1}^{k} in_i - 1$$

$$= k - 1 - (\nu - 1) \sum_{i=1}^{k} in_i + 2(\nu - 1) \left( k^2 + 2 + \sum_{i=1}^{k} i^2 n_i \right)$$

$$= k - 1 + (\nu - 1) \left( 2k^2 + 4 + \sum_{i=1}^{k} (2i - 1) in_i \right) \geq 18(\nu - 1) + 1,$$

and the proof is complete.

**3. Proof of the Theorem and Its Corollary**

The function $\psi$ must be nonconstant by Lemma 5. Let $a \in \mathbb{C} \setminus \{0\}$. The second main theorem (see [6, p. 18]) then yields

$$T(r, \psi) \leq \overline{N}(r, \psi) + \overline{N}\left( r, \frac{1}{\psi} \right) + \overline{N}\left( r, \frac{1}{\psi - a} \right) + N_{\Sigma}(r, \psi) + S(r, \psi).$$
By Lemma 2, we may replace $S(r, \psi)$ by $S(r, w)$. Using also Lemma 3 and noticing that any pole of $\psi$ must be a branch point of $w$, we get

$$T(r, \psi) \leq \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}\left(r, \frac{1}{\psi - a}\right) + 2N_\beta(r, w) + S(r, w). \quad (6)$$

Next we estimate the term $\overline{N}(r, 1/\psi)$. Let $w$ have an expansion

$$w(z) = w(z_0) + c_r(z - z_0)^{\tau/\lambda} + \cdots \quad (7)$$

around a point $z_0$. Then $w'$ has an expansion

$$w'(z) = \frac{\tau}{\lambda}c_r(z - z_0)^{(\tau - \lambda)/\lambda} + \cdots.$$ 

Thus we obtain

$$\overline{N}\left(r, \frac{1}{\psi}\right) \leq \overline{N}\left(r, \frac{1}{w}\right) + \sum_{\omega \neq 0} \left(\tau - \lambda\right)^+ + \sum_{i=2}^k \overline{N}\left(r, \frac{1}{w^{(i)}}\right) .$$

Here

$$\sum_{\omega \neq 0} \left(\tau - \lambda\right)^+ \leq \sum_{\omega \neq 0} \left(\tau - 1\right) = n_{\omega}(r, w) - \sum_{\omega = 0} \left(\tau - 1\right)$$

$$= \overline{N}\left(r, \frac{1}{w}\right) + n_{\omega}(r, w) - n\left(r, \frac{1}{w}\right) .$$

Integrating logarithmically we get

$$\overline{N}\left(r, \frac{1}{\psi}\right) \leq 2\overline{N}\left(r, \frac{1}{w}\right) + N_{\omega}(r, w) - N\left(r, \frac{1}{w}\right) + \sum_{i=2}^k \overline{N}\left(r, \frac{1}{w^{(i)}}\right) .$$

The identity (3) now gives

$$\overline{N}\left(r, \frac{1}{\psi}\right) \leq 2\overline{N}\left(r, \frac{1}{w}\right) + N_\beta(r, w) + \sum_{i=2}^k \overline{N}\left(r, \frac{1}{w^{(i)}}\right) + S(r, w) . \quad (8)$$

At each zero of $w$, the function $\psi$ also has a zero. In fact, if $w(z_0) = 0$ in (7), the expansion of $\psi$ becomes

$$\psi(z) = c(z - z_0)^{(\tau - \mu\lambda)/\lambda} + \cdots, \quad c \in \mathbb{C},$$
where $\gamma \tau - \bar{\mu} \lambda \geq \gamma - \bar{\mu} \nu = \alpha + 1 \geq 2$ by Lemma 6. We thus obtain

$$\alpha \bar{n}\left(r, \frac{1}{w}\right) = \sum_{w=0}^{\alpha} (\gamma - \bar{\mu} \nu - 1)$$

$$\leq \sum_{w=0}^{\alpha} (\gamma \tau - \bar{\mu} \lambda - 1) \leq n\left(r, \frac{1}{\psi}\right) - \bar{n}\left(r, \frac{1}{\psi}\right)$$

and, integrating logarithmically,

$$\alpha \overline{N}\left(r, \frac{1}{w}\right) \leq N\left(r, \frac{1}{\psi}\right) - \overline{N}\left(r, \frac{1}{\psi}\right).$$

