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# A Classification of Involutive Automorphisms of an Affine Kac–Moody Lie Algebra

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In this thesis we classify the conjugacy classes of involutions in Aut $\mathbf{g}$, where $\mathbf{g}$ is an affine Kac–Moody Lie Algebra. We distinguish between two kinds of involutions, those which preserve the conjugacy class of a Borel subalgebra and those which don't.

We give a complete and non-redundant list of representatives of involutions of the first kind and we compute their fixed points sets. We prove that any involution of the first kind has a conjugate which leaves invariant the components of the Gauss decomposition $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$. We also give a complete list of representatives of the conjugacy classes of involutions of the second kind. © 1988 Academic Press, Inc.

*Contents.* *Introduction.* *Part I.* 1. Coxeter systems and Tits systems. 2. Affine Weyl groups. *Part II.* 1. Kac–Moody Lie algebras. 2. The Tits system associated with a Kac–Moody algebra. 3. Affine Lie algebras. *Part III.* 1. Existence of a $\sigma$-invariant pair $\{\mathbf{h} \subset \mathbf{b}\}$. 2. Involutive automorphisms of the first kind. 3. Involutive automorphisms of the second kind. 4. Realizations of classical involutions. 5. Fixed points sets. Appendix A: Tits buildings. Appendix B: Fixed point lemma.

## INTRODUCTION

In 1968 V. Kac and R. Moody introduced a new class of infinite-dimensional Lie algebras that generalized the finite-dimensional semisimple Lie algebras. An important subclass of infinite-dimensional Lie algebras among Kac–Moody algebras is the so-called affine Lie algebras. In this thesis we consider the problem of classifying the automorphisms of order two of an affine Lie algebra.

In the finite-dimensional case this problem was solved by E. Cartan, who used it as a main tool for the classification of symmetric spaces.

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An important fact in the finite-dimensional case is that given an automorphism $\sigma$, there exist a Borel subalgebra $\mathbf{b}$ and a Cartan subalgebra $\mathbf{h}$ contained in $\mathbf{b}$ that are stable under $\sigma$ [D, 21.18, Problem 3]. In our case we have the concepts of Borel subalgebra and Cartan subalgebra but the claim above is not true in general.

We say that $\sigma \in \operatorname{Aut} \mathbf{g}$ is of the first or the second kind depending on whether $\sigma$ leaves stable the conjugacy class of a Borel subalgebra.

Associating a Tits system to a Kac–Moody algebra $\mathbf{g}$, we can use a result of Bruhat and Tits on the existence of a fixed point for any bounded group of isometries of a building to prove that when $\sigma$ is of the first kind and of finite order we can find a $\sigma$-invariant pair $\{\mathbf{h} \subset \mathbf{b}\}$. We can then use this result to elaborate a list of representatives of the conjugacy classes of involutive automorphisms of the first kind. Finally, looking at the fixed points sets for the different classes of automorphisms, we conclude that they are not conjugated under $\operatorname{Aut} \mathbf{g}$.

In computing the fixed points set we use a realization of the automorphisms determined by a symmetry of the Dynkin diagram which shows that the restriction of $\sigma$ to $\mathbf{n}_+$, ($\mathbf{b}_+ = \mathbf{n}_+ \oplus \mathbf{h}$) can be considered as an automorphism over the polynomial algebra $\mathbb{C}[t]$.

Thanks to a result by Peterson and Kac on the conjugacy classes of Borel subalgebras we can find, for any involution $\sigma$ of the second kind, a conjugate of the Cartan subalgebra $\mathbf{h}$ which is invariant under $\sigma$. A complete list of involutions of the second kind is given, but because it was not possible to give a description of the fixed points sets, we could not check whether this list was redundant.

In Part I we give the definitions of Coxeter systems, Tits systems, and affine Weyl groups.

Part II contains the elements of the theory of Kac–Moody algebras, the connection between them and Tits systems, and a realization of the affine Lie algebras.

In Part III we prove the theorem on the existence of an invariant pair $\{\mathbf{h} \subset \mathbf{b}\}$, and we proceed to classify the involutions of the first kind and list a set of representatives together with their fixed points sets in Table V. We also classify the involutions of the second kind and list them in Table VI.

Finally, for the convenience of the reader, we added appendixes with the necessary background on the theory of buildings.

# PART I

## 1. Coxeter Systems and Tits Systems

We will recall some facts about Coxeter and Tits systems. The proofs can be found in [B, Chap. IV] (denoted in this chapter by [B]).

1.1. DEFINITION. A *Coxeter system* is a pair $(W, S)$ where $W$ is a group and $S$ is a part of $W$ satisfying the following axioms [B, Sect. 1, #3]:

(i) $S$ generates $W$ and the elements of $S$ have order two;

(ii) let $m(s, s')$ be the order of the element $ss' \in W$ and let $I$ be the set of pairs $(s, s')$ such that $m(s, s')$ is finite, then the set of generators $S$ and the relations $ss'^{m(s,s')} = 1$ for every $(s, s')$ in $I$ form a presentation of $W$.

1.2. For every subset $X \subset S$ we denote by $W_X$ the subgroup of $W$ generated by $X$. We have $W_X \cap S = X$ and $W_X \cap W_Y = W_{X \cap Y}$ for all $X, Y \subset S$ [B, Sect. 1, #8, Theorem 2].

1.3. DEFINITION. The *Coxeter graph*, $\operatorname{Cox}(W, S)$, of the Coxeter system $(W, S)$ is the pair $(G, f)$ obtained as follows: $G$ is the graph whose vertices are the elements of $S$, two different vertices being joined by a lace, if and only if the corresponding elements do not commute; $f$ is the map $\{s, s'\} \to m(s, s')$ from the set of laces to the set formed by $\infty$ and the integers $>2$ [B, Sect. 1, #9].

1.4. DEFINITION. A *Tits system* is a quadruplet $(G, B, N, S)$ where $G$ is a group, $B$ and $N$ are subgroups of $G$, and $S$ is a subset of $N/(B \cap N)$, satisfying the following axioms [B, Sect. 2, #1]:

(T1) the set $B \cup N$ generates $G$ and $B \cap N$ is a normal subgroup of $N$;

(T2) the set $S$ generates the group $W = N/(B \cap N)$ and it consists of elements of order two;

(T3) we have $sBwB \subset BwB \cup BswB$, for all $s$ in $S$, $w$ in $W$;

(T4) for every $s$ in $S$ we have $sBs \neq B$.

The group $W$ is called the *Weyl group* of the Tits system. The pair $(W, S)$ is a Coxeter system [B, Sect. 2, #4, Theorem 2].

In the sequel we will denote by $(G, B, N, S)$ a Tits system, by $W$ its Weyl group, and by $\nu\colon N \to W$ the canonical homomorphism.

1.5. The map $w \to BwB$ is a bijection between $W$ and the set $B \backslash G/B$ of double classes of $G$ modulo $B$ [B, Sect. 2, #3, Theorem 1].

1.6. For any part $X$ of $S$ let $B_X = BW_XB = \bigcup_{w \in W_X} BwB$. The map $X \to B_X$ is a bijection between the set of subsets of $S$ and the set of subgroups of $G$ containing $B$, and we have

$$B_X \cap B_Y = B_{X \cap Y}, \qquad \text{for all} \quad X, Y \subset S \text{ [B, Sect. 2, \#5, Theorem 3].}$$

1.7. DEFINITION. A subgroup $P$ of $G$ is called *parabolic* if it contains a conjugate of $B$.

For such a $P$ there exists a well-defined subset $X \subset S$ such that $P$ is conjugated to $B_X$ [B, Sect. 2, #6, Proposition 4]. We call $X$ the type of $P$.

1.8. DEFINITION. A homomorphism $\varphi$: $G \to \hat{G}$ ($\hat{G}$ a group) is called *B-adapted* if it satisfies the following two conditions [BT, (1.2.13)]:

(i) the kernel of $\varphi$ is contained in $B$;

(ii) for every $g \in \hat{G}$, there exists $h \in G$ such that $\varphi(hBh^{-1}) = g\varphi(B)g^{-1}$.

1.9. Let $\varphi$ denote a $B$-adapted homomorphism $\varphi$: $G \to \hat{G}$. For every parabolic subgroup $P \subset G$ and for every $g \in \hat{G}$ the preimage $\varphi^{-1}(g\varphi(P)g^{-1})$ is a parabolic subgroup of $G$ denoted by ${}^gP$ [BT, (1.2.15)].

1.10. For every $g$ in $\hat{G}$ there exists a permutation $\xi(g)$ of $W$ such that $\varphi(B\xi(g)wB) = \varphi(h^{-1})g\varphi(BwB)g^{-1}\varphi(h)$, for all $w$ in $W$, $h$ in $G$ satisfying 1.8(ii). In fact $\xi(g)$ is an automorphism of the Coxeter system $(W, S)$ and is a homomorphism from $\hat{G}$ into the group of automorphisms of $(W, S)$ acting on the Coxeter graph of $(W, S)$ [BT, (1.2.16)].

If $P$ is a parabolic subgroup of type $X \subset S$, the subgroup ${}^gP$ is of type $\zeta(g)X$ [BT, (1.2.18)].

## 2. AFFINE WEYL GROUPS

2.1. DEFINITIONS AND NOTATIONS. Let $A$ be an affine space, together with a distance that comes from a positive definite scalar product on the space of translations of $A$.

Let $W$ be a discrete subgroup of the group of affine transformations of $A$ generated by orthogonal reflections with respect to affine hyperplanes of $A$. We will assume that $W$ acts irreducibly on $A$.

If $L$ is an affine hyperplane of $A$ we denote by $s_L$ the orthogonal reflection that leaves fixed each point of $L$. Reciprocally, if $s$ is an orthogonal reflection of $A$ we denote its fixed point set by $L_s$.

A *wall* in $A$, with respect to $W$, will be a hyperplane $L$ such that $s_L$ belongs to $W$.

An *affine root* of $A$ is any closed half-space of $A$ bounded by a wall which is called the root's wall. The set of affine roots will be denoted by $\Sigma$.

If $\alpha \in \Sigma$ we write $r_\alpha = s_{\partial\alpha}$, where $\partial\alpha$ is the wall of $\alpha$.

A *facet* $F$ of $A$ is an equivalence class in $A$ for the relation $x \sim y$ if and only if $x$ and $y$ are contained in the same affine roots. A facet $F$ is then a convex set, open in the affine subspace that it generates, called the support of $F$.

The set of facets $\mathscr{F}$ has the order relation $F < F'$ if $F \subset \bar{F}'$ were $\bar{F}'$ is the closure of $F'$ in $A$.

The *chambers* of $A$ are the connected components of the complement in $A$ of the union of walls, and we call *faces* of $A$ the facets of $A$ whose support is an affine hyperplane of $A$ [BT, (1.3.3)].

2.2. Fix a chamber $C$ of $A$, and $\bar{C}$ is a fundamental domain for the action of $W$ on $A$ [B, Chap. V, Sect. 3, #3, Theorem 2]. Furthermore, $W$ is generated by the set $S$ of reflections corresponding to the walls that are the support of the faces of $\bar{C}$.

2.3. The pair $(W, S)$ is a Coxeter system [B, Chap. V, Sect. 3, #2, Theorem 1].

2.4. We will assume that the Coxeter graph $(G, f)$ of $(W, S)$ is connected.

2.5. If $W$ is infinite we say that $W$ is an *affine Weyl group.*

2.6. $A$, together with $\mathscr{F}$, the order relation, and the canonical affine structure on each of the sets $F$, is a simplicial complex.

2.7. For each facet $F$ there is exactly one facet $\bar{C}$ transformed to $F$ by an element of $W$ [BT, (1.3.5)].

2.8. For each proper subset $X$ of $S$, the set

$$C_X = \{a \in \bar{C} \colon X = \{s \in S \colon a \in L_s\}\} \text{ is a facet of } \bar{C}.$$

Furthermore, the map $X \to C_X$ is a bijection of the set

$$T = \{X \subset S \colon X \neq S\} = \{X \subset S \colon W_X \text{ is finite}\}$$

onto the set of facets of $\bar{C}$ ($\varnothing \to C$).

We say that a facet $F$ has type $X$, for $X \subset T$, if $F$ is a $W$-translate of $C_X$. Then $T$ is the set of types of facets [BT, (1.3.5)].

2.9. The stabilizer of $C_X$ in $W$, i.e., $\{w \text{ in } W \colon w(C_X) \subset C_X\}$, is the subgroup $W_X$ [B, Chap. V, Sect. 3, #3, Proposition 1].

In the sequel $W$ will denote an affine Weyl group acting irreducibly on the affine space $A$.

# PART II

## 1. Kac–Moody Lie Algebras

1.1. Let $A=(a_{ij})_{i,j\in I}$ be an integral $n\times n$ matrix of rank $P$ indexed by the finite set $I$. We will associate with it a Lie algebra $\mathbf{g}(A)$. The matrix $A$ is called a *generalized Cartan matrix* if it satisfies the following:

(C1) $a_{ii}=2$ for $i=1, ..., n$;

(C2) $a_{ij}$ are non-positive integers for $i\neq j$;

(C3) $a_{ij}=0$ implies $a_{ji}=0$.

A *realization* of $A$ is a triple $(\mathbf{h}, \pi, \pi^\vee)$, where $\mathbf{h}$ is a finite-dimensional complex vector space and $\pi=\{\alpha_i\}_{i\in I}\subset\mathbf{h}^*$ and $\pi^\vee=\{\alpha_i^\vee\}_{i\in I}\subset\mathbf{h}$ are indexed subsets in $\mathbf{h}^*$ and $\mathbf{h}$, respectively, and satisfy

1.1.1. both sets $\pi$ and $\pi^\vee$ are linearly independent;

1.1.2. $\langle\alpha_i, \alpha_j^\vee\rangle=a_{ji}$, $i, j=1, , n$;

1.1.3. $n-\operatorname{rank} A=\dim\mathbf{h}-n$.

1.2. For any $n\times n$ matrix $A$ there exists a unique, up to isomorphism, realization [K, 1].

1.2.1. Given two matrices $A$ and $A'$ and their realizations $(\mathbf{h}, \pi, \pi^\vee)$ and $(\mathbf{h}', \pi', \pi^{\vee\prime})$, we obtain a realization of the direct sum of two matrices $(\mathbf{h}\oplus\mathbf{h}', \pi\times\{0\}\cup\{0\}\times\pi', \pi^\vee\times\{0\}\cup\{0\}\times\pi^{\vee\prime})$, which is called the *direct sum* of the realizations.

