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A note on nonautonomous logistic equation with random perturbation ☆

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Abstract

This paper discusses a randomized nonautonomous logistic equation

$$dN(t) = N(t)\big[\big(a(t) - b(t)N(t)\big)\,dt + \alpha(t)\,dB(t)\big],$$

where $B(t)$ is 1-dimensional standard Brownian motion. We show that $E[1/N(t)]$ has a unique positive T-periodic solution $E[1/N_p(t)]$ provided $a(t)$, $b(t)$, and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and $\int_0^T [a(s) - \alpha^2(s)]\,ds > 0$.

Keywords: Randomized logistic equation; Periodic solution; Itô's formula

1. Introduction

A simple nonautonomous logistic equation, based on ordinary differential equations, is usually denoted by

$$\dot{N}(t) = N(t)\big[a(t) - b(t)N(t)\big], \tag{1.1}$$

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on $t \geqslant 0$ with initial value $N(0) = N_0 > 0$, and models the population density N of a single species whose members compete among themselves for a limited amount of food and living space, where $a(t)$ is the rate of growth and $a(t)/b(t)$ the carrying capacity at time t, both $a(t)$ and $b(t)$ are positive continuous functions. We refer the reader to May [1] for a detailed model construction. For an autonomous system (1.1), there is a stable equilibrium point of the population. Many authors have obtained a lot of interesting results about the stability of positive solutions for the above system (1.1) with its general case, for example, see Globalism [2]. When parameters $a(t)$ and $b(t)$ are positive T-periodic functions, Eq. (1.1) has a stable positive T-periodic solution $N_p(t)$,

$$1/N_p(t) = \frac{\int_t^{t+T} \exp\{\int_t^s a(\tau)\,d\tau\} b(s)\,ds}{\exp\{\int_0^T a(\tau)\,d\tau\} - 1}, \quad t \geqslant 0.$$

The existence of a stable periodic solution is of fundamental importance biologically since it concerns the long time survival of species. The study of such phenomena has become an essential part of the qualitative theory of differential equations. For historical background, and the basic theory of periodicity, and discussions of applications of (1.1) to a variety of dynamical models, we refer to the reader to, for example, the work of Burton [3] and the references therein. In contrast, if we now let parameters $a(t) > 0$ and $b(t) < 0$, then Eq. (1.1) has only the local solution

$$N(t) = \frac{\exp\{\int_0^t a(s)\,ds\}}{1/N_0 - \int_0^t |b(s)| \exp\{\int_0^s a(\tau)\,d\tau\}\,ds} \quad (0 \leqslant t < T_e),$$

which explodes to infinity at the finite time T_e, where T_e is determined by the equation

$$1/N_0 = \int_0^{T_e} \left|b(s)\right| \exp\left\{ \int_0^s a(\tau)\,d\tau \right\} ds.$$

However, given that population systems are often subject to environmental noise (cf. Mao et al. [4]), it is important to discover whether the presence of a such noise affects these results. Suppose that parameter $a(t)$ is stochastically perturbed, with

$$a(t) \to a(t) + \alpha(t)\dot{B}(t),$$

where $\dot{B}(t)$ is white noise and $\alpha^2(t)$ represents the intensity of the noise. Then this environmentally perturbed system may be described by the Itô equation

$$dN(t) = N(t)\big[\big(a(t) - b(t)N(t)\big)\,dt + \alpha(t)\,dB(t)\big], \quad t \geqslant 0, \tag{1.2}$$

where $B(t)$ is the 1-dimensional standard Brownian motion with $B(0) = 0$, $N(0) = N_0$ and N_0 is a positive random variable. Here $a(t)$, $b(t)$, and $\alpha(t)$ are bounded continuous functions defined on $[0, \infty)$, $a(t) > 0$ and $b(t) > 0$. It is reasonable to assume that N_0 is independent of $B(t)$.

Remark 1.1. Mao et al. [4] consider the environmentally perturbed system

$$dN(t) = N(t)\big[\big(a + bN(t)\big)\,dt + \alpha N(t)\,dB(t)\big], \quad t \geqslant 0,$$

where $a, b, \alpha > 0$ with $N(0) = N_0 > 0$. No matter how small $\alpha > 0$, they show that the solution will not explode in a finite time. This result reveals the important property that the environmental noise suppresses the explosion.

In order to for a stochastic differential equation to have a unique global (i.e., no explosion in a finite time) solution for any initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (cf. Arnold [5] and Freedman [6]). However, the coefficients of Eq. (1.2) do not satisfy the linear growth condition, though they are local Lipschitz continuous, so the solution of Eq. (2.1) may explode at a finite time.

