JOURNAL OF FUNCTIONAL ANALYSIS 13, 13–19 (1973)

Projections of Polynomial Hulls

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Communicated by John Wermer

Received February 15, 1972

The following theorem is discussed. Let X be a compact subset of the unit sphere in $\mathbb{C}^n$ whose polynomially convex hull, $\hat{X}$, contains the origin, then the sum of the areas of the n coordinate projections of $\hat{X}$ is bounded below by π. This applies, in particular, when $\hat{X}$ is a one-dimensional analytic subvariety V containing the origin, and in this case generalizes the fact that the "area" of V is at least π; in fact, the area of V is the sum of the areas of the n coordinate projections when these areas are counted with multiplicity. A convex analog of the theorem is obtained. Hartog's theorem that separate analyticity implies analyticity, usually proved with the use of subharmonic functions (Hartog's lemma), will be derived as a consequence of the theorem, the proof of which is based upon the elements of uniform algebras.

1.

Let B denote the open unit ball in $\mathbb{C}^n$, $B = \{z \in \mathbb{C}^n : \| z \| < 1\}$; $\partial B = \{z \in \mathbb{C}^n : \| z \| = 1\}$ where $\| z \| = \|(z_1, z_2, ..., z_n)\| = (\sum_1^n | z_i |^2)^{1/2}$. For $S \subseteq \mathbb{C}^n$, $z_j(S)$ will be the jth coordinate projection of S; λ will be planar Lebesgue measure in $\mathbb{C}$. Our main result is the following theorem.

THEOREM 1. *Let X be a compact subset of ∂B and suppose that $\hat{X}$, the polynomially convex hull of X, contains the origin. Then*

$$\sum_{j=1}^{n} \lambda(z_j(\hat{X})) \geqslant \pi.$$

The constant π is best possible and is attained when $\hat{X}$ is a complex line. In [2] Theorem 1 was obtained for the case when $\hat{X}$ is an analytic subvariety of B. For a 1-variety V through 0 in B, this generalizes the fact that the area of V is at least π; in fact, the area of V is just the sum of the areas of the n coordinate projections, when these areas are counted with multiplicity. In general, $\hat{X}$ need not contain any subvarieties, and, moreover, by an example of Stolzenberg ([6], cf. [8]), the sets $z_j(\hat{X})$ need not have interior. Stolzenberg's hull is a limit of one-dimensional varieties, and it is an open question whether every hull is such. If this were so, Theorem 1 would follow from the special case of a variety.

As an application we shall indicate a proof of a classical theorem of Hartog's (on the analyticity of a function analytic in each variable) which avoids the use of subharmonic functions. Other applications can be found in [2]. We shall be using the elements of uniform algebras, with its standard terminology and notation as found in the books of Gamelin [4] and Stout [7]; in particular, for X compact in $\mathbb{C}^n$, $P(X)$ and $R(X)$ will denote the uniform closure in $C(X)$ of the polynomials and the rational functions analytic on a neighborhood of X, respectively.

2.

We shall need a quantitative version of the Hartog–Rosenthal theorem. If $(E, \|\cdot\|)$ is a normed linear space, $x \in E$, $A \subseteq E$, then define $\operatorname{dist}(x, A) = \inf\{\|x - a\| : a \in A\}$.

Lemma 2. *Let $K \subseteq \mathbb{C}$ be compact. Then considering $\bar{z}$ as a function in $C(K)$ and $R(K)$ as a subset of $C(K)$, we have*

$$\operatorname{dist}(\bar{z}, R(K)) \leqslant (\lambda(K)/\pi)^{1/2}.$$

Proof. Let ψ be a C^∞ function with compact support in $\mathbb{C}$ such that $\psi(z) = \bar{z}$ on a neighborhood of K. By the generalized Cauchy integral formula

$$\psi(z) = -\frac{1}{\pi}\int \frac{\partial\psi}{\partial\bar{\zeta}}\frac{du\,dv}{\zeta - z}; \qquad z \in \mathbb{C}, \qquad \zeta = u + iv.$$