A matrix $A$ is called *decomposable* if after reordering the indices, $A$ decomposes into a non-trivial direct sum. We can always decompose a matrix $A$ into a direct sum of indecomposable matrices and the corresponding realization into a direct sum of the corresponding realizations.

1.2.2. $\pi$ is called the *root basis*, $\pi^\vee$ the *dual root basis*, and elements from $\pi$ (resp. $\pi^\vee$) are called *simple roots* (resp. *simple co-roots*). We also set

$$Q=\bigoplus_{i\in I}\mathbb{Z}\alpha_i,\qquad Q_+=\bigoplus_{i\in I}\mathbb{Z}_+\alpha_i,\qquad Q^\vee=\bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee.$$

The lattice $Q$ is called the *root lattice*.

For $\alpha=\sum k_i\alpha_i\in Q_+$, the number $\operatorname{ht}\alpha=\sum k_i$ is called the *height* of $\alpha$.

1.3.1. Let $A=(a_{ij})$ be a generalized Cartan matrix and let $(\mathbf{h}, \pi, \pi^\vee)$ be a realization of $A$. Let $\mathbf{g}(A)$ be a Lie algebra with generators $e_i$, $f_i$ $(i=1, ..., n)$, and $\mathbf{h}$ and the following defining relations:

$$\begin{aligned}
[e_i, f_j] &= \delta_{i,j}\alpha_i^\vee, && i, j = 1, ..., n\\
[h, h'] &= 0, && h, h' \text{ in } \mathbf{h}\\
[h, e_i] &= \langle \alpha_i, h\rangle\, e_i, && i = 1, ..., n, h \text{ in } \mathbf{h}\\
[h, f_i] &= -\langle \alpha_i, h\rangle\, f_i, && i = 1, ..., n, h \text{ in } \mathbf{h}\\
(\mathrm{ad}\, e_i)^{1-a_{ij}}(e_j) &= 0, && i \neq j\\
(\mathrm{ad}\, f_i)^{1-a_{ij}}(f_j) &= 0, && i \neq j.
\end{aligned}$$

1.4. Definition. The Lie algebra $\mathbf{g}(A)$ is called a Kac–Moody algebra. The subalgebra $\mathbf{h}$ of $\mathbf{g}(A)$ is called the *Cartan subalgebra*. The matrix $A$ is called the *Cartan matrix* of the Lie algebra $\mathbf{g}(A)$. $n$ is called the rank of $\mathbf{g}(A)$.

The elements $e_i, f_i$ $(i = 1, ..., n)$ are called *Chevalley generators*, and they generate the subalgebra $\mathbf{g}'(A) = [\mathbf{g}(A), \mathbf{g}(A)]$. One has $\mathbf{g}(A) = \mathbf{g}'(A) + \mathbf{h}$, $\mathbf{g}(A) = \mathbf{g}'(A)$ iff $\det A \neq 0$.

We set $\mathbf{h}' = \bigoplus_{i \in I} \mathbb{C}\alpha_i$. One has

$$\mathbf{g}'(A) \cap \mathbf{h} = \mathbf{h}', \qquad \mathbf{g}'(A) \cap g_\alpha = g_\alpha \qquad \text{if} \quad \alpha \neq 0.$$

With respect to $\mathbf{h}$, we have the root space decomposition

$$\mathbf{g}(A) = \bigoplus_{\alpha \in Q} g_\alpha. \tag{1.4.1}$$

$g_\alpha$ is the *root space* attached to $\alpha$. The number $\mathrm{mult}\,\alpha = \dim g_\alpha$ is the *multiplicity* of $\alpha$. An element $\alpha$ in $Q$ is called a *root* if $\alpha \neq 0$ and $\mathrm{mult}\,\alpha > 0$.

A root $\alpha$ in $Q_+$ (resp. $-\alpha$ in $Q_+$) is called *positive* (resp. *negative*). Denote by $\Delta$, $\Delta_+$, $\Delta_-$ the set of all roots, positive roots, and negative roots, respectively, and then $\Delta = \Delta_+ \cup \Delta_-$, $\Delta_+ \cap \Delta_- = \emptyset$.

Let $\mathbf{n}_+$ (resp. $\mathbf{n}_-$) be the subalgebra of $\mathbf{g}(A)$ generated by $e_1, ..., e_n$ (resp. $f_1, ..., f_n$) and then we have the *Gauss decomposition*

$$\mathbf{g}(A) = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+. \tag{1.4.2}$$

The map $e_i \to -f_i$, $f_i \to -e_i$ $(i = 1, ..., n)$, $h \to -h$, $h$ in $\mathbf{h}$, can be uniquely extended to an involution $\omega$ of the Lie algebra $\mathbf{g}(A)$. $\omega$ is called the *Cartan involution of* $\mathbf{g}(A)$.

1.4.3. Let $\mathrm{Aut}(A)$ be the group of all permutations $\sigma$ of $I$ satisfying $a_{\sigma(i),\sigma(j)} = a_{i,j}$. We regard $\mathrm{Aut}(A)$ as a subgroup of $\mathrm{Aut}(\mathbf{g}')$ by requiring $\sigma(e_i) = e_{\sigma(i)}$, $\sigma(f_i) = f_{\sigma(i)}$. We define the outer automorphism group $\mathrm{Out}(A)$ of $\mathbf{g}'$ to be $\mathrm{Aut}(A)$ if $\dim \mathbf{g}' < \infty$ and $\{\mathrm{id}, \omega\} \times \mathrm{Aut}(A)$ otherwise.

## 1.5. *The Symmetric Bilinear Form*

An $n\times n$ matrix $A=(a_{ij})$ is called *symmetrizable* if there exists a non-degenerate diagonal matrix $D=\operatorname{diag}(\varepsilon_1, ..., \varepsilon_n)$ and a symmetric matrix $B=(b_{ij})$ such that $A=DB$.

Let $A$ be symmetrizable and let $(\mathbf{h}, \pi, \pi^\vee)$ be a realization of $A$. Fix a complementary subspace $\mathbf{h}''$ to $\mathbf{h}'=\bigoplus_{i\in I}C\alpha_i^\vee$ in $\mathbf{h}$, then

$$(1.5.1)\qquad \begin{aligned} &(\alpha_i^\vee, h)=\langle\alpha_i, h\rangle\,\varepsilon_i, &&h \text{ in } \mathbf{h},\ i=1, ..., n\\ &(h', h'')=0, &&h', h'' \text{ in } \mathbf{h}''\end{aligned}$$

defines a non-degenerate bilinear $\mathbb{C}$ valued form ( , ) on $\mathbf{h}$.

This gives an isomorphism $v$: $\mathbf{h}\to\mathbf{h}^*$ defined by $\langle v(h), h_i\rangle=(h, h_i)$, $h$, $h_i$ in $\mathbf{h}$, as well as the induced bilinear form on $\mathbf{h}$.

## 1.6. *The Weyl Group of a Kac–Moody Algebra*

For each $i=1, ..., n$ we define a fundamental reflection $r_i$ of the space $\mathbf{h}^*$ by

$$(1.6.1)\qquad r_i(\alpha)=\alpha-\langle\alpha, \alpha_i^\vee\rangle\,\alpha_i \qquad \forall\alpha\in\mathbf{h}^*.$$

The subgroup $W\subset GL(\mathbf{h}^*)$ generated by $r_i$, $i=1, ..., n$, is called the Weyl group of $\mathbf{g}(A)$.

1.6.2. The bilinear form in 1.5.1 is $W$ invariant [K, Proposition 3.9].

1.6.3. We define the set of real roots as $\Delta^{\mathrm{re}}=\{\alpha\in\Delta\colon \alpha=\pm w(\beta),\ w$ in $W$, $\beta$ in $\pi\}$ and the set of imaginary roots as $\Delta^{\mathrm{im}}=\Delta\backslash\Delta^{\mathrm{re}}$.

Let $(\mathbf{h}_{\mathbb{R}}, \pi, \pi^\vee)$ be a realization of the matrix $A$ over $\mathbb{R}$, i.e., $\mathbf{h}_{\mathbb{R}}^*$ is a real vector space of dimension $2n-\operatorname{rank} A$, so that $\mathbf{h}=\mathbb{C}\otimes\mathbf{h}_{\mathbb{R}}$, then $Q^\vee\subset\mathbf{h}_{\mathbb{R}}$ and therefore $W$ actos on $\mathbf{h}_{\mathbb{R}}^*$.

1.6.4. The set $C=\{h$ in $\mathbf{h}_{\mathbb{R}}\colon \langle\alpha_i, h\rangle\geqslant 0,\ i=1, ..., n\}$ is called a fundamental chamber. The sets $w(C)$ ($w$ in $W$) are called chambers and the set $X=\bigcup_{w\in W}w(C)$ is called the Tits cone. We have the dual notions of $C^\vee$ and $X^\vee$ in $\mathbf{h}_{\mathbb{R}}^*$.

1.6.5. The group $W$ is a Coxeter group [K, Proposition 3.13].

# 2. The Tits System Associated with a Kac–Moody Algebra

Here we will construct the adjoint group $G$ associated to a Kac–Moody algebra $\mathbf{g}(A)$, and we will associate to it a Tits system $(G, B, N, S)$.

2.1. Let $(\mathbf{g}(A), \mathrm{ad})$ be the adjoint representation of $\mathbf{g}(A)$. Because of the defining relations of $\mathbf{g}(A)$ (1.3.1) we have that the expressions

$\exp \mathrm{ad}(te_i)v = \sum_{n \geq 0} (1/n!)\, t^n (\mathrm{ad}\, e_i)^n v$ and, $\exp \mathrm{ad}(tf_i)v = \sum_{n \geq 0} (1/n!)\, t^n (\mathrm{ad}\, f_i)^n v$ are well-defined automorphisms for any $t$ in $\mathbb{C}$, $i$ in $I$, $v$ in $\mathbf{g}(A)$. This implies that $\exp(\mathrm{ad}\, x)v = \sum_{n \geq 0} (1/n!)(\mathrm{ad}\, x)^n v$ is well defined for all $v$ in $\mathbf{g}(A)$, $x$ in $g_\alpha$, $\alpha$ in $\Delta^{\mathrm{re}}$.

2.2. Let $G$ be the subgroup of $\mathrm{Aut}(\mathbf{g}(A))$ generated by

$$\{\exp \mathrm{ad}(te_i), \exp \mathrm{ad}(tf_i), i \in I, t \in \mathbb{C}\}.$$

2.3. Definition. $G$ is called the *adjoint group* associated to the Kac–Moody Lie algebra $\mathbf{g}(A)$.

2.4. The subgroup $U_\alpha = \exp g_\alpha \subset G$ is an additive one-parameter subgroup of $G$. The $U_\alpha$, $\alpha \in \pm\pi$, generate $G$, and $G$ is its own derived subgroup [PK, 2].

Denote by $U_+$ (resp. $U_-$) the subgroup of $G$ generated by $U_\alpha$ (resp. $U_{-\alpha}$), $\alpha \in \Delta^{\mathrm{re}}_+$.

2.5. For each $i$ in $I$ we have a unique homomorphism

$$\varnothing_i : SL(\mathbb{C}) \to G$$

satisfying

$$\varnothing_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp(te_i), \qquad \varnothing_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp(tf_i), \qquad t \in C.$$

2.6. Let $G_i = \varnothing_i(SL(\mathbb{C}))$, $H_i = \varnothing_i(\{\mathrm{diag}(t, t^{-1}) \colon t \in C\})$ and let $N_i$ be the normalizer of $H_i$ in $G$. Let $H$ (resp. $N$) be the subgroup of $G$ generated by the $H_i$ (resp. $N_i$) $H$ is an abelian normal subgroup of $N$, and it is the direct product of the $H_i$, $i \in I$ [PK, 2]. There is an isomorphism $\varnothing$: $W \to N/H$ such that $\varnothing(r_i) = N_i H/H$. We identify $W$ and $N/H$ by $\varnothing$ and put $B_+ = HU_+$, $B_- = HU_-$, $S = \{\varnothing(r_i)\}_{i \in I}$.

*Remark.* In [PK] the simple connected group associated to $\mathbf{g}(A)$ is defined, and its quotient by the center gives our group. It follows that all the results listed below are an immediate consequence of the corresponding results from [PK].

2.7. The quadruple $(G, B_+, N, S)$ is a Tits system.

$$G = \bigcup_{w \in W} B_+ w B_+ \qquad \text{(Bruhat decomposition)}$$

$$G = \bigcup_{w \in W} B_+ w B_- \qquad \text{(Birkhoff decomposition) [PK, Corollary 2].}$$

2.8. $\tilde{H} = \mathrm{Hom}(Q, C)$ can be viewed as a subgroup of Aut $(\mathbf{g})$ in the following way: $h \cdot v = h(\alpha)v$, $h$ in $\tilde{H}$, $v$ in $g_\alpha$, $\alpha \in \Delta$. Therefore $\tilde{H}$ acts on $G$ and determines the group $\tilde{H} \propto G$.

2.9. $\mathrm{Aut}(\mathbf{g}'(A)/\mathbf{c}) = \mathrm{Out}(\mathbf{g}'(A)) \propto (\tilde{H} \propto G)$ [PK, Theorem 2].

## 3. Affine Lie Algebras

3.1. We consider the case of a Kac–Moody algebra $\mathbf{g}(A)$, where $A$ is a generalized Cartan matrix with all its proper principal minors positive and $\det A = 0$. They are called affine Lie algebras and Tables I, II, and III list the corresponding Dynkin diagrams [K, Theorem 4.8].

3.2. We will proceed to construct a realization of them:

Let $L = C[t, t^{-1}]$ be the algebra of Laurent polynomials in $t$. Recall that the residue of a Laurent polynomial $P = \sum_{r > j > s} c_j t^j$ is defined as $\mathrm{Res}\, P = c_{-1}$.