Since $B(t)$ is not periodic, we cannot expect the solution $N(t)$ to Eq. (1.2) is periodic even if $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions. In fact, as far as authors know, there are few work on periodic solutions of stochastic differential equations. In this paper, we show that $E[1/N(t)]$ has a unique positive T-periodic solution $E[1/N_p(t)]$ provided $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and $\int_0^T [a(s) - \alpha^2(s)]\,ds > 0$. Here, and in the sequel, "$E[f]$" will mean the mathematical expectation of f.

The remain part of this paper is as follows. In Section 2, we represent the unique solution of Eq. (1.2) and show that the solution will not explode in a finite time. In Section 3, we represent the unique positive T-periodic solution $E[1/N_p(t)]$ and in Section 4, similar to the arguments for dealing with Eq. (1.2), we consider a general randomized model with intensity $\alpha^2(t)$,

$$dN(t) = N(t)\big[\big(a(t) - b(t)N^\theta(t)\big)\,dt + \alpha(t)\,dB(t)\big].$$

2. Representation of global solution

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space on which an increasing and right continuous family $\{\mathcal{F}_t\}_{t\in[0,T]}$ of complete sub-σ-algebras of $\mathcal{F}$ is defined. Let $B(t)$ be a given 1-dimensional standard Brownian motion defined on the probability space. In this section, we shall represent the unique solution of Eq. (1.2) and show that the solution will not explode in a finite time.

Lemma 2.1.

$$E\left[\exp\left\{\int_{t_0}^{t} \alpha(s)\,dB(s)\right\}\right] = \exp\left\{\frac{1}{2}\int_{t_0}^{t} \alpha^2(s)\,ds\right\}, \quad 0 \leqslant t_0 \leqslant t.$$

Proof. Let

$$y(t) = \exp\left\{\int_{t_0}^{t} \alpha(s)\,dB(s)\right\}$$

and apply Itô's formula

$$dy(t) = \exp\left\{\int_{t_0}^{t} \alpha(s)\,dB(s)\right\}\alpha(t)\,dB(t) + \frac{1}{2}\exp\left\{\int_{t_0}^{t} \alpha(s)\,dB(s)\right\}\alpha^2(t)\,dt.$$

Thus

$$y(t) = y(t_0) + \int_{t_0}^{t} \exp\left\{\int_{t_0}^{s} \alpha(\tau)\,dB(\tau)\right\}\alpha(s)\,dB(s) + \frac{1}{2}\int_{t_0}^{t} \exp\left\{\int_{t_0}^{s} \alpha(\tau)\,dB(\tau)\right\}\alpha^2(s)\,ds.$$

So we get

$$E\big[y(t)\big] = E\big[y(t_0)\big] + \frac{1}{2}\int_{t_0}^{t} E\big[y(s)\big]\alpha^2(s)\,ds,$$

i.e.,

$$\frac{d}{dt}E\big[y(t)\big] = \frac{1}{2}\alpha^2(t)E\big[y(t)\big], \quad E\big[y(t_0)\big] = 1.$$

So,

$$E\big[y(t)\big] = \exp\left\{\frac{1}{2}\int_{t_0}^{t} \alpha^2(s)\,ds\right\}.$$

The proof of Lemma 2.1 is completed. □

We have the following main result in this section.

Theorem 2.2. *Assume that $a(t)$, $b(t)$ and $\alpha(t)$ are bounded continuous functions defined on $[0,\infty)$, $a(t) > 0$ and $b(t) > 0$. Then there exists a unique continuous positive solution $N(t)$ to Eq.* (1.2) *for any initial value $N(0) = N_0 > 0$, which is global and represented by*

$$N(t) = \frac{\exp\{\int_0^t [a(s) - \frac{\alpha^2(s)}{2}]\,ds + \alpha(s)\,dB(s)\}}{1/N_0 + \int_0^t b(s)\exp\{\int_0^s [a(\tau) - \frac{\alpha^2(\tau)}{2}]\,d\tau + \alpha(\tau)\,dB(\tau)\}\,ds}, \quad t \geqslant 0. \tag{2.1}$$

Proof. Since the coefficients of the equation are local Lipschitz continuous for any initial value $N_0 > 0$, there is a unique local solution $N(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time (cf. Arnold [5] and Freedman [6]).