Restricting attention to points in K and using $(\partial\psi/\partial\bar{\zeta}) \equiv 1$ on K we get

$$\bar{z} = -\frac{1}{\pi}\int_K \frac{du\,dv}{\zeta - z} - \frac{1}{\pi}\int_{\mathbb{C}\setminus K} \frac{\partial\psi}{\partial\bar{\zeta}}\frac{du\,dv}{\zeta - z}.$$

The second integral on the right represents a function in $R(K)$, and, therefore,

$$\operatorname{dist}(\bar{z}, R(K)) \leqslant \left\| \frac{1}{\pi}\int_K \frac{du\,dv}{\zeta - z} \right\|_K. \tag{2.1}$$

By an elegant computation, Ahlfors and Beurling [1, pp. 106–107] have found that the right side of (2.1) is dominated by $(\lambda(K)/\pi)^{1/2}$.
Q.E.D.

Proof of Theorem 1. Let $\epsilon > 0$. For each j, $1 \leqslant j \leqslant n$, we can approximate $\bar{z}$ on $z_j(\hat{X})$ to within $(\lambda(z_j(\hat{X})) + \epsilon)/\pi)^{1/2}$ by a rational function r_j with poles off $z_j(\hat{X})$. Define $f_j(z_1, z_2, ..., z_n) = r_j(z_j)$. Then f_j is analytic on a neighborhood of $\hat{X}$ and, hence, is in $P(\hat{X})$ by the Oka–Weil theorem. Also,

$$\| \bar{z}_j - f_j \|_{\hat{X}} \leqslant ((\lambda(z_j(\hat{X})) + \epsilon)/\pi)^{1/2}. \tag{2.2}$$

Set $f = \sum_1^n z_j f_j \in P(\hat{X})$. Since $0 \in \hat{X}$, evaluation at 0 is a continuous homomorphism φ on $P(\hat{X})$. As $\varphi(z_j) = 0$ for $1 \leqslant j \leqslant n$, it follows that $\varphi(f) = 0$, and, hence, f is not invertible in the Banach algebra $P(\hat{X})$. Consider for points z in X the expression

$$\sum_1^n z_j(\bar{z}_j - f_j). \tag{2.3}$$

Because $\sum | z_j |^2 = 1$ on X, the expression of (2.3) equals $1 - f$ on X. Estimating (2.3) by Schwarz's inequality and applying (2.2) gives

$$\| 1 - f \|_X \leqslant \left(\left(\sum_1^n \lambda(z_j(\hat{X})) + n\epsilon \right) \Big/ \pi \right)^{1/2}. \tag{2.4}$$

Now as f is not invertible in $P(\hat{X})$, $1 \leqslant \| 1 - f \|_{\hat{X}} = \| 1 - f \|_X$. Hence, the right side of (2.4) is $\geqslant 1$. Letting $\epsilon \to 0$ gives the desired result.
Q.E.D.

Remark 1. The conclusion can be slightly improved to read

$$\sum_1^n \lambda(z_j(\hat{X} \cap B)) \geqslant \pi. \tag{2.5}$$

In fact, if $0 < r < 1$, let $X_r = \hat{X} \cap \{z : \| z \| = r\}$. By Rossi's local maximum modulus principle, $\hat{X}_r = \hat{X} \cap \{z : \| z \| \leqslant r\}$. Hence, by applying the theorem (with a scale change) to X_r, we get

$$\sum_1^n \lambda(z_j(\hat{X} \cap \{z : \| z \| \leqslant r\})) \geqslant \pi r^2.$$

Now letting $r \nearrow 1$ gives (2.5).

Remark 2. For our application we need the following form of Theorem 1. Let V be an analytic subvariety of B which contains 0 as a nonisolated point. Then $\sum \lambda(z_j(V)) \geqslant \pi$. To see this, observe that we may assume that V extends to be analytic in a neighborhood of $\bar{B}$. In this case, take $X = \bar{V} \cap \partial B$ and it follows that $0 \in \hat{X}$ and $\hat{X} \cap B = V$. Now we apply Remark 1.

Remark 3. Theorems in several complex variables often have convexity analogs [3]; Shields suggested that this may be the case for Theorem 1 and indeed we have the following.