Let $\mathring{\mathbf{g}}$ be a finite-dimensional simple Lie algebra, then $L(\mathring{\mathbf{g}}) = L \otimes \mathring{\mathbf{g}}$ is an infinite-dimensional Lie algebra with the bracket

$$[P \otimes X, Q \otimes Y] = PQ \otimes [X, Y], \qquad P, Q \text{ in } L,\ X, Y \text{ in } \mathring{\mathbf{g}}.$$

Fix a non-degenerate, invariant, symmetric bilinear form in $\mathring{\mathbf{g}}$ $(\cdot,\cdot)$, and we extend this form to an $L$ valued form on $L(\mathring{\mathbf{g}})$, $(\cdot,\cdot)_t$ by

$$(P \otimes X, Q \otimes Y)_t = PQ(X, Y), \qquad P, Q \text{ in } L,\ X, Y \text{ in } \mathring{\mathbf{g}}.$$

The derivation $t^j(d/dt)$ of $L$ extends to $L(\mathring{\mathbf{g}})$ by

$$t^j \frac{d}{dt}(P \otimes X) = t^j \frac{dP}{dt} \otimes X, \qquad P \text{ in } L,\ X \text{ in } \mathring{\mathbf{g}}.$$

Therefore $\psi(a, b) = (da/dt, b)$ $a$, $b$ in $L(\mathring{\mathbf{g}})$ defines a cocycle on $L(\mathring{\mathbf{g}})$ [K, 7, Corollaries 1 and 2].

Denote by $\tilde{L}(\mathring{\mathbf{g}})$ the central extension of the Lie algebra $L(\mathring{\mathbf{g}})$ associated to the cocycle $\psi$. That is, $\tilde{L}(\mathring{\mathbf{g}}) = L(\mathring{\mathbf{g}}) \oplus Cc$ with the bracket $[a + \lambda c, b + \mu c] = [a, b] + \psi(a, b)c$ $a$, $b$ in $L(\mathring{\mathbf{g}})$, $\lambda$, $\mu$ in $\mathbb{C}$.

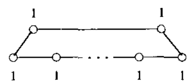
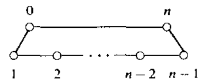
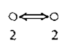
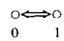
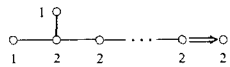
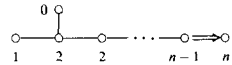
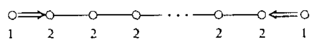
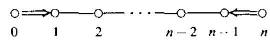
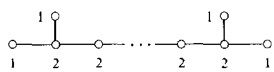
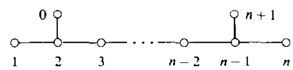
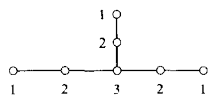
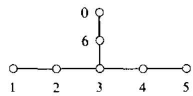
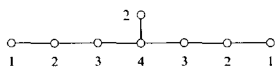
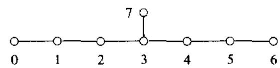
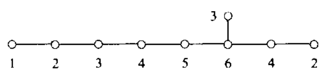
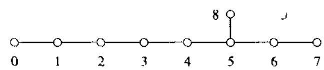
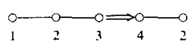
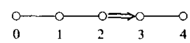
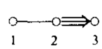
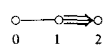
Finally, denote by $\hat{L}(\mathring{\mathbf{g}})$ the Lie algebra which is obtained by adjoining to $\tilde{L}(\mathring{\mathbf{g}})$ a derivation $d$ which acts on $L(\mathring{\mathbf{g}})$ as $t(d/dt)$ and kills $c$.

Explicitly we have $\hat{L}(\mathring{\mathbf{g}}) = L(\mathring{\mathbf{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ with the bracket defined by ($X$, $Y$ in $\mathring{\mathbf{g}}$; $\lambda$, $\mu$, $\lambda_1$, $\mu_1$, in $\mathbb{C}$, $k$, $j$ in $\mathbb{Z}$) [K, (7.2.1)]:

$$\begin{aligned}&[t^k \otimes X + \lambda c + \mu d, t^j \otimes Y + \lambda_1 c + \mu_1 d]\\ &\quad = t^{k+j} \otimes [X, Y] + \mu j t^j \otimes Y - \mu_1 k t^k \otimes X + k\, \delta_{j,-k}(X, Y)c.\end{aligned}$$

TABLE I

Affine

| Algebra | Dynkin Diagram | Vertices Numeration |
|---|---|---|
| $a_n^{(1)}$ |  |  |
| $a_1^{(1)}$ |  |  |
| $b_n^{(1)}$ |  |  |
| $c_n^{(1)}$ |  |  |
| $d_{n+1}^{(1)}$ |  |  |
| $e_6^{(1)}$ |  |  |
| $e_7^{(1)}$ |  |  |
| $e_8^{(1)}$ |  |  |
| $f_4^{(1)}$ |  |  |
| $g_2^{(1)}$ |  |  |

$\hat{L}(\mathring{g})$ is an affine Lie algebra and every Kac–Moody algebra from Table I is obtained in this way [K, Theorem 7.4].

### 3.3. *Twisted Affine Lie Algebras*

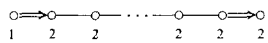
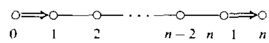
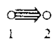
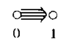
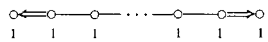
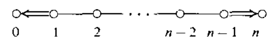
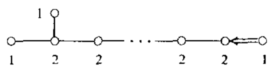
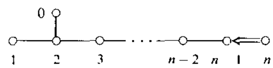
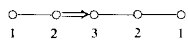
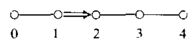
An affine Lie algebra from Tables II and III is realized as the fixed point set of the automorphism $\tilde{\rho}_0$ of $L(\mathring{g})$ determined by the conditions $\tilde{\rho}_0(t^k \otimes Y) = (-t)^k \otimes \rho(Y)$, $\tilde{\rho}_0|_{\mathbb{C}c+\mathbb{C}d} = \mathrm{id}$, where $\rho$ is in Aut$(X)$ and $X$ is the Cartan matrix of $\mathring{g}$ [K, Theorem 8.3]. The Lie algebra thus obtained is denoted by $\hat{L}(\mathring{g}, \rho)$ and it is called a twisted affine Lie algebra.

The bilinear form from Section 1.5 gives a positive semidefinite symmetric bilinear form ( , ), on the real vector space $E$ generated by the roots in $\Delta \subset h^*$. We know that the subspace $E_0 = \{x \text{ in } E\text{: } (x, y) = 0 \text{ for all } y \text{ in } E\}$ is generated by an imaginary root $\delta$. Consider the spaces in $E^*$,

$$A = \{x \text{ in } E^*\text{: } \langle x, \delta \rangle = 1\} \qquad \text{and} \qquad T = \{x \text{ in } E^*\text{: } \langle x, \delta \rangle = 0\},$$

TABLE II

Affine

| Algebra | Dynkin Diagrams | Numeration |
|---|---|---|
| $a_{2n}^{(2)}, n>1$ |  |  |
| $a_2^{(2)}$ |  |  |
| $d_{n+1}^{(2)}, n>1$ |  |  |
| $a_{2n-1}^{(2)}, n>2$ |  |  |
| $e_6^{(2)}$ |  |  |

then $T$ acts on $A$ by translations and the induced bilinear form on $E/E_0$ is non-degenerate, hence it gives a positive definite symmetric bilinear form on $(E/E_0)^* \simeq T$ and this gives $A$ the structure of an affine Euclidean space.

The Weyl group $W$ fixes $\delta$ and therefore its action on $E$ induces an action on $A$ by affine transformations; as $W$ is infinite, it is an affine Weyl group. This is why the Kac–Moody algebras described above are called affine Lie algebras.

We can generalize the construction of $L(\mathring{\mathbf{g}}, \rho)$ to the cases where $\mathring{\mathbf{g}} = \mathbf{g}_1 + \mathbf{g}_2 + \cdots + \mathbf{g}_s + \mathbf{c}$ is a direct sum of simple Lie algebras $\mathbf{g}_i$, and $\mathbf{c}$, the center of $\mathring{\mathbf{g}}$, is at most one dimensional, $\mathbf{c} = \mathbb{C}c$, and $\rho = \rho_1 + \rho_2 + \cdots + \rho_s$ is in $\mathrm{Aut}(M)$, where $M$ is the Cartan matrix of $\mathbf{g}'$. Indeed, $L(\mathring{\mathbf{g}})$ and $d$ are defined as before and we use a bilinear form for which the $\mathbf{g}_i$'s are pairwise orthogonal to define $\psi$, if $\mathbf{c} = \{0\}$, then $\tilde{L}(\mathring{\mathbf{g}}) = L(\mathring{\mathbf{g}})$. The bracket on $\tilde{L}(\mathring{\mathbf{g}})$ is still defined as

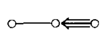
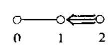
$$[t^k \otimes X, t^j \otimes Y] = t^{k+j} \otimes [X, Y] + \psi(t^k \otimes X, t^j \otimes Y)c,$$
$$j, k \in \mathbb{Z},\ X, Y \in \mathring{\mathbf{g}}.$$

We extend $\tilde{L}(\mathring{\mathbf{g}})$ to $\hat{L}(\mathring{\mathbf{g}})$ by adding the derivation $d$.

Finally, we denote by $\hat{L}(\mathring{\mathbf{g}}, \rho)$ the fixed point set in $\hat{L}(\mathring{\mathbf{g}})$ of the automorphism $\tilde{\rho}$ defined as $\tilde{\rho}(t^k \otimes X) = (-t^k) \otimes \rho(X)$, $\tilde{\rho}\,|_{\mathbb{C}c + \mathbb{C}d} = \mathrm{id}$.

TABLE III

Affine

| | | |
|---|---|---|
| $d_4^{(3)}$ |  |  |

# PART III

## 1. Existence of a $\sigma$-Invariant Pair $\{\mathbf{h} \subset \mathbf{b}\}$

We will distinguish two kinds of automorphisms, those $\sigma$ for which $[\mathbf{b}_+ : \sigma(\mathbf{b}_+) \cap \mathbf{b}_+] < \infty$ and those for which $[\mathbf{b}_+ : \sigma(\mathbf{b}_-) \cap \mathbf{b}_+] < \infty$. From the description of $\mathrm{Aut}(\mathbf{g}'(A))$ in 2.9, it follows that any automorphism of $g$ falls into one of these classes. In the following they will be referred to as automorphisms of the first kind and second kind, respectively.

We will use the results from the previous chapters to prove the following:

1.1. Theorem. *Let* $\mathbf{g}$ *be an affine Lie algebra. Let* $\sigma \in \mathrm{Aut}\,\mathbf{g}$ *be a finite order automorphism of the first kind, then there is a Borel subalgebra* $\mathbf{b} \subset \mathbf{g}$ *and a Cartan subalgebra* $\mathbf{h} \subset \mathbf{b}$ *that are stable under* $\sigma$.

*Proof.* $\sigma$ induces an automorphism $\tilde{\sigma} = \mathrm{Ad}\,\sigma$ of $\mathrm{Aut}\,\mathbf{g}$ leaving stable $G$. Consider the Tits system $(G, B_+, N, S)$ and its building $\mathscr{B}$ (see Appendix A).

The hypothesis $[\mathbf{b}_+ : \sigma(\mathbf{b}_+) \cap \mathbf{b}_+] < \infty$ means $\sigma(\mathbf{b}_+) = \mathrm{Ad}\, g(\mathbf{b}_+)$ for some $g$ in $G$. Then $\tilde{\sigma} \in \mathrm{Aut}_{B_+} G$ (Appendix A.5.1) and it is of order $m$. Indeed, we have

$$\tilde{\sigma}(b)(\sigma(\mathbf{b}_+)) = \sigma\, \mathrm{Ad}\, b\sigma^{-1}(\sigma(\mathbf{b}_+)) = \sigma\, \mathrm{Ad}\, b(\mathbf{b}_+) = \sigma(\mathbf{b}_+) \qquad \forall b \in B_+,$$

therefore $\tilde{\sigma}(b) \in \mathrm{Stab}_G\, \sigma(\mathbf{b}_+) = \mathrm{Stab}_G\, \mathrm{Ad}\, g(\mathbf{b}_+) = gB_+g^{-1}$, proving the claim.

So $\tilde{\sigma}$ determines an isometry $\hat{\sigma}$ of $\mathscr{B}$ (Appendix A.5.1) and has the same order as $\sigma$. Therefore $\hat{\sigma}$ has a fixed point $(P, x) \in \mathscr{B}$, $P \in \mathscr{P}$, $x \in C_{\tau(p)}$. $\hat{\sigma}(P, x) = (P, x)$ implies $\mathrm{Ad}(P) = \tilde{\sigma}\, \mathrm{Ad}(P)\, \tilde{\sigma}^{-1}$, hence $P$ is the normalizer of $\tilde{\sigma}(P)$ in $G$, so $P \subset \tilde{\sigma}(P)$ and vice versa. Thus $\tilde{\sigma}(P) = P$.

But $P = \mathrm{Stab}_G\, \mathbf{p}$, where $\mathbf{p}$ is a parabolic subalgebra in $\mathbf{g}$ and $\tilde{\sigma}(P) = \mathrm{Stab}_G\, \sigma(\mathbf{p})$, hence $\sigma(\mathbf{p}) = \mathbf{p}$.

$\sigma$ leaves the radical $R$ of $\mathbf{p}$ stable, therefore it induces $\bar{\sigma}$: $\mathbf{p}/R \to \mathbf{p}/R$, a finite order automorphism of $\mathbf{p}/R$ which is a finite-dimensional semisimple Lie algebra, and we know in this case there is a Borel $\bar{\mathbf{b}} \subset \mathbf{p}/R$ stable under $\bar{\sigma}$, therefore $\pi^{-1}(\bar{\mathbf{b}})$ is a Borel invariant under $\sigma$ ($\pi$ is the canonical homomorphism).