To show this solution is global, we will represent the solution. Let

$$x(t) := \exp\left\{ \int_0^t \left[\frac{\alpha^2(s)}{2} - a(s) \right] ds - \alpha(s)\, dB(s) \right\}$$
$$\times \left[1/N_0 + \int_0^t b(s) \exp\left\{ \int_0^s \left[a(\tau) - \frac{\alpha^2(\tau)}{2} \right] d\tau + \alpha(\tau)\, dB(\tau) \right\} ds \right]; \quad (2.2)$$

then $x(t)$ satisfies the equation

$$dx(t) = \left[\left(\alpha^2(t) - a(t)\right) dt - \alpha(t)\, dB(t) \right] x(t) + b(t)\, dt. \quad (2.3)$$

Let $N(t) := 1/x(t)$, then $N(t) > 0$ and $N(t)$ is continuous and global on $t \in [0, \infty)$. By Itô's formula

$$\begin{aligned} dN(t) &= -\frac{dx(t)}{x^2(t)} + \frac{(dx(t))^2}{x^3(t)} \\ &= -\left[\left(\alpha^2(t) - a(t)\right) dt - \alpha(t)\, dB(t) \right] N(t) - b(t) N^2(t)\, dt + N(t)\alpha^2(t)\, dt \\ &= N(t)\left[\left(a(t) - b(t)N(t)\right) dt + \alpha(t)\, dB(t) \right]. \end{aligned}$$

Thus $N(t)$ is a continuous positive solution of Eq. (1.2) and global on $t \in [0, \infty)$ (i.e., $\tau_e = \infty$). This completes the proof of Theorem 2.1. □

Remark 2.3. If $b(t) < 0$, Eq. (1.2) has only the local solution

$$N(t) = \frac{\exp\{\int_0^t [a(s) - \frac{\alpha^2(s)}{2}]\, ds + \alpha(s)\, dB(s)\}}{1/N_0 - \int_0^t |b(s)| \exp\{\int_0^s [a(\tau) - \frac{\alpha^2(\tau)}{2}]\, d\tau + \alpha(\tau)\, dB(\tau)\}\, ds},$$
$$0 \leqslant t < t_e, \quad (2.4)$$

which explodes to infinity at the time t_e:

$$t_e = \inf\{t\colon\ f(t) = 1/N_0\},$$

where

$$f(t) = \int_0^{t_e} |b(s)| \exp\left\{ \int_0^s \left[a(\tau) - \frac{\alpha^2(\tau)}{2} \right] d\tau + \alpha(\tau)\, dB(\tau) \right\} ds.$$

For each fixed $t > 0$, since N_0 is independent of $B(t)$ then $1/N_0$ is independent of $f(t)$, thus $f(t)(\omega) \neq 1/N_0(\omega)$ for some $\omega \in \Omega$. It follows by Lemma 2.1 that

$$E\left[f(t)\right] = \int_0^t |b(s)| \exp\left\{ \int_0^s a(\tau)\, d\tau \right\} ds, \qquad E\left[f'(t)\right] = |b(t)| \exp\left\{ \int_0^t a(\tau)\, d\tau \right\}$$

and

$$E\left[f(t) f'(t)\right] = |b(t)| E\left[\int_0^t |b(s)| \exp\left\{ \int_0^s \left[2a(\tau) - \alpha^2(\tau)\right] d\tau + 2\alpha(\tau)\, dB(\tau) \right\}\right.$$

$$\times \exp\left\{\int_s^t \left[a(\tau) - \frac{\alpha^2(\tau)}{2}\right] d\tau + \alpha(\tau)\, dB(\tau)\right\} ds\Bigg]$$
$$= \left|b(t)\right| \int_0^t \left|b(s)\right| \exp\left\{\int_0^s \left[2a(\tau) + \alpha^2(\tau)\right] d\tau\right\} \exp\left\{\int_s^t a(\tau)\, d\tau\right\} ds$$
$$= \left|b(t)\right| \exp\left\{\int_0^t a(\tau)\, d\tau\right\} \int_0^t \left|b(s)\right| \exp\left\{\int_0^s \left[a(\tau) + \alpha^2(\tau)\right] d\tau\right\} ds.$$

Let

$$g(t) = 1/N_0 - f(t), \quad t \geqslant 0.$$

Then we have

$$\frac{d}{dt} E\left[g^2(t)\right] = 2\left(E\left[f(t) f'(t)\right] - E[1/N_0] E\left[f'(t)\right]\right)$$
$$= 2\left|b(t)\right| \exp\left\{\int_0^t a(\tau)\, d\tau\right\}$$
$$\times \left[\int_0^t \left|b(s)\right| \exp\left\{\int_0^s \left[a(\tau) + \alpha^2(\tau)\right] d\tau\right\} ds - E[1/N_0]\right].$$