THEOREM 3. *Let X be a subset of the unit sphere $S^{n-1} = \{p \in \mathbb{R}^n: \| p \| = 1\}$ in $\mathbb{R}^n$. Suppose that* Ch X, *the convex hull of X, contains* 0. *Let l_j = the length of the interval $x_j(\text{Ch } X) \subseteq \mathbb{R}$ (where x_j is the* jth *coordinate projection). Then*

$$\left(\sum_{j=1}^{n} l_j^2\right)^{1/2} \geqslant 2. \tag{2.6}$$

The proof of Theorem 3 is directly analogous to that of Theorem 1 and begins with a real analog of Lemma 2.

LEMMA 4. *Let J be a finite interval in $\mathbb{R}$ of length l. Then there is a real constant c such that*

$$\| x - c \|_J \leqslant \tfrac{1}{2} l.$$

Proof. Choose c to be the midpoint of J. Q.E.D.

Proof of Theorem 3. Let J_j be $x_j(\text{Ch } X)$ and c_j the corresponding constant from Lemma 4. Note $\| x_j - c_j \|_{\text{Ch} X} \leqslant \frac{1}{2} l_j$. Let $f(x) = 1 - \sum_1^n c_j x_j$. Since f is an affine function and $0 \in \text{Ch } X$, it follows that $1 = |f(0)| \leqslant \| f \|_X$. For $x \in X$, $\sum x_j^2 = 1$ and so $f(x) = \sum x_j(x_j - c_j)$. Hence,

$$| f(x)| \leqslant \left(\sum x_j^2\right)^{1/2} \left(\sum (x_j - c_j)^2\right)^{1/2} \leqslant \left(\sum \tfrac{1}{4} l_j^2\right)^{1/2}$$

for $x \in X$. That is $1 \leqslant \| f \|_X \leqslant \frac{1}{2} (\sum l_j^2)^{1/2}$. Q.E.D.

Remark. Examination of the proof shows that equality holds in (2.6) if and only if there is $\alpha = (a_1, a_2, \ldots, a_n) \in S^{n-1}$ such that X is a subset of $\{(\epsilon_1 a_1, \epsilon_2 a_2, \ldots, \epsilon_n a_n): \epsilon_j = \pm 1\}$.

3.

Our proof of Hartog's theorem will depend upon the following proposition. The open unit disc, $\{z \in \mathbb{C}: |z| < 1\}$ will be denoted by U; its n-fold product in $\mathbb{C}^n$, the unit polydisc, by U^n; $\{rz: z \in U\}$ by rU; and the jth coordinate projection in $\mathbb{C}^n$ by z_j. Hence, if $\alpha \in \mathbb{C}$, $z_j^{-1}(\alpha) = \{(\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{C}^n: \zeta_j = \alpha\}$.

PROPOSITION 5. *Let $\{V_k\}$ be a family of analytic subvarieties of U^n without isolated points. Let $0 < r < 1$ be such that $V_k \cap (U^{n-1} \times (rU)) = \varnothing$ for all k. Suppose that for $\alpha \in U$ and $1 \leqslant s \leqslant n-1$, the family $\{V_k \cap z_s^{-1}(\alpha)\}$ of subsets of U^n is locally finite. Then $\{V_k\}$ is locally finite.*

Remark. A special case of this result was obtained by Nishino [5].

Proof. By shrinking the polydisc we may assume, for every $\alpha \in U$ and $1 \leqslant s \leqslant n-1$, that $V_k \cap z_s^{-1}(\alpha)$ is empty for large enough k. We argue by contradiction and assume that there is $x_0 \in U^n$ and points $x_k \in V_k$, $k = 1, 2, \ldots$, converging to x_0. Let L_k, $k = 0, 1, 2, \ldots$, be a biholomorphism of U^n which takes x_k to 0 and which is of the form $L_k(z_1, z_2, \ldots, z_n) = (L_k^1(z_1), L_k^2(z_2), \ldots, L_k^n(z_n))$ where L_k^s is the linear fractional transformation given by $L_k^s(z) = (z - x_k^s)/(1 - \overline{x_k^s} z)$ where $x_k = (x_k^1, x_k^2, \ldots, x_k^n)$. Let $W_k = L_k(V_k)$, an analytic subvariety of U^n containing 0. Therefore, as $B \subseteq U^n$, we get