Doing this with $B_-$ instead of $B_+$ we would get another Borel $\mathbf{b}'$ invariant under $\sigma$ and conjugated to $\mathbf{b}_-$. Therefore $\mathbf{b} \cap \mathbf{b}'$ is a finite-dimensional solvable subalgebra stable under $\sigma$ and contains a conjugate of $\mathbf{h}$. Indeed, it follows from the fact that given any pair of elements $x$, $y$ in $G$ there exists a $z$ in $G$ such that $xB_+x^{-1} \cap yB_-y^{-1} \supset zHz^{-1}$; to see this we can reduce to the case $x = 1$, and by II.2.7 we can write $y = b^+wb^-$, $b^\pm \in B_\pm$, $w \in N$, thus $B_+ \cap yBy^{-1} = b^+(B_+ \cap wBw^{-1})\, b^{+^{-1}} \supset b^+Hb^{+^{-1}}$. So to complete the proof of the theorem we only need to prove the following:

1.2. LEMMA. *Let $S$ be a complex solvable Lie algebra of finite dimension. Let $\sigma$ be an automorphism of $S$ of order $m$. Then there exists a Cartan subalgebra $\mathbf{h} \subset S$ stable under $\sigma$.*

*Proof.* By induction on dim $S$.

dim $S = 1$ is obvious.

dim $S > 1$. If $\mathbf{z} =$ center of $S$ is non-trivial then we apply the inductive hypothesis on $S/\mathbf{z}$ to get $\mathbf{h}$ stable under $\bar{\sigma}$: $S/\mathbf{z} \to S/\mathbf{z}$, then $\pi^{-1}(\mathbf{h}) \subset S$ is a Cartan subalgebra $\sigma$-invariant.

If $\mathbf{z} = 0$ then $I = \{y = \sum x_i$, in $S$: $[S, x_i] \subset \mathbb{C}x_i\}$ is a $\sigma$-invariant ideal, hence $\sigma$ induces $\bar{\sigma}$: $S/I \to S/I$. $I$ is non-trivial so we can apply the inductive hypothesis to get $\tilde{\mathbf{h}} = \{x \in S/I$: $[x, x^0] = 0\}$ stable under $\sigma$. Thus $\pi^{-1}(\tilde{\mathbf{h}}) = \{x \in S$: $[x, x^0] \in I\}$ is stable under $\sigma$ and contains a Cartan subalgebra of $S$. If $\dim \pi^{-1}(\tilde{\mathbf{h}}) < \dim S$ we can apply again the inductive hypothesis and we are done.

If $\pi^{-1}(\tilde{\mathbf{h}}) = S$ then we have to consider the case where $S = \mathbf{h} \oplus I$, where $\mathbf{h}$ is any CSA of $S$. Using the fact that all the CSA of $S$ are conjugate [W, Theorem 4.4.1.1], we have $\sigma(\mathbf{h}) = \exp \operatorname{ad} x(\mathbf{h})$ and we want to find $y \in I$ such that $\sigma(\exp(\operatorname{ad} y)(\mathbf{h})) = \exp(\operatorname{ad} y)(\mathbf{h})$, that is,

$$\exp(\operatorname{ad}(y - \sigma(y) - x))(\mathbf{h}) = \mathbf{h} \qquad (I \text{ is abelian})$$

or

$$(*) \qquad y - \sigma(y) = x.$$

If we write $x = \sum_{j=0}^{m-1} x_j$ $x_j \in I_j$, $I = \bigoplus_{j=0}^{m-1} I_j$, $I_j$ eigenspaces of $\sigma$,

$$y = \sum_{j=0}^{m-1} y_j,$$

then $y - \sigma(y) = \prod_{j=0}^{m-1} (1 - \varepsilon^j) y_j$, $\varepsilon = \exp(2\pi i/m)$. Thus Eq. $(*)$ has a solution if and only if $x_0 = 0$. But using the relation $\mathbf{h} = \sigma^m(\mathbf{h}) = \exp \operatorname{ad} \sum_{n=0}^{m-1} \sigma^n(x)(\mathbf{h})$ we have $\sum_{n=0}^{m-1} \sigma^n(x) \in \mathbf{h} \cap I = \{0\}$, hence, $mx_0 = 0$. Therefore $\sigma$ leaves $\exp(\operatorname{ad} \sum_{j=1}^{m-1} x_j/(1 - \varepsilon^j))(\mathbf{h})$ stable.

## 2. INVOLUTIVE AUTOMORPHISMS OF THE FIRST KIND

We are now able to compute a list of representatives of involutive automorphisms.

To give the list we will use a list of representatives of involutions for the finite-dimensional Lie algebras, which is shown in Tables IV and V. The elements $\{p_i\} \subset \mathring{\mathbf{h}}$ that appear on the expression of the automorphisms correspond to a basis of $\mathring{\mathbf{h}}$ dual to $\mathring{\pi} = \{\alpha_i\}_{i \in I} \subset \mathring{\mathbf{h}}^*$; $\mathbf{t}$ denotes a one-dimensional center.

2.1. Let $\mathbf{g}^{(i)}$ be a Kac–Moody algebra from Table I or II and $\tau$ an automorphism of $\mathring{\mathbf{g}}$. We denote by $1\times\tau$: $\mathbf{g}^{(i)}\to\mathbf{g}^{(i)}$, $i=1,2,3$, the automorphism defined by $1\times\tau(t^k\otimes X)=t^k\otimes\tau(X)$ and it is the identity on $\mathbb{C}c\oplus\mathbb{C}d$.

2.2. We also define $\tau_0$: $\mathbf{g}^{(i)}\to\mathbf{g}^{(i)}$ by $\tau_0(t^k\otimes X)=(-1)^k\,t^k\otimes X$ and it is the identity on $\mathbb{C}c\oplus\mathbb{C}d$.

2.3. We will denote by $\{\hat{p}_i\}$ a basis of $\mathring{\mathbf{h}}\oplus\mathbb{C}d$ dual to $\pi$.

2.4. We list in Table VI a complete and non-redundant set of representatives of involutive automorphisms of the first kind of all affine Kac–Moody algebras $\mathbf{g}^{(i)}$, $i=1,2,3$, together with their fixed points sets.

2.5. THEOREM. *Any involutive automorphism of an affine Kac–Moody Lie algebra is conjugated to an automorphism from Table* VI.

*Proof.* We start by proving the following

2.6. LEMMA. *If an automorphism $\sigma$ of* $\mathbf{g}$ *fixes* $\mathbf{h}$ *pointwise then it is of the form* exp ad $x$, *for some element $x$ in* $\mathbf{h}$.

TABLE IV

Inner Automorphisms of finite-dimensional Kac–Moody Algebras

| Algebra | Automorphism | | Fixed Point Set |
|---|---|---|---|
| $a_n$ | $\tau_j=\exp\operatorname{ad}(\pi i p_j)$, | $0<j<1+n/2$, | $t+a_j+a_{n-j+1}$ |
| $b_n$ | $\tau_j=\exp\operatorname{ad}(\pi i p_j)$, | $1<j<n$, | $d_j+b_{n-j}$ |
| | $\tau_1=\exp\operatorname{ad}(\pi i p_1)$ | | $t+b_{n-1}$ |
| $c_n$ | $\tau_j=\exp\operatorname{ad}(\pi i p_j)$, | $0<j<1+n/2$, | $c_j+c_{n-j}$ |
| | $\tau_n=\exp\operatorname{ad}(\pi i p_n)$ | | $t+a_{n-1}$ |
| $d_n$ | $\tau_j=\exp\operatorname{ad}(\pi i p_j)$, | $1<j<1+n/2$, | $d_j+d_{n-j}$ |
| | $\tau_n=\exp\operatorname{ad}(\pi i p_n)$ | | $t+d_{n-1}$ |
| | $\tau_1=\exp\operatorname{ad}(\pi i p_1)$ | | $t+a_{n-1}$ |
| $e_6$ | $\tau_1=\exp\operatorname{ad}(\pi i p_1)$ | | $t+d_5$ |
| | $\tau_6=\exp\operatorname{ad}(\pi i p_6)$ | | $t+a_5$ |
| $e_7$ | $\tau_1=\exp\operatorname{ad}(\pi i p_1)$ | | $a_1+d_6$ |
| | $\tau_6=\exp\operatorname{ad}(\pi i p_6)$ | | $t+e_6$ |
| | $\tau_7=\exp\operatorname{ad}(\pi i p_7)$ | | $a_7$ |
| $e_8$ | $\tau_1=\exp\operatorname{ad}(\pi i p_1)$ | | $d_8$ |
| | $\tau_7=\exp\operatorname{ad}(\pi i p_7)$ | | $a_1+e_7$ |
| $f_4$ | $\tau_1=\exp\operatorname{ad}(\pi i p_1)$ | | $a_1+c_3$ |
| | $\tau_4=\exp\operatorname{ad}(\pi i p_4)$ | | $b_4$ |
| $g_2$ | $\tau_1=\exp\operatorname{ad}(\pi i p_1)$ | | $a_1+a_1$ |

*Proof.* $\sigma$ leaves stable the root spaces $g_\alpha$, $\alpha \in \Delta$, and it is determined by the action on the generators $\sigma(e_i) = c_i e_i$, $i$ in $I$, hence we can choose $x$ in **h** such that $\exp \alpha_i(x) = c_i$, therefore $\sigma = \exp \operatorname{ad} x$, proving the lemma.

We consider first the case of an automorphism in $\tilde{H} \propto G$, which will be referred to as an inner automorphism.

$\sigma$ is of the first kind so we can assume that $\sigma$ leaves stable **h** and $\mathbf{b}_+ = \mathbf{n}_+ \oplus \mathbf{h}$. Therefore $\sigma$ is in $\tilde{H} \propto (B \cap N) = \tilde{H} \propto H$, it leaves **h**, pointwise fixed, and it is of the form $\exp \operatorname{ad}(h)$, for some $h$ in **h**, and we can write $\sigma = \exp \operatorname{ad}(h) \exp \operatorname{ad}(s \cdot d)$, $h$ in $\mathring{\mathbf{h}} \subset \mathring{\mathbf{g}}$, $s$ in $\mathbb{C}$. Using the fact that $\sigma$ is an involution we have $\tau = \exp \operatorname{ad}(h)$ is an inner involution of $\mathring{\mathbf{g}}$ and $s \in \pi i \mathbb{Z}$, hence we can write $\sigma = (1 \times \tau) \cdot \tau_0^s$, $s \in \{0, 1\}$. Conjugating by $1 \times \gamma$ for a certain automorphism $\gamma$ of $\mathring{\mathbf{g}}$ we can assume that $\tau$ is one of the involutions from Table IV.

The only thing that remains to prove is that when $s = 1$ we have $\sigma \sim \tau_0$. We will show this by actually giving the automorphism that conjugates $\sigma$ to $\tau_0$. Recall the definition of $\dot{r}_i = \exp \operatorname{ad} f_i \exp \operatorname{ad} - e_i \exp \operatorname{ad} f_i$. We have the following:

| | | |
|---|---|---|
| $a_n^{(1)}$ | $\tau_0 = \rho_j \cdot \tau_j \cdot \tau_0 \cdot \rho_j^{-1}$, | where $\rho_j$ is determined by $\rho_j(\alpha_i) = \alpha_{i-j}$ |
| $b_n^{(1)}$ | $\tau_j \tau_0 = g \cdot \tau_{j+1} \cdot \tau_0 \cdot g^{-1}$, | $j > 1$, where $g = \rho \cdot \dot{r}_j \cdot \dot{r}_{j-1} \cdots \dot{r}_2 \cdot \dot{r}_0$ and $\rho$ is determined by $\rho(\alpha_0) = \alpha_1$ |
| | $\tau_0 = \dot{r}_0 \cdot \tau_2 \cdot \tau_0 \cdot r_0^{-1}$ | |
| | $\tau_0 = \rho \cdot \tau_1 \cdot \tau_0 \cdot \rho^{-1}$ | |
| $c_n^{(1)}$ | $\tau_j \tau_0 = g \cdot \tau_{j+1} \cdot \tau_0 \cdot g^{-1}$, | $j < n-1$ where $g = \dot{r}_j \cdot \dot{r}_{j-1} \cdots \dot{r}_1 \cdot \dot{r}_0$ |
| | $\tau_0 = \rho \cdot \tau_n \cdot \tau_0 \cdot \rho^{-1}$, | $\rho$ is determined by $\rho(\alpha_i) = \alpha_{n-i}$ |
| $d_n^{(1)}$ | $\tau_j \cdot \tau_0 = g \cdot \tau_{j+1} \tau_0 \cdot g^{-1}$, | $1 < j < [n/2]$ where $g$ is as for $b_n^{(1)}$ |
| | $\tau_0 = \rho_i \cdot \tau_i \cdot \tau_0 \cdot \rho_i^{-1}$, | $i \in \{1, n\}$, $\rho_i$ satisfies $\rho_i(\alpha_i) = \alpha_i$ |
| $e_6^{(1)}$ | $\tau_0 = g \cdot \tau_2 \cdot \tau_0 \cdot g^{-1}$, | $g = \rho \cdot \dot{r}_5 \cdot \dot{r}_4 \cdot \dot{r}_3 \cdot \dot{r}_6 \cdot \dot{r}_0$, $\rho(\alpha_0) = \alpha_5$ |
| | $\tau_0 = \rho \cdot \tau_1 \cdot \tau_0 \cdot \rho^{-1}$, | $\rho(\alpha_0) = \alpha_1$ |
| $e_7^{(1)}$ | $\tau_0 = \dot{r}_0 \cdot \tau_1 \cdot \tau_0 \cdot \dot{r}_0$ | |
| | $\tau_0 = \rho \cdot \tau_6 \cdot \tau_0 \cdot \rho^{-1}$, | $\rho(\alpha_i) = \alpha_{6-i}$ |
| | $\tau_6 \cdot \tau_0 = g \cdot \tau_7 \cdot \tau_0 \cdot g^{-1}$, | $g = \dot{r}_6 \cdot \dot{r}_5 \cdot \dot{r}_4 \cdot \dot{r}_3 \cdot \dot{r}_2 \cdot \dot{r}_1 \cdot \dot{r}_0$ |
| $e_8^{(1)}$ | $\tau_0 = \dot{r}_0 \cdot \tau_1 \cdot \tau_0 \cdot \dot{r}_0^{-1}$ | |
| | $\tau_0 = g \cdot \dot{r} \cdot g^{-1} \cdot \tau_7 \cdot \tau_0 \cdot g \cdot \dot{r}^{-1} \cdot g^{-1}$, | $g = \dot{r}_0 \cdot \dot{r}_1 \cdot \dot{r}_1 \cdot \dot{r}_3 \cdot \dot{r}_4 \cdot \dot{r}_5$, $\dot{r} = \dot{r}_6 \cdot \dot{r}_8$ |
| $f_4^{(1)}$ | $\tau_0 = \dot{r}_0 \cdot \tau_1 \cdot \tau_0 \cdot \dot{r}_0^{-1}$ | |
| | $\tau_0 = g^{-1} \cdot \dot{r}_3 \cdot g \cdot \tau_4 \cdot \tau_0 \cdot g^{-1} \cdot \dot{r}_3^{-1} \cdot g$, | $g = \dot{r}_0 \cdot \dot{r}_1 \cdot \dot{r}_2$ |
| $g_2^{(1)}$ | $\tau_0 = \dot{r}_0 \cdot \tau_1 \cdot \tau_0 \cdot \dot{r}_0^{-1}$. | |