Let $0 < T_1 < \infty$ satisfy

$$\int_0^{T_1} \left|b(s)\right| \exp\left\{\int_0^s a(\tau)\, d\tau\right\} ds = E[1/N_0].$$

(Notice that $T_1 = T_e$ provided $N_0 \equiv$ constant.) Then $E[g(T_1)] = 0$, $E[g(t)] > 0$ for $0 \leqslant t < T_1$ and $E[g(t)] < 0$ for $t > T_1$. We can see $g(t)(\omega) < 0$ on $t \in [T_1, \infty)$ for some $\omega \in \Omega$. Let $0 < T_2 < T_1$ satisfy

$$\int_0^{T_2} \left|b(s)\right| \exp\left\{\int_0^s \left[a(\tau) + \alpha^2(\tau)\right] d\tau\right\} ds = E[1/N_0].$$

Then $\frac{d}{dt} E[g^2(t)] < 0$ for $0 \leqslant t < T_2$ and $\frac{d}{dt} E[g^2(t)] > 0$ for $t > T_2$. Thus $E[g^2(t)]$ is decreasing on $[0, T_2]$ and increasing on $[T_2, \infty)$, and $E[g^2(T_2)] = \min_{t \in [0,\infty)} E[g^2(t)] > 0$. Since $g(t)$ is decreasing for $t \geqslant 0$ and $g(0) > 0$, we can see $g(t)$ may be positive on $[0, T_2)$.

3. Positive periodic solution of $E[1/N(t)]$

In this section, we assume that $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and $\int_0^T [a(s) - \alpha^2(s)]\, ds > 0$.

By Lemma 2.1, we obtain from (2.3) that

$$E[x(t)] = E[1/N(t)] = \exp\left\{\int_0^t [\alpha^2(s) - a(s)]\,ds\right\} E[1/N_0] + \int_0^t b(s)\exp\left\{\int_s^t [\alpha^2(\tau) - a(\tau)]\,d\tau\right\} ds, \quad t \geqslant 0. \tag{3.1}$$

In fact, $E[1/N(t)] = E[x(t)]$ satisfies the following equation:

$$\frac{d}{dt}E[x(t)] = [\alpha^2(t) - a(t)]E[x(t)] + b(t), \quad t \geqslant 0. \tag{3.2}$$

We can see that Eq. (3.2) has a unique positive T-periodic solution

$$E[1/N_p(t)] = E[x_p(t)] = \frac{\int_t^{t+T} \exp\{\int_t^s [a(\tau) - \alpha^2(\tau)]\,d\tau\} b(s)\,ds}{\exp\{\int_0^T [a(\tau) - \alpha^2(\tau)]\,d\tau\} - 1}, \quad t \geqslant 0. \tag{3.3}$$

Thus we have the following result.

Theorem 3.1. *Suppose $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and $\int_0^T [a(s) - \alpha^2(s)]\,ds > 0$. Then $E[1/N(t)]$ of Eq.* (1.2) *has a unique positive T-periodic solution $E[1/N_p(t)]$ which is defined by* (3.3)*. In addition,*

$$\lim_{t\to+\infty}\{E[1/N(t)] - E[1/N_p(t)]\} = 0,$$

where $N(t)$ is the solution of Eq. (1.2) *for any initial value $N(0) = N_0 > 0$.*

Proof. We only need to show that every solution of Eq. (3.1) tends to Eq. (3.3), as $t \to +\infty$. In Eq. (3.1),

$$E[x_0]\exp\left\{\int_0^t \left(\alpha^2(s) - a(s)\right) ds\right\} \to 0. \tag{3.4}$$

So it is only necessary to verify that

$$\int_0^t b(s)\exp\left\{-\int_s^t r(\tau)\,d\tau\right\} ds - \frac{1}{A-1}\int_t^{t+T} \exp\left\{-\int_s^t r(\tau)\,d\tau\right\} b(s)\,ds \to 0 \tag{3.5}$$

as $t \to +\infty$, where $r(t) = a(t) - \alpha^2(t)$, $A = \exp\{\int_0^T r(\tau)\,d\tau\}$. We can rewrite (3.5) as

$$\int_0^t b(s)\exp\left\{-\int_s^t r(\tau)\,d\tau\right\} ds - \frac{1}{A-1}\int_t^{t+T} b(s)\exp\left\{-\int_s^t r(\tau)\,d\tau\right\} ds$$

$$= \exp\left\{-\int_0^t r(\tau)\,d\tau\right\}\left(\int_0^t b(s)\exp\left\{\int_0^s r(\tau)\,d\tau\right\} ds\right.$$