$$\sum_{j=1}^{n} \lambda(z_j(W_k)) \geqslant \pi, \tag{3.1}$$

for each k. For $1 \leqslant j \leqslant n-1$, the sets $\{z_j(V_k)\} \subseteq U$ eventually omit every point of U as $k \to \infty$. Hence, $\lambda(z_j(W_k)) \to 0$ as $k \to \infty$. It follows from (3.1) that

$$\liminf_{k\to\infty} \lambda(z_n(W_k)) \geqslant \pi. \tag{3.2}$$

On the other hand, as $L_k \to L_0$ uniformly on compact subsets of U^n and as $L_0^n(rU)$ is a neighborhood of $-x_0^n \in U$, it follows (after possibly omitting a finite number of V_k's) that there is a nonempty open subset Ω of U which contains $-x_0^n$ and is such that $L_k^n(rU) \supseteq \Omega$ for all k. Therefore, $z_n(W_k) \cap \Omega = \varnothing$ for all k. This implies that $\lambda(z_n(W_k)) \leqslant \pi - \lambda(\Omega)$, in contradiction to (3.2). Q.E.D.

HARTOG'S THEOREM. *A complex valued function f which is defined on an open subset Ω of $\mathbb{C}^n$ and which is analytic in each variable separately, is analytic.*

Remark. We recall the usual reductions: First, by induction, we may assume the theorem for functions of $n - 1$ variables. We note that it is enough to show that f is locally bounded; for this implies continuity by a simple 1-variable Cauchy integral argument and continuity implies analyticity by expanding the kernel in the iterated Cauchy integral. Next observe that, as analyticity is a local property, it suffices to show that f is locally bounded in a polydisc Δ such that $\bar{\Delta} \subseteq \Omega$. Without loss of generality we may take Δ to be U^n. Setting $M(z_n) = \sup\{|f(z', z_n)|: (z', z_n) \in U^{n-1} \times U\}$ for $z_n \in U$ and applying the Baire category argument, it follows that $M(z_n)$ is uniformly bounded on some nonempty open subset of $\{z_n: |z_n| < 1\}$. By making a change of variable in z_n, we may assume that there exists r with $0 < r < \frac{1}{2}$ and $A > 0$ such that $|f(z', z_n)| < A$ if $z' \in U^{n-1}$ and $|z_n| \leqslant 2r$. It follows that f is analytic on $Q = U^{n-1} \times (2rU)$. For fixed $z' \in U^{n-1}$, $z \to f(z', z)$ is analytic on U and so there is a Taylor series,

$$f(z', z_n) = \sum_{j=0}^{\infty} a_j(z')\, z_n{}^j.$$

As f is analytic on Q, the a_k's are analytic on U^{n-1}.

Proof. In order to show that f is locally bounded on U^n we argue by contradiction; i.e., we suppose that there is $x_0 \in U^n$ and $\{x_k\} \subseteq U^n$ such that $x_k \to x_0$ and $f(x_k) \to \infty$. Let $f_N(z', z_n) = \sum_0^N a_j(z')\, z_n{}^j$. The f_N are analytic on U^n and converge pointwise to f there. As $f(x_k) \to \infty$, there are $N_k \to \infty$ such that $c_k = f_{N_k}(x_k) \to \infty$. Let $V_k = \{z \in U^n: f_{N_k}(z) - c_k = 0\}$, a subvariety of U^n. Since the f_N's are uniformly bounded on $U^{n-1} \times (rU)$ and since $c_k \to \infty$, it follows that $V_k \cap (U^{n-1} \times (rU))$ is empty for large k and by passing to a subsequence it is no loss of generality to assume that these sets are empty for all k. For fixed $\alpha \in U$, $z' \to f(\alpha, z')$ is, by induction, analytic on U^{n-1}. It follows that $\{f_N(\alpha, z')\}$ is uniformly bounded on compact subsets of U^{n-1} and, consequently, that $\{V_k \cap z_1^{-1}(\alpha)\}$ is locally finite. In the same way, for $1 \leqslant s \leqslant n - 1$, $\{V_k \cap z_s^{-1}(\alpha)\}$ is locally finite. By Proposition 5, $\{V_k\}$ is locally finite. But $x_k \in V_k$ and $x_k \to x_0 \in U^n$, a contradiction. Q.E.D.

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