TABLE V

Outer Automorphisms of Finite-Dimensional Kac–Moody Algebras

| Algebra | Automorphism | Fixed Point Set |
| --- | --- | --- |
| $a_{2k}$ | $\rho(\alpha_j) = \alpha_{2k-j+1}$ | $b_k$ |
| $a_{2k-1}$ | $\rho(\alpha_j) = \alpha_{2k-j}$ | $c_k$ |
| | $\rho' = \rho \cdot \tau_k$ | $d_k$ |
| $d_n$ | $\rho(\alpha_n) = \alpha_{n-1}$, $\quad \rho(\alpha_j) = \alpha_j$, $\quad j = n, n-1$ | $b_{n-1}$ |
| | $\rho'_j = \rho \cdot \tau_j$, $\quad 0 < j < [n/2]$ | $b_j + b_{n-j+1}$ |
| $e_6$ | $\rho(\alpha_6) = \alpha_6$, $\quad \rho(\alpha_j) = \alpha_{6-j}$, $\quad j \nleq 6$ | $f_4$ |
| | $\rho' = \rho \cdot \tau_6$ | $c_4$ |

The verification for the algebras from Table II is based on the identity $\dot{w} \cdot \exp \mathrm{ad}(h) \cdot \dot{w}^{-1} = \exp \mathrm{ad}(w(h))$ for $\dot{w}$ in $W \subset \mathrm{Int}\, G$ and $h \in \mathbf{h}$.

When $\sigma$ in an outer automorphism of the first kind, i.e., is an element of $(\mathrm{Aut}(A) \propto (\tilde{H} \propto G)) \backslash (\tilde{H} \propto G)$, we still can assume that $\sigma$ leaves $\{\mathbf{h} \subset \mathbf{b}_+\}$ stable, that is, we can write $\sigma$ as $D \cdot \exp \mathrm{ad}(h)$, $h$ in $\mathbf{h}$, and $D$ in $\mathrm{Aut}(A)$. $\sigma^2 = \mathrm{id}$ implies $D^2 = \mathrm{id}$. Conjugating by an element of $\mathrm{Aut}(A)$ we can assume that $D$ is one of the following:

| | | |
| --- | --- | --- |
| $a_{2k-1}^{(1)}$ | $D = \rho_0$, | $\rho_0(\alpha_i) = \alpha_{2k-i}$, $i$ in $I$ |
| | $D = \rho_1$, | $\rho_1(\alpha_i) = \alpha_{2k-i-1}$, $i$ in $I$ |
| | $D = \rho_2$, | $\rho_2(\alpha_i) = \alpha_{i+k}$, $i$ in $I$ |
| $a_{2k}^{(1)}$ | $D = \rho_0$ | |
| $b_n^{(1)}$ | $D = \rho_1$, | $\rho_1(\alpha_0) = \alpha_1$ |
| $c_n^{(1)}$ | $D = \rho_1$, | $\rho_1(\alpha_i) = \alpha_{n-i}$ |
| $d_n^{(1)}$ | $D = \rho_0$, | $\rho_0(\alpha_n) = \alpha_{n-1}$, and fixes the rest |
| | $D = \rho_1$, | $\rho_1(\alpha_i) = \alpha_{n-i}$ |
| | $D = \rho_2 =$ | $\rho_0 \cdot \rho_1 \cdot \rho_0 \cdot \rho_1$ |
| $e_6^{(1)}$ | $D = \rho_0$, | $\rho_0(\alpha_i) = \alpha_{6-i}$, $0 < i < 6$, and fixes the rest |
| $e_7^{(1)}$ | $D = \rho_1$, | $\rho_1(\alpha_i) = \alpha_{6-i}$, $0 < i < 7$, and fixes $\alpha_7$ |
| $a_{2n-1}^{(2)}$ | $D = \rho_1$, | $\rho_1(\alpha_0) = \alpha_1$, and fixes the rest |
| $d_{n+1}^{(2)}$ | $D = \rho_1$, | $\rho_1(\alpha_i) = \alpha_{n-i}$. |

We can write $h = h^+ + h^-$, so that $D(h^+) = h^+$ and $D(h^-) = -h^-$. Conjugating by $\exp \mathrm{ad}(h^-/2)$ we can assume $D(h) = h$. The fact that $\sigma$ is an

TABLE VI

Involutions of the First Kind of Affine Lie Algebras

| Algebra | | Automorphism | Fixed Point Set |
|---|---|---|---|
| $a_{2k-1}^{(1)}$ | | $\sigma = 1 \times \tau_j$ | $\hat{L}(a_{2k-1}^{\tau_j})$ |
| | | $\sigma = \tau_0$ | $a_{2k-1}^{(1)}$ |
| | | $\sigma = \rho_2$ | $\hat{L}(a_{k-1} + \mathbf{c})$ |
| | | $\sigma = 1 \times \rho$ | $\hat{L}(c_k)$ |
| | | $\sigma = \rho_1$ | $c_{k-1}^{(1)}$ |
| | | $\sigma = 1 \times (\rho \cdot \tau_k)$ | $\hat{L}(d_k)$ |
| | | $\sigma = \tau_0 \cdot (1 \times \rho)$ | $a_{2k-1}^{(2)}$ |
| $a_1^{(1)}$ | | $\sigma = \rho_1$ | $\hat{L}(\mathbf{c})$ |
| $a_{2k}^{(1)}$ | | $\sigma = 1 \times \tau_j$ | $\hat{L}(a_{2k}^{\tau_j})$ |
| | | $\sigma = \tau_0$ | $a_{2k}^{(1)}$ |
| | | $\sigma = 1 \times \rho$ | $\hat{L}(b_k)$ |
| | | $\sigma = \tau_0 \cdot (1 \times \rho)$ | $a_{2k}^{(2)}$ |
| $b_n^{(1)}$ | | $\sigma = 1 \times \tau_j$ | $\hat{L}(b_n^{\tau_j})$ |
| | | $\sigma = \tau_0$ | $b_n^{(1)}$ |
| | | $\sigma = \rho_1$ | $d_n^{(2)}$ |
| | | $\sigma = \rho_1 \cdot (1 \times \tau_j), \quad 3 < j < n$ | $\hat{L}(b_{j-1} + d_{n-j-1}, 1 \times \rho)$ |
| | | $\sigma = \rho_1 \cdot (1 \times \tau_2)$ | $\hat{L}(a_1 + d_{n-1}, 1 \times \rho)$ |
| | | $\sigma = \rho_1 \cdot (1 \times \tau_3)$ | $\hat{L}(c_2 + d_{n-2}, 1 \times \rho)$ |
| | | $\sigma = \rho_1 \cdot (1 \times \tau_n)$ | $\hat{L}(b_{n-1} + \mathbf{c})$ |
| $c_{2k-1}^{(1)}$ | | $\sigma = 1 \times \tau_j$ | $\hat{L}(c_{2k-1}^{\tau_j})$ |
| | | $\sigma = \tau_0$ | $c_{2k-1}^{(1)}$ |
| | | $\sigma = \rho_1$ | $\hat{L}(a_{2k-2} + \mathbf{c}, \rho)$ |
| $c_{2k}^{(1)}$ | | $\sigma = 1 \times \tau_j$ | $\hat{L}(c_{2k}^{\tau_j})$ |
| | | $\sigma = \tau_0$ | $c_{2k}^{(1)}$ |
| | | $\sigma = \rho_1$ | $c_k^{(1)}, k > 1, a_1^{(1)}, k = 1$ |
| | | $\sigma = \rho_1 \cdot (1 \times \tau_j)$ | $\hat{L}(a_{2k-1} + \mathbf{c}, \rho)$ |
| $d_{2k-1}^{(1)}$ | | $\sigma = 1 \times \tau_j$ | $\hat{L}(d_{2k-1}^{\tau_j})$ |
| | | $\sigma = \tau_0$ | $d_{2k-1}^{(1)}$ |
| | | $\sigma = 1 \times \rho$ | $\hat{L}(b_{2k-2})$ |
| | | $\sigma = 1 \times \rho'_j$ | $\hat{L}(b_j + b_{2k-j-2})$ |
| | | $\sigma = \rho_1$ | $b_{k-1}^{(1)}$ |
| | | $\sigma = \rho_2$ | $d_{2k-2}^{(2)}$ |
| | $k > 2$ | $\sigma = \rho_2 \cdot (1 \times \tau_2)$ | $\hat{L}(a_1 + d_{2k-3}, 1 \times \rho)$ |
| | $k > 3$ | $\sigma = \rho_2 \cdot (1 \times \tau_j), \quad j > 2$ | $\hat{L}(d_j + d_{2k-j-1}, \rho \times \rho)$ |
| $d_{2k+2}^{(1)}$ | | $\sigma = 1 \times \tau_j$ | $\hat{L}(d_{2k+2}^{\tau_j})$ |
| | | $\sigma = \tau_0$ | $d_{2k}^{(1)}$ |
| | | $\sigma = 1 \times \rho$ | $\hat{L}(b_{2k-1})$ |
| | | $\sigma = 1 \times \rho'_j$ | $\hat{L}(b_j + b_{2k-j+1})$ |
| | | $\sigma = \rho_1$ | $\hat{L}(a_{2k-1} + \mathbf{c}, \rho)$ |
| | | $\sigma = \rho_1 \cdot (1 \times \tau_j)$ | $\hat{L}(d_k + \mathbf{c})$ |
| | | $\sigma = \rho_2$ | $\hat{L}(d_{2k-1} + \mathbf{c}, \rho)$ |
| | $k > 1$ | $\sigma = \rho_2 \cdot (1 \times \tau_2)$ | $\hat{L}(a_1 + d_{2k-2}, 1 \times \rho)$ |
| | $k > 2$ | $\sigma = \rho_2 \cdot (1 \times \tau_j), \quad j > 2$ | $\hat{L}(d_j + d_{2k-j}, \rho \times \rho)$ |
| | $k = 1$ | $\sigma = \rho_2 \cdot (1 \times \tau_2)$ | $\hat{L}(a_1 + a_1)$ |

*Table continued*

TABLE VI (*continued*)

| | | |
|---|---|---|
| $e_6^{(1)}$ | $\sigma = 1 \times \tau_j$ | $\hat{L}(e_6^{\tau_j})$ |
| | $\sigma = \tau_0$ | $e_6^{(1)}$ |
| | $\sigma = 1 \times \rho$ | $\hat{L}(f_4)$ |
| | $\sigma = 1 \times (\rho \cdot \tau_6)$ | $\hat{L}(c_4)$ |
| | $\sigma = (1 \times \rho) \cdot \tau_0$ | $e_6^{(2)}$ |
| $e_7^{(1)}$ | $\sigma = 1 \times \tau_j$ | $\hat{L}(e_7^{\tau_j})$ |
| | $\sigma = \tau_0$ | $e_7^{(1)}$ |
| | $\sigma = \rho_1$ | |
| | $\sigma = \rho_1 \cdot (1 \times \tau_3)$ | |
| $e_8^{(1)}$ | $\sigma = 1 \times \tau_i$ | $\hat{L}(e_8^{\tau_j})$ |
| | $\sigma = \tau_0$ | $e_8^{(1)}$ |
| $f_4^{(1)}$ | $\sigma = 1 \times \tau_i$ | $\hat{L}(f_4^{\tau_j})$ |
| | $\sigma = \tau_0$ | $f_4^{(1)}$ |
| $g_2^{(1)}$ | $\sigma = 1 \times \tau_1$ | $\hat{L}(g_2^{\tau_1})$ |
| | $\sigma = \tau_0$ | $g_2^{(1)}$ |
| $a_2^{(2)}$ | $\sigma = \exp \operatorname{ad}(\pi i p_1)$ | $\hat{L}(a_1 + \mathbf{c})$ |
| | $\sigma = \tau_0$ | $a_1^{(1)}$ |
| $a_{2k}^{(2)}$ | $\sigma = \exp \operatorname{ad}(\pi i p_1)$ | $a_{2k-1}^{(2)}$ |
| | $\sigma = \exp \operatorname{ad}(\pi i p_j), \quad j > 1$ | $\hat{L}(a_{2j-1} + a_{2(k-j)} + \mathbf{c}, \rho\tau_j \times \rho \times (-1))$ |
| | $\sigma = \tau_0$ | $b_k^{(1)}$ |
| $a_{2k-1}^{(2)}$ | $\sigma = \exp \operatorname{ad}(\pi i p_k)$ | $d_k^{(1)}$ |
| | $\sigma = \exp \operatorname{ad}(\pi i p_{k-1})$ | $\hat{L}(a_1 + a_{2k-3} + \mathbf{c}, 1 \times \rho \times (-1))$ |
| | $\sigma = \exp \operatorname{ad}(\pi i p_j), \quad 1 < j < [(k+1)/2]$ | $\hat{L}(a_{2j-1} + a_{2(k-j)-1} + \mathbf{c}, \rho \times \rho \times (-1))$ |
| | $\sigma = \exp \operatorname{ad}(\pi i (p_k + p_0))$ | $\hat{L}((a_{k-1} + a_{k-1}) + \mathbf{c}, \rho \times 1)$ |
| | $\sigma = \tau_0$ | $c_k^{(1)}$ |
| | $\sigma = \rho_1$ | $a_{2k-2}^{(2)}$ |
| | $\sigma = \rho_i \cdot \exp \operatorname{ad}(\pi i p_j)$ | $\hat{L}(a_{2(j-1)} + a_{2(k-j)}, \rho \times \rho)$ |
| $d_{n+1}^{(2)}$ | $\sigma = \exp \operatorname{ad}(\pi i (p_0 + p_j)), \quad 1 < j < n$ | $\hat{L}(d_j + d_{n-j-1}, 1 \times \rho)$ |
| | $\sigma = \exp \operatorname{ad}(\pi i p_j)$ | $\hat{L}(b_j + b_{n-j})$ |
| | $\sigma = \exp \operatorname{ad}(\pi i (p_0 + p_1))$ | $\hat{L}(d_n + \mathbf{c}, \rho \times 1)$ |
| | $\sigma = \exp \operatorname{ad}(\pi i (p_0 + p_n))$ | $\hat{L}(d_n + \mathbf{c}, 1 \times (-1))$ |
| | $\sigma = \tau_0$ | $b_n^{(1)}$ |
| $d_{2n+1}^{(2)}$ | $\sigma = \rho_1$ | $a_n^{(2)}$ |
| | $\sigma = \rho_1 \cdot \exp \operatorname{ad}(\pi i p_n)$ | $b_n^{(1)}$ |
| $d_{2n}^{(2)}$ | $\sigma = \rho_1$ | $d_n^{(2)}$ |
| $e_6^{(2)}$ | $\sigma = \exp \operatorname{ad}(\pi i p_1)$ | $\hat{L}(d_5 + \mathbf{c}, \rho \times (-1))$ |
| | $\sigma = \exp \operatorname{ad}(\pi i p_3)$ | $\hat{L}(a_1 + a_5, 1 \times \rho)$ |
| | $\sigma = \exp \operatorname{ad}(\pi i p_4)$ | $c_4^{(1)}$ |
| | $\sigma = \tau_0$ | $f_4^{(1)}$ |
| $d_4^{(3)}$ | $\sigma = \exp \operatorname{ad}(\pi i p_2)$ | $\hat{L}(a_1 + a_1)$ |
| | $\sigma = \tau_0$ | $d_4^{(3)}$ |