Substituting this into (8) and recalling Lemma 1, we obtain

$$\overline{N}\left(r, \frac{1}{\psi}\right) \leq \frac{2}{\alpha} \left( N\left(r, \frac{1}{\psi}\right) - \overline{N}\left(r, \frac{1}{\psi}\right) \right) + N_\delta(r, w)$$

$$+ \sum_{i=1}^{k} (T(r, w) + (2i - 1)N_\delta(r, w)) + S(r, w).$$

Hence we have

$$\frac{\alpha + 2}{\alpha} \overline{N}\left(r, \frac{1}{\psi}\right) \leq \frac{2}{\alpha} T(r, \psi) + (k - 1)T(r, w)$$

$$+ \sum_{i=1}^{k} (2i - 1)N_\delta(r, w) + S(r, w),$$

i.e.,

$$\overline{N}\left(r, \frac{1}{\psi}\right) \leq \frac{2}{\alpha + 2} T(r, \psi)$$

$$+ \frac{\alpha}{\alpha + 2} ((k - 1)T(r, w) + k^2 N_\delta(r, w)) + S(r, w).$$

The inequality (6) now gives

$$\frac{\alpha}{\alpha + 2} T(r, \psi) \leq \frac{\alpha}{\alpha + 2} ((k - 1)T(r, w) + k^2 N_\delta(r, w))$$

$$+ \overline{N}\left(r, \frac{1}{\psi - \alpha}\right) + 2N_\delta(r, w) + S(r, w).$$
so that
\[
T(r, \psi) \leq (k - 1)T(r, w) + k^2N_{\beta}(r, w) + 2N_{\beta}(r, w) + \frac{4}{\alpha}N_{\beta}(r, w)
+ \alpha + \frac{2}{\alpha}N\left(r, \frac{1}{\psi - a}\right) + S(r, w)
\leq \left(k - 1 + 2(\nu - 1)(k^2 + 2) + \frac{8(\nu - 1)}{\alpha}\right)T(r, w)
+ \alpha + \frac{2}{\alpha}N\left(r, \frac{1}{\psi - a}\right) + S(r, w).
\]
Combining this with (5), we have
\[
\left(\gamma - \overline{m} - 2(\nu - 1)\sum_{i=1}^{k} i^2n_i - k + 1 - 2(\nu - 1)(k^2 + 2) - \frac{8(\nu - 1)}{\alpha}\right)
\times T(r, w)
= \left(n_0 - \sum_{i=2}^{k} (i - 1)n_i - 2(\nu - 1)(k^2 + 2 + \sum_{i=1}^{k} i^2n_i) - k\right)T(r, w)
+ \left(1 - \frac{8(\nu - 1)}{\alpha}\right)T(r, w)
\leq \frac{\alpha + 2}{\alpha}N\left(r, \frac{1}{\psi - a}\right) + S(r, w).
\tag{9}
\]
By Lemma 6 and the hypothesis on \(n_0\), the left-hand side of (9) is greater than \(\frac{5}{\alpha}T(r, w)\), and we have
\[
pT(r, w) \leq N\left(r, \frac{1}{\psi - a}\right) + S(r, w),
\]
where
\[
p := \frac{\alpha(n_0 - m)}{\alpha + 2} + \frac{\alpha - 8(\nu - 1)}{\alpha + 2} \geq \frac{1}{3}.
\]
This completes the proof.

**Proof of the Corollary.** By Lemma 2, we have
\[
T(r, \psi) \leq (\gamma + 2(\nu - 1)\sigma)T(r, w) + S(r, w).
\tag{10}
\]
Let \( a \in \mathbb{C} \setminus \{0\} \). By (10) and the theorem, there exists a set \( E \) of finite linear measure such that

\[
\lim_{{r \to \infty}} \frac{\bar{N}(r, 1/(\psi - \alpha))}{\tau(r, \psi)} \geq \lim_{{r \to \infty}} \frac{(p - \alpha(1))T(r, \omega)}{(\gamma + 2(\nu - 1)\sigma + o(1))T(r, \omega)} = \frac{p}{\gamma + 2(\nu - 1)\sigma}.
\]

The corollary is proved.

REFERENCES