$$- \frac{1}{A-1} \int_t^{t+T} b(s) \exp\left\{ \int_0^s r(\tau)\, d\tau \right\} ds \Bigg)$$

$$:= \exp\left\{ - \int_0^t r(\tau)\, d\tau \right\} F(t), \tag{3.6}$$

where

$$F'(t) = b(t) \exp\left\{ \int_0^t r(\tau)\, d\tau \right\}$$

$$- \frac{1}{A-1} \left(b(t+\tau) \exp\left\{ \int_0^{t+T} r(\tau)\, d\tau \right\} - b(t) \exp\left\{ \int_0^t r(\tau)\, d\tau \right\} \right)$$

$$= b(t) \exp\left\{ \int_0^t r(\tau)\, d\tau \right\} \left(1 - \frac{\exp\{\int_t^{t+T} r(\tau)\, d\tau\} - 1}{A-1} \right) = 0,$$

since $r(\tau)$ is periodic with periodic T. Hence $F(t) \equiv$ constant. So (3.4) and (3.6) imply (3.5) and the proof is completed. □

4. Related results

A general nonautonomous logistic equation, based on ordinary differential equations, is usually denoted by

$$\dot{N}(t) = N(t)\big[a(t) - b(t)N^{\theta}(t)\big] \quad (\theta > 0). \tag{4.1}$$

Some detailed studies about the model may be found in Gilpin and Ayala [7,8]. Similarly, we can consider a randomized model based on (4.1) with intensity $\alpha^2(t)$,

$$dN(t) = N(t)\big[\big(a(t) - b(t)N^{\theta}(t)\big)\, dt + \alpha(t)\, dB(t)\big], \quad t \geqslant 0, \tag{4.2}$$

where $\theta > 0$ is an odd number, $B(t)$ is the 1-dimensional standard Brownian motion with $B(0) = 0$, $N(0) = N_0$ and N_0 is a positive random variable. Here $a(t)$, $b(t)$ and $\alpha(t)$ are bounded continuous functions defined on $[0, \infty)$, $a(t) > 0$, $b(t) > 0$ and N_0 is independent of $B(t)$.

Let $N(t)$ be a solution of Eq. (4.2), by Itô's formula

$$dN^{\theta}(t) = N^{\theta}(t)\left[\left(\theta a(t) + \frac{\theta(\theta-1)}{2}\alpha^2(t) - \theta b(t)N^{\theta}(t) \right) dt + \theta\alpha(t)\, dB(t) \right]. \tag{4.3}$$

Similarly to the proof of Theorems 2.1 and 3.1, we have the following results.

Theorem 4.1. *Assume that $a(t)$, $b(t)$ and $\alpha(t)$ are bounded continuous functions defined on $[0,\infty)$, $a(t)>0$ and $b(t)>0$. Then there exists a unique continuous solution $N(t)$ to Eq.* (4.2) *for any initial value $N(0)=N_0>0$, which is global and represented by*

$$N^\theta(t)=\frac{\exp\{\theta(\int_0^t[a(s)-\frac{\alpha^2(s)}{2}]\,ds+\alpha(s)\,dB(s))\}}{1/N_0^\theta+\theta\int_0^t b(s)\exp\{\theta(\int_0^s[a(\tau)-\frac{\alpha^2(\tau)}{2}]\,d\tau+\alpha(\tau)\,dB(\tau))\}\,ds},\quad t\geqslant 0.$$

Theorem 4.2. *Suppose $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t)>0$, $b(t)>0$ and $\int_0^T[a(s)-\frac{\theta+1}{2}\alpha^2(s)]\,ds>0$. Then $E[1/N^\theta(t)]$ of Eq.* (4.2) *has a unique positive T-periodic solution $E[1/N_p^\theta(t)]$ which is represented by*

$$E\big[1/N_p^\theta(t)\big]=\frac{\theta\int_t^{t+T}\exp\{\int_t^s[\theta a(\tau)-\frac{\theta(\theta+1)}{2}\alpha^2(\tau)]\,d\tau\}b(s)\,ds}{\exp\{\int_0^T[\theta a(\tau)-\frac{\theta(\theta+1)}{2}\alpha^2(\tau)]\,d\tau\}-1},\quad t\geqslant 0.$$

In addition,

$$\lim_{t\to+\infty}\big\{E\big[1/N^\theta(t)\big]-E\big[1/N_p^\theta(t)\big]\big\}=0,$$

where $N(t)$ is the solution of Eq. (4.2) *for any initial value $N(0)=N_0>0$.*

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