involution implies $D(h)+h \in 2\pi iL$, where $L$ is the lattice generated by $\{\hat{p}_i\}_{i=0}^n$. Therefore $h$ is in $\pi iL$ and we can write $h=\pi i \sum a_j \hat{p}_j$, where $a_j = a_{D(j)}$, $a_j \in \{0, 1\}$. So $h=\pi i(\sum_{j\in J} a_j \hat{p}_j + D(\sum_{j\in J} a_j \hat{p}_j) + \sum_{j=D(j)} a_j \hat{p}_j)$, where $J \cap D(J) \neq 0$ and

$$J \cup D(J) = \{s \in I: s \neq D(s)\} \quad (D(e_i) = e_{D(i)}).$$

Conjugating by exp ad$(\pi i \sum_{j\in J} a_j \hat{p}_j)$ we have

$$D\cdot \exp \operatorname{ad}(h) \sim D\cdot \exp \operatorname{ad}\left(\pi i \sum_{j=D(j)} a_j \hat{p}_j\right).$$

We further have the following equivalences:

$a_{2k-1}^{(1)}$ $\rho_0 \cdot \exp \operatorname{ad}(\pi i \hat{p}_k) = \rho_2 \cdot \rho_0 \cdot \exp \operatorname{ad}(\pi i \hat{p}_0)\cdot \rho_2^{-1}$.

$b_n^{(1)}$ Using $\langle \{r_i\}_{i=2}^n \rangle \subset W$ we can reduce to the cases shown in the list as if we were in the finite-dimensional case.

$d_n^{(1)}$ Again we use $\langle \{\dot{r}_i\}_{i=2}^{n-2} \rangle$ for $\rho_2$ and $\langle \{r_i\}_{i=2}^n \rangle$ for $\rho_0$.

$e_6^{(1)}$ Here we use $\langle \dot{r}_0, \dot{r}_6, \dot{r}_3 \rangle$ as in $a_3$.

$e_7^{(1)}$ We use $\langle \dot{r}_7, \dot{r}_3 \rangle$.

## 3. Involutive Automorphisms of the Second Kind

3.1. In this case, $\sigma(\mathbf{b}_+) = g(\mathbf{b}_-)$ for some $g$ in $G$, writing $g = b^+ w b^-$, $b^\pm$ in $B_\pm$, we have $\sigma(\mathbf{b}_+) \cap \mathbf{b}_+ = b^+(\mathbf{b}_+ \cap w(\mathbf{b}_-)) \supset b^+(\mathbf{h})$, then $\sigma(\mathbf{b}_+) \cap \mathbf{b}_+$ is a finite-dimensional solvable Lie algebra stable under $\sigma$, which contains a conjugate of $\mathbf{h}$. By Lemma III.1.2 we can assume that $\mathbf{h}$ is stable under $\sigma$.

So $\sigma$ induces an automorphism of the root system and, using the fact that it is an involution on $\mathbf{h}$, we have $\sigma|_{\mathbf{h}} = \omega \cdot D^q \cdot r$, $q=0, 1$; $r \in W$; $D$ is determined by a symmetry of the Dynkin diagram of $\mathbf{g}$ and $\omega = -\mathrm{id}$; $D^q \cdot r$ is of order two and we have the following:

3.2. Lemma. *Given $D$, $r$, and $q$ as above and $(D^q \cdot r)^2 = \mathrm{id}$, there exist an element $s$ in $W$ and orthogonal simple roots $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_m}$ fixed by $(D^q)^*$ such that $s\cdot D \cdot r \cdot s^{-1} = D\cdot r_{\alpha_{i_1}} \cdot r_{\alpha_{i_2}} \cdots r_{\alpha_{i_m}}$.*

*Proof.* We consider the case $q=1$. When $q=0$, the proof is similar.

We know that $l(w) = \#\{w(\Delta_+) \cap \Delta_-\} = \#\{D\cdot w(\Delta_+)\cap \Delta_-\}$ for all $w$ in $W$. Assume that $\alpha_i \in \pi$ is such that $D\cdot r(\alpha_i)$ is in $\Delta_-$. Then $r_i \cdot D \cdot r \cdot r_i = D \cdot r_{D(i)} \cdot r \cdot r_i$ and

$$\begin{aligned} l(r_{D(i)}\cdot r \cdot r_i) &= l(r) - 2 \qquad \text{if} \quad r(\alpha_i) \neq -D(\alpha_i) \\ &= l(r) \qquad\qquad \text{if} \quad r(\alpha_i) = -D(\alpha_i). \end{aligned}$$

Now we will prove the lemma by induction on $l(r)$:

If $l(r)=0$ it is clear. Suppose $r \in \{D \cdot s \cdot D \cdot r \cdot s^{-1}: s \in W\}$ has minimal length $l(r)>0$, then there is $\alpha_i \in \pi$ such that $r(\alpha_i) \in \Delta_-$ and we have $r_{D(i)} \cdot r = r \cdot r_i$ because $r \cdot r_i(\alpha_j) = r(\alpha_j) + 2((\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))\, D(\alpha_i)$,

$$r_{D(i)} \cdot r(\alpha_j) = r(\alpha_j) - 2((D(\alpha_i), r(\alpha_j))/(D(\alpha_i), D(\alpha_i)))\, D(\alpha_i),$$

and $(\alpha_i, \alpha_j) = -(D(\alpha_i), r(\alpha_j))$, $(\alpha_i, \alpha_j) = (D(\alpha_i), D(\alpha_j))$. Therefore $D \cdot r$ commutes with $r_i$, $D \cdot r \cdot r_i$ satisfies the inductive hypothesis, and $l(r \cdot r_i) = l(r) - 1$. Thus $D \cdot r \cdot r_i = D \cdot r_{i_1} \cdot r_{i_2} \cdots r_{i_m}$.

We only need to show that $r_i$ commutes with $D$ and $r_{i_j}$ for all $j$. We know $r_{D(i)} \cdot r_{i_1} \cdots r_{i_m} = r_{i_1} \cdots r_{i_m} \cdot r_i$, and applying both sides to $\alpha_{i_j}$ we find $(\alpha_i, \alpha_j) = 0$, therefore $r_i$ commutes with $r_{i_j}$ for all $j$ and this in turn implies that $r_i$ commutes with $D$.

3.3. Using the lemma we have an automorphism $\omega \cdot D \cdot \dot{r}_{i_1} \cdots \dot{r}_{i_m}$ of $\mathbf{g}$, whose restriction to $\mathbf{h}$ coincides with $\sigma$. Therefore the composition $\sigma \cdot \omega \cdot D \cdot \dot{r}_{i_1} \cdots \dot{r}_{i_m}$ is an automorphism leaving $\mathbf{h}$ pointwise fixed, hence it has to be of the form $\exp \operatorname{ad}(-h)$ for some $h$ in $\mathbf{h}$, that is, $\sigma = \omega \cdot D \cdot \dot{r}_{i_1} \cdots \dot{r}_{i_m} \cdot \exp \operatorname{ad}(h)$. Writing $h = h^+ + h^-$ with $\sigma(h^+) = h^+$, $\sigma(h^-) = -h^-$, and conjugating by $\exp \operatorname{ad}(h^-/2)$, we can assume $\sigma(h) = h$.

The condition $\sigma^2 = \mathrm{id}$ reads $\exp \operatorname{ad}(2h) \cdot \dot{r}_{i_1}^2 \cdots \dot{r}_{i_m}^2 = \mathrm{id}$ and we know that $\dot{r}_i^2 = \exp \operatorname{ad}(\pi i \alpha_i^\vee)$. Hence we have

$$2h = \pi i \sum_{j=1}^{m} \alpha_{i_j}^\vee + 2\pi i \sum_{j=1}^{n} b_j \hat{p}_j, \qquad b_j \in \{0, 1\}$$

or

$$h = (\pi i/2) \sum_{j=1}^{m} \alpha_{i_j}^\vee + \pi i \sum_{j=1}^{n} b_j \hat{p}_j.$$

The condition $\sigma(h) = h$ means

$$h = \pi i \sum_{j=1}^{m} (1 + b_{i_j}) \hat{p}_{i_j} + \pi i \sum_{\alpha_j \in M} (b_{D(j)} - b_j)(\hat{p}_{D(j)} - \hat{p}_j)/2,$$
$$M \cap D(M) = \varnothing, \; M \cup D(M) = \pi.$$

If $b_{i_j} = 0$ then $\sigma(e_{i_j}) = -e_{i_j}$ and

$$\dim(\exp \operatorname{ad}(-f_{i_j}/2) \cdot \sigma \cdot \exp \operatorname{ad}(f_{i_j}/2)(\mathbf{b}_+) \cap \mathbf{b}_+) < \dim(\sigma(\mathbf{b}_+) \cap \mathbf{b}_+).$$

Hence if we assume $\dim(\sigma(\mathbf{b}_+) \cap \mathbf{b}_+)$ minimal for $\sigma$ in its conjugacy class we have $b_{i_j} = 1$, $1 < j < m$, that is, $h = \pi i \sum_{j \in M} (b_{D(j)} - b_j)(p_{D(j)} - p_j)/2$.

The condition $2h + \pi i \sum_{j=1}^{m} \alpha_{i_j}^\vee \in L$ leaves us with the following cases;

TABLE VII

Involutions of the Second Kind of Affine Lie Algebras

| Algebra | | Automorphism |
|---|---|---|
| $a_1^{(1)}$ | | $\sigma_1 = \omega$ |
| | | $\sigma_2 = \omega \cdot \rho_1$ |
| $a_n^{(1)}$ | | $\sigma_1 = \omega$ |
| | | $\sigma_2 = \omega \cdot \rho_0$ |
| | | $\sigma_3 = \omega \cdot \rho_0 \cdot \dot{r}_0 \cdot \exp \operatorname{ad}(\pi i(\hat{p}_1 - \hat{p}_n)/2)$ |
| | $n = 2k+1$ | $\sigma_4 = \omega \cdot \rho_0 \cdot \dot{r}_0 \cdot \dot{r}_{k+1} \cdot \exp \operatorname{ad}(\pi i(\hat{p}_1 - \hat{p}_n + \hat{p}_{k+2} - \hat{p}_k)/2)$ |
| | | $\sigma_5 = \omega \cdot \rho_1$ |
| | | $\sigma_6 = \omega \cdot \rho_2$ |
| $b_n^{(1)}$ | | $\sigma_1 = \omega$ |
| | | $\sigma_2 = \omega \cdot \rho_1$ |
| $c_n^{(1)}$ | | $\sigma_1 = \omega$ |
| | | $\sigma_2 = \omega \cdot \rho_1$ |
| | $n = 2k$ | $\sigma_3 = \omega \cdot \rho_1 \cdot \dot{r}_k \cdot \exp \operatorname{ad}(\pi i(\hat{p}_{k-1} - \hat{p}_{k+1})/2)$ |
| $d_n^{(1)}$ | | $\sigma_1 = \omega$ |
| | | $\sigma_2 = \omega \cdot \rho_0$ |
| | | $\sigma_3 = \omega \cdot \rho_1$ |
| | | $\sigma_4 = \omega \cdot \rho_2$ |
| | $n = 2k$ | $\sigma_5 = \sigma_4 \cdot \dot{r}_2 \cdot \dot{r}_4 \cdots \dot{r}_{n-4} \cdot \dot{r}_{n-2} \cdot \exp \operatorname{ad}(\pi i(\hat{p}_0 - \hat{p}_1 + \hat{p}_n - \hat{p}_{n-1})/2)$ |
| | | $\sigma_6 = \sigma_3 \cdot \dot{r}_k \cdot \exp \operatorname{ad}(\pi i(\hat{p}_{k-1} - \hat{p}_{k+1})/2)$ |
| | $n = 2k+1$ | $\sigma_7 = \sigma_2 \cdot \dot{r}_1 \cdot \dot{r}_3 \cdots \dot{r}_{n-4} \cdot \dot{r}_{n-2} \cdot \exp \operatorname{ad}(\pi i(\hat{p}_n - \hat{p}_{n-1})/2)$ |
| $e_6^{(1)}$ | | $\sigma_1 = \omega$ |
| | | $\sigma_2 = \omega \cdot \rho_0$ |
| | | $\sigma_3 = \omega \cdot \rho_0 \cdot \dot{r}_0 \cdot \dot{r}_3 \cdot \exp \operatorname{ad}(\pi i(\hat{p}_2 - \hat{p}_4)/2)$ |
| $e_7^{(1)}$ | | $\sigma_1 = \omega$ |
| | | $\sigma_2 = \omega \cdot \rho_1$ |
| $e_8^{(1)}$ | | $\sigma_1 = \omega$ |
| $f_4^{(1)}$ | | $\sigma_1 = \omega$ |
| $g_2^{(1)}$ | | $\sigma_1 = \omega$ |
| $a_2^{(2)}$ | | $\sigma_1 = \omega$ |
| $a_{2n}^{(2)}$ | | $\sigma_1 = \omega$ |
| $a_{2n-1}^{(2)}$ | | $\sigma_1 = \omega$ |
| | | $\sigma_2 = \omega \cdot \rho_1$ |
| | | $\sigma_3 = \omega \cdot \rho_1 \cdot \dot{r}_2 \cdot \dot{r}_4 \cdots \dot{r}_{2[n/2]-2} \cdot \dot{r}_{2[n/2]} \cdot \exp \operatorname{ad}(\pi i(\hat{p}_0 - \hat{p}_1)/2)$ |
| $d_{n+1}^{(2)}$ | | $\sigma_1 = \omega$ |
| | | $\sigma_2 = \omega \cdot \rho_1$ |
| | $n = 2k$ | $\sigma_3 = \omega \cdot \rho_1 \cdot \dot{r}_k \cdot \exp \operatorname{ad}(\pi i(\hat{p}_{k-1} - \hat{p}_{k+1})/2)$ |
| $e_6^{(2)}$ | | $\sigma_1 = \omega$ |
| $d_4^{(3)}$ | | $\sigma_1 = \omega$ |

besides $\omega$ and $\omega \cdot D$, the notation is the same as in the preceding section (see Table VII):

$$a_{2k}^{(1)} = \omega \cdot \rho_0 \cdot \dot{r}_0 \cdot \exp \mathrm{ad}(\pi i(\hat{p}_1 - \hat{p}_{2k})/2)$$

$$a_{2k+1}^{(1)} = \omega \cdot \rho_0 \cdot \dot{r}_0 \cdot \exp \mathrm{ad}(\pi i(\hat{p}_1 - \hat{p}_{2k+1})/2)$$

$$= \omega \cdot \rho_0 \cdot \dot{r}_0 \cdot \dot{r}_{k+1} \cdot \exp \mathrm{ad}(\pi i(\hat{p}_1 - \hat{p}_{2k+1} - (\hat{p}_k - \hat{p}_{k+2}))/2)$$

$$c_{2k}^{(1)} = \omega \cdot \rho_1 \cdot \dot{r}_k \cdot \exp \mathrm{ad}(\pi i(\hat{p}_{k-1} - \hat{p}_{k+1})/2)$$

$$d_{2k}^{(1)} = \omega \cdot \rho_2 \cdot \dot{r}_2 \cdots \dot{r}_{2k-4} \cdot \dot{r}_{2k-2} \cdot \exp \mathrm{ad}(\pi i(\hat{p}_0 - \hat{p}_1 + \hat{p}_{2k} - \hat{p}_{2k-1})/2)$$

$$= \omega \cdot \rho_1 \cdot \dot{r}_k \cdot \exp \mathrm{ad}(\pi i(\hat{p}_{k-1} - \hat{p}_{k+1})/2)$$

$$d_{2k+1}^{(1)} = \omega \cdot \rho_0 \cdot \dot{r}_1 \cdot \dot{r}_3 \cdots \dot{r}_{2k-3} \cdot \dot{r}_{2k-1} \cdot \exp \mathrm{ad}(\pi i(\hat{p}_{2k} - \hat{p}_{2k+1})/2)$$

$$e_6^{(1)} = \omega \cdot \rho_0 \cdot \dot{r}_0 \cdot \dot{r}_3 \cdot = \omega \cdot \rho_2 - \hat{p}_4)/2).$$

## 4. Realizations of Classical Involutions

We consider the case when $\mathring{\mathfrak{g}}$ is a classical Lie algebra viewed as a subalgebra of $M(n, \mathbb{C})$, $n \in \mathbb{N}$. Then we can realize the automorphisms of $L(\mathring{\mathfrak{g}})$ in $\mathrm{Aut}(A)$, in one of the following ways:

$$A \xrightarrow{\rho} CAC^{-1} + \varphi(A)c$$

or

$$A \xrightarrow{\rho} -CA^TC^{-1} + \varphi(A)c,$$

where $A$ is in $L(\mathring{\mathfrak{g}})$; $C$ is a certain element of $gl(\mathbb{C}[t, t^{-1}])$ and $\varphi(A) = \lambda\,\mathrm{Res}(\mathrm{tr}(C(dC/dt)A))$, for $A$ in $L(\mathring{\mathfrak{g}})$, $\langle X, Y\rangle = \lambda\,\mathrm{tr}(XY)$, $X, Y \in \mathring{\mathfrak{g}}$. Indeed, $\varphi(A)$ is determined by the condition

$$\rho([A, B]) = [\rho(A), \rho(B)],$$

which implies

$$\varphi([A, B]) = \psi(\rho(A) - \varphi(A)c, \rho(B) - \varphi(B)c) - \psi(A, B)$$

$$= \mathrm{Res}\left(\lambda\,\mathrm{tr}\left(C\frac{dC}{dt}[A, B]\right)\right).$$

We consider only the cases when $\rho$ does not preserve $\mathring{\mathfrak{g}}$. We will use the notation $J_s = (a_{ij})_{i,j=1}^{s} = ((-1)^{i+1}\delta_{i,s+1-j})$; $I_s = \mathrm{Id}_{\mathbb{C}^s}$. We have the following:

$$\hat{sl}_2(\mathbb{C})\ \rho_1(\alpha_0) = \alpha_1, \qquad \rho_1(A) = CAC^{-1} + \varphi(A)c,$$

where $C = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$;

$$\hat{sl}_{2k}(\mathbb{C})\ \rho_1(\alpha_i) = \alpha_{2k+i-1}, \qquad \rho_1(A) = -CA^TC^{-1} + \varphi(A)c,$$

where $C = \begin{pmatrix} -t^{-1} & 0 \\ 0 & J \end{pmatrix}$;

$$\rho_2(\alpha_i) = \alpha_{i+k}, \qquad \rho_2(A) = CAC^{-1} + \varphi(A)c,$$

where $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where $A = I_k$, $B = tI_k$;

$$s\hat{o}_{2k+1}(\mathbb{C})\ \rho_1(\alpha_0) = \alpha_1, \qquad \rho_1(A) = CAC^{-1} + \varphi(A)c,$$

where $C = \begin{pmatrix} 1 & & -t^{-1} \\ & I_{s-1} & \cdot \\ t & & I_{s-1} \end{pmatrix}$;

$$s\hat{p}_n(\mathbb{C})\ \rho_1(\alpha_i) = \alpha_{n-i}, \qquad \rho_1(A) = -CA^TC^{-1} + \varphi(A)c,$$

where $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where $A = J_n$, $B = -tJ_n$;

$$s\hat{o}_{2k}(\mathbb{C})\ \rho(\alpha_i) = \alpha_{k-i}, \qquad \rho(A) = CAC^{-1} + \varphi(A)c,$$

where $C = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$, where $A = J_k$, $B = -tJ_k$;

$$\rho(\alpha_0) = \alpha_1, \qquad \rho(\alpha_k) = \alpha_{k-1}, \qquad \rho(A) = CAC^{-1} + \varphi(A)c,$$

where $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where $A = \begin{pmatrix} & & t^{-1} \\ & -I_{k-1} & \\ t & & \end{pmatrix}$, $B = \begin{pmatrix} & & -1 \\ & -I_{k-1} & \\ -1 & & \end{pmatrix}$.

## 5. Fixed Points Sets

5.1. To compute the fixed points set of an involution we need to make the following observations:

(1) If $\sigma$ is of the form $1 \times \tau$, where $\tau$ is an involution of the finite-dimensional Lie algebra $\mathring{\mathbf{g}}$, then $\hat{L}(\mathring{\mathbf{g}})^\sigma \simeq \hat{L}(\mathring{\mathbf{g}}^\tau)$.

(2) If $\sigma = \tau_0$, then $\hat{L}(\mathring{\mathbf{g}}, \rho)^\sigma \simeq \hat{L}(\mathring{\mathbf{g}}, \rho^2)$, $\rho$ in $\mathrm{Aut}(A)$.

(3) If $\sigma = \tau_0 \cdot (1 \times \rho)$, then $\hat{L}(\mathring{\mathbf{g}})^\sigma \simeq \hat{L}(\mathring{\mathbf{g}}, \rho)$, and this in turn is isomorphic to $g^{(i)}$ for $i = 1$ or 2, depending on whether $\rho$ is an inner or an outer automorphism of $\mathring{\mathbf{g}}$.

The remaining cases are handled case by case. The results are shown in Table VI.

5.2. We show in two examples how the fixed points set is computed. The idea is the following: we consider the Kac–Moody algebra $\mathbf{g}_0 = \langle e_i + \sigma(e_i); f_i + \sigma(f_i), i \text{ in } I \rangle$, clearly $\mathbf{g}_0 \subset \mathbf{g}^\sigma$, and we are going to look for a complement to $\mathbf{g}_0$ in $\mathbf{g}^\sigma$; using a gradation on $\mathbf{g}^\sigma$, we find the first element $Y$

in $\mathbf{g}^\sigma \backslash \mathbf{g}_0$, form $\mathbf{g}_0' = \langle \mathbf{g}_0, Y \rangle$, and repeat the procedure with $\mathbf{g}_0'$ instead of $\mathbf{g}_0$.

EXAMPLE I. $\mathbf{g}$ of type $A_{2k-1}^{(1)}$, $\sigma = \rho_1$, in $\mathrm{Aut}(A)$, satisfying $\rho_1(e_i) = e_{2k-i-1}$, $i \in I$.

We have the Kac–Moody algebra $\mathbf{g}_0$ generated by

$$\{X_{\alpha_i} + \sigma(X_{\alpha_i}); X_{-\alpha_i} + \sigma(X_{-\alpha_i}); h_i + \sigma(h_i); c\}, \qquad i = 0, ..., k-1.$$

From the realization it follows that $\sigma$ commutes with multiplication by $t$ in $\mathbf{n}_+$.

We have that $\mathbf{g}^\sigma = \mathbf{n}_+^\sigma + \mathbf{h}^\sigma + \mathbf{n}_-^\sigma$; $\mathbf{g}_0 = \mathbf{n}_+^0 + \mathbf{h}^0 + \mathbf{n}_-^0$ and we also have

$$\mathbf{n}_+^\sigma = \bigoplus_{j \geqslant 0} t^j(\mathring{\mathbf{n}}_+ + (t \otimes \mathring{\mathbf{h}}) + (t \otimes \mathring{\mathbf{n}}_-))^\sigma.$$

We want to find a complement $Q_+$ to $\mathbf{n}_+^0$ in $\mathbf{n}_+^\sigma$; as $\mathbf{n}_+^0$ is graded and $\sigma$ is $C[t]$ linear on $\mathbf{n}_+$, we will have $Q_+ = \bigoplus_{j \geqslant 0} t^j Q_0$, where $Q_0$ is a complement to $\mathbf{n}_+^0 \cap (\mathring{\mathbf{n}}_+ + (t \otimes \mathring{\mathbf{h}}) + (t \otimes \mathring{\mathbf{n}}_-))$ in $(\mathring{\mathbf{n}}_0 + (t \otimes \mathring{\mathbf{h}}) + (t \otimes \mathring{\mathbf{n}}_-))^\sigma$.

As we have $\mathring{\mathbf{n}}_+^\sigma \cup (t \otimes \mathring{\mathbf{h}})^\sigma \cup (t \times \mathring{\mathbf{n}}_-) \subset \mathbf{g}_0$, we need only to consider the subspace $\bigoplus_{g_\alpha \subset \mathring{n}_+} (g_\alpha + g_{\sigma(\alpha)})^\sigma$. Take $X_\alpha + \sigma(X_\alpha) \in \mathbf{g}^\sigma$, $X_\alpha \in \mathring{\mathbf{n}}_+ \backslash \mathbf{n}_+^0$, $\mathrm{ht}(\alpha)$ minimal. Write $X_\alpha = [X_{\alpha_i}, X_\beta]$, for $0 < i < k$ smallest possible. Consider

$$[X_{\alpha_i} + \sigma(X_{\alpha_i}), X_\beta + \sigma(X_\beta)] = [X_{\alpha_i}, X_\beta] + \sigma([X_{\alpha_i}, X_\beta]) + [X_{\alpha_i}, \sigma(X_\beta)] + \sigma([X_{\alpha_i}, \sigma(X_\beta)]).$$

By the minimality assumption on $\mathrm{ht}(\alpha)$, we have that $X_\beta + \sigma(X_\beta)$ belongs to $\mathbf{n}_+^0$, there fore the only cases when $X_\alpha + \sigma(X_\alpha)$ might not be in $\mathbf{n}_+^0$ are when $(*)\, \alpha_i + \beta$ and $\sigma(\alpha_i) + \beta$ belong to $\Delta$.

Let $\beta = \sum_{j=m}^n \alpha_j$, the condition $(*)$ implies $2k - m = n - 1$, $\sigma(\alpha_{m-1}) = \alpha_{n+1}$, and $\sigma(\beta) = \beta$, therefore $\sigma(X_\beta) = -\chi_\beta$ and $X_\alpha + \sigma(X_\alpha) = [X_{\alpha_i} - \sigma(X_{\alpha_i}), X_\beta]$, but $X_\beta = [X_{\alpha_m} - \sigma(X_{\alpha_m}), X_{\beta - \alpha_m} + (X_{\beta - \alpha_m})]$ for $X_{\beta - \alpha_m}$ in $g_{\beta - \alpha_m}$. Hence

$$\begin{aligned} X_\alpha + \sigma(X_\alpha) &= [X_{\alpha_{m-1}} - \sigma(X_{\alpha_{m-1}}), [X_{\alpha_m} - \sigma(X_{\alpha_m}), X_{\beta - \alpha_m} + \sigma(X_{\beta - \alpha_m}]] \\ &= a[[X_{\alpha_{n-1}} + \sigma(X_{\alpha_{n-1}}), X_{\alpha_n} + \sigma(X_{\alpha_n})], X_{\beta - \alpha_n} + \sigma(X_{\beta - \alpha_n})] \\ &\quad + b[X_{\alpha n} + \sigma(X_{\alpha_n}), [X_{\alpha_{n-1}} + \sigma(X_{\alpha_{n-1}}), X_{\beta - \alpha_n} + \sigma(X_{\beta - \alpha_n})]], \\ &\qquad a, b \text{ in } \mathbb{C}, \end{aligned}$$

The right-hand side is in $\mathbf{n}_+^0$, therefore in this case we have $(g_\alpha + g_{\pi\alpha})^\sigma \subset \mathbf{g}_0$, which implies $Q_0 = \{0\}$ and then $\mathbf{g}_0 = \mathbf{g}^\sigma$.

EXAMPLE II. Consider the involution $\rho_2$ of $a_{2k-1}$, $\rho_2(e_i)=e_{i+k}$. Take $\beta=\sum_{j=n}^{0}\alpha_j$ as before. In this case the condition (*) implies

$$i=m-1, \qquad m+k-1=n+1.$$

Again $\mathrm{ht}(\alpha)$ minimal implies $X_\beta+\sigma(X_\beta)$ is in $\mathbf{g}_0$ and $[X_{\alpha_i}+\sigma(X_{\alpha_i}), X_\beta+\sigma(X_\beta)]=X_\alpha+\sigma(X_\alpha)+X_{\alpha'}+\sigma(X_{\alpha'})$ is in $g$, where $\alpha'=\alpha_{n+1}+\beta$. Taking $m=2, ..., k+1$ we have a linearly dependent family of rank $k-1$. In fact there is a linear combination $Y=\sum_{\mathrm{ht}\,\gamma=k}c_\gamma(X_\gamma+\sigma(X_\gamma))$ such that $[Y, X_{\alpha_i}+\sigma(X_{\alpha_i})]=0$, $i=0, ..., k-1$, and the space $\mathbf{H}'=\bigoplus_{j\in\mathbb{Z}}t^j\otimes Y$ is complementary to $\mathbf{g}_0$ in $\mathbf{g}^\sigma$ witht he additional property that $\mathbf{H}=\mathbf{H}'\oplus Cc$ is an infinite-dimensional Heisenberg Lie algebra.

5.3. In other cases we find that $\langle X_\alpha+\sigma(X_\alpha)\notin\mathbf{g}_0 : \alpha>0$ and $\mathrm{ht}\,\alpha$ is minimal$\rangle$ is a one-dimensional space generated by an element $Y_+$ such that $[X_{-\alpha_i}+\sigma(X_{\alpha_i}), Y_+]=0$ for all $i$ in $I$, and there exists $\alpha_j\in\pi$, such that $[X_{\alpha_j}+\sigma(X_{\alpha_j}), Y_+]\neq 0$. In this case we define $\mathbf{g}_0'=$ the subalgebra generated by $\{\mathbf{g}_0, Y_+, Y_-, [Y_+, Y_-]\}$, where $Y_-$ is a generator for the corresponding space in $\mathbf{n}_-^\sigma$. We then repeat the process with $\mathbf{g}_0'$ instead of $\mathbf{g}_0$.

## APPENDIX A: TITS BUILDINGS

*Introduction*

In the process of defining buildings we should keep in mind the following procedure to construct complicated geometrical objects from simpler ones [T, 1].

Take an object $C$ and to each element $x$ of $C$ attach a subgroup $G_x$ of the group $G$. Then there exists a unique minimal object $\mathscr{B}$, extending $C$, on which $G$ acts in such a way that no two elements of $C$ are equivalent under $G$ and $G_x$ is the stability group of $x$ in $G$. This can be achieved by taking the quotient of the product $G\times C$ by the equivalence relation $(g, x)\sim(g', x')$ if and only if $x=x'$ and $g^{-1}g'\in G_x$.

Now let $(G, B, N, S)$ be a Tits system of affine type, that is, $W=N/(B\cap N)$ is an affine Weyl group. We further assume that $\bigcap_{n\in N} nBn^{-1}=B\cap N$.

We are going to associate to $(G, B, N, S)$ a set $\mathscr{B}$ endowed with a structure of simplical complex, a set of morphisms of simplicial complexes from the affine space $A$ into $\mathscr{B}$, a distance making $\mathscr{B}$ a complete metric space for which the morphisms above are isometries, and an action of $G$ on $\mathscr{B}$ preserving these structures.

In this case $C$ is going to be a fundamental chamber for the irreducible action of $W$ on an affine space. For each $x$ in $C$ we call $W_x$ the stabilizer of $x$ in $W$. Finally, we associate to $x$ the group $G_x = BW_xB$.

A.1. *The Simplicial Complex*

Let $\mathscr{P}$ be the set of parabolic subgroups of $G$. For $P \in \mathscr{P}$ we denote by $\tau(P) \subset T$ the type of $P$. Let $\mathscr{B}$ be the set of pairs $(P, x)$ with $P \in \mathscr{P}$, $x \in C_{\tau(P)}$. For $P \in \mathscr{P}$, the set $F = F(P) = \{(P, x): x \in C_{\tau(P)}\}$ is called a *facet* of $\mathscr{B}$ of type $\tau(F) = \tau(P)$ and codimension card$(\tau(P))$. If $P' \in \mathscr{P}$ and $P \subset P'$ we say that $F(P')$ is a facet of $F(P)$, and we have then $\tau(P') \supset \tau(P)$. Reciprocally, the facet $F(P)$ has for any type $Y \subset \tau(P)$ exactly one facet of type $Y$ [BT, (2.1.1)].

The map $(P, x) \to x$ is called the *canonical map* from $\mathscr{B}$ onto $\bar{C}$. If $F$ (resp. $\bar{F}$) is a facet of type $X$ of $\mathscr{B}$, the restriction of this map to $F$ (resp. $\bar{F}$) is a bijection of $F$ (resp. $\bar{F}$) onto $C_X$ (resp. $\bar{C}_X$) whose inverse is called the canonical map from $C_X$ (resp. $\bar{C}_X$) onto $F$ (resp. $\bar{F}$). Then we can give $F$ (resp. $\bar{F}$) the structure of an open (resp. closed) affine simplex that comes from the structure in $C$ (resp. $\bar{C}$) [BT, (2.1.2)].

A.1.1. $G$ acts on $\mathscr{B}$ by $g \cdot (P, x) = (gPg^{-1}, x)$, $P \in \mathscr{P}$, $x \in C_{\tau(P)}$ [BT, (2.1.4)].

A.2. *The Structural Mappings and the Apartments of* $\mathscr{B}$

There exists a unique map $j$: $A \to \mathscr{B}$ satisfying the following two conditions:

(i) the restriction of $j$ to $\bar{C}$ is the canonical bijection from $\bar{C}$ onto $\overline{F(B)}$;

(ii) $j(v(n) \cdot x) = n \cdot j(x)$ for all $n \in N$, $x \in A$.

The map $j$ is injective. If $F$ is a facet of $A$ then $j(F)$ is a facet of $\mathscr{B}$ of the same type, we have $j(\bar{F}) = \overline{j(F)}$, and the restriction of $j$ to $\bar{F}$ is an affine bijection of $\bar{F}$ onto $\overline{j(F)}$ [BT, Proposition 2.2.1].

A.2.1. Definition. The map $j$ is called the *canonical map* from $A$ into $\mathscr{B}$. A map $\varphi$ from $A$ into $\mathscr{B}$ is called a *structural map* of $\mathscr{B}$ if there is an element $g \in G$ such that $\varphi(x) = g \cdot j(x)$ for all $x$ in $A$.

We call the *apartment* (resp. *half apartment*, *wall*) of $\mathscr{B}$ any subset of $\mathscr{B}$ that is the image of $A$ (resp. an affine root, a wall of $A$) under a structural map.

We can transport the structure of affine euclidean space from $A$ to $j(A)$ and to any apartment by requiring the maps to be isometries. In particular, we will denote the distance in an apartment $M$ by $d_M$. For any $g$ in $G$, the map $x \to g \cdot x$ is an isomorphism of $A$ onto the apartment $g \cdot A$.

A.3. *The Metric of the Building*

There is a unique function $d$: $\mathscr{B} \times \mathscr{B} \to \mathbb{R}_+$ such that on every apartment $A$, $d$ coincides with $d_A$ [BT, Lemma 2.5.1]. The function $d$ is invariant under $G$.

Furthermore, $d$ is a distance on $\mathscr{B}$ and for any $x$, $y$ in $\mathscr{B}$, the subset $D = \{z \in \mathscr{B}\colon d(x, z) + d(z, y) = d(x, y)\}$ is contained in every apartment $A$ containing $x$ and $y$, and coincides with the segment $[x, y]$ of the affine space $A$ [BT, Proposition 2.5.4].

A.3.1. The metric space $\mathscr{B}$ is complete [BT, Theorem 2.5.12].

A.4. Definition. The set $\mathscr{B}$ endowed with the structure of simplicial complex, the family of structural maps, and the distance $d$ defined above is called the building of the Tits system $(G, B, N, S)$.

A.5. *B-Adapted Homomorphisms*

Let $\varphi$: $G \to \hat{G}$ be a $B$-adapted homomorphism. Let $(P, x) \in \mathscr{B}$, $P \in \mathscr{P}$, $x \in C_{\tau(P)}$; as ${}^gP$ is of type $\zeta(g)(\tau(P))$, the pair $({}^gP, \xi(g) \cdot x)$ is an element of $\mathscr{B}$ and this determines an action of $\hat{G}$ on $\mathscr{B}$.

The action satisfies [BT, Proposition 2.7.2]:

(i) If $g \in G$, $y \in \mathscr{B}$, then $\varphi(g) \cdot y = g \cdot y$.

(ii) For every $g$ in $\hat{G}$, the map $y \to g \cdot y$ is an isometry of the simplicial complex.

A.5.1. In particular, we can take $\hat{G} = \operatorname{Aut}_B G$, that is, the group of automorphisms of $G$ that preserve the conjugacy class of $B$, and we take Ad: $G \to \operatorname{Aut}_B G$ as the map $\varphi$. Then $\gamma \cdot g \cdot x = \gamma(g) \cdot \gamma \cdot x$ for every $\gamma$ in $\hat{G}$, $g$ in $G$, and $x$ in $\mathscr{B}$ [BT, 2.7.4].

## APPENDIX B: Fixed Point Lemma

B.1. Lemma. *Let $x$, $y \in \mathscr{B}$. There is a unique point $m$ in $\mathscr{B}$ such that $d(x, m) = d(y, m) = d(x, y)/2$. It is called the middle of $[x, y]$. Let $x$, $y$, $z \in \mathscr{B}$ and let $m$ be the middle of $[x, y]$, then*

$$(*) \qquad d(x, z)^2 + d(z, y)^2 \geqslant 2d(m, z)^2 + d(x, y)^2/2.$$

*This can be shown by using the fact that for any apartment $A$ of $\mathscr{B}$ and any chamber $C$ of $A$, there exists a retraction $\rho_C$: $\mathscr{B} \to A$ such that $\rho_C^{-1}(\bar{C}) = \bar{C}$* [BT, (2.3.4] *and such that*

$$d(\rho_C(u), \rho_C(v)) \leqslant d(u, v) \qquad \textit{for all} \quad u, v \textit{ in } \mathscr{B},$$

*having equality when there is an apartment that contains u, v, and* $\bar{C}$ [BT, (2.5.3)]. *Taking an apartment A that contains x and y, and a chamber* $\bar{C}$ *in A that contains m, we have* $d(\rho_C(z), \rho_C(m)) = d(z, m)$ *and we are reduced to a planar geometry inequality.*

B.2. LEMMA. *Let E be a complete metric space and let E′ be a part of E having the following property:*

*For any pair of points x, y in E′, there exists a point m in E such that* (*) *holds for every z in E.*

*If M is a bounded set in E′, then the stabilizer of M in* Isom *E, i.e.,* $\{\sigma \in \text{Isom } E: \sigma(M) \subset M\}$, *has a fixed point in the closure of E′ in E.*

*Proof.* If $X$, $Y$ are two sets in $E$ we call $\text{diam}(X, Y) = \sup d(x, y)$ and $\text{diam}(X) = \text{diam}(X, X)$.

Let $k \in \mathbb{R}$, $0 < k < 1$. For every set $X \subset E'$ define

$$f(X) = \{m \in E' : \exists x, y \in X \text{ s.t. } (*) \text{ is satisfied } \forall z \in E$$

and

$$d(x, y) > k \cdot \text{diam}(X)\}.$$

Then $f(X) \neq \emptyset$ if $X \neq \emptyset$.

For $m$, $x$, $y$ as in the definition of $f(X)$ we have

$$\begin{aligned} d(m, z)^2 &\leqslant (d(x, z)^2 + d(y, z)^2)/2 - d(x, y)^2/4 \\ &< \text{diam}(X, z)^2 - k^2 \, \text{diam}(X)^2/4. \end{aligned}$$

Taking $z$ in $X$,

$$\text{diam}(f(X), X) < k_1 \, \text{diam}(X), \qquad \text{where} \quad k_1 = \sqrt{1 - k^2/4}.$$

Therefore taking $z$ in $f(X)$,

$$\text{diam } f(X) < k_2 \, \text{diam}(X), \qquad k_2 = \sqrt{1 - k^2/2}.$$

If $M$ is a bounded set of $E'$, then

$$\text{diam}(f^q(M)) < k_2^q \, \text{diam}(M), \qquad q > 1;$$

for every $q$ in $\mathbb{N}$ pick $x$ in $f^q(M)$, then

$$d(x_q, x_{q+1}) < k_1 k_2^q \, \text{diam}(M)$$

$$d(x_q, x_{q+m}) < k_1 k_2^q \, \text{diam}(M)/(1 - k_2) = c k_2^q, \qquad c \text{ a constant.}$$

Hence $\{x_q\}$ is a Cauchy sequence in $E$ and therefore there is a limit point $x \in \bar{E}'$, independent of the choice of $\{x_q\}_{q \in \mathbb{N}}$. As the stabilizer of $M$ in Isom$(E)$ leaves each one of the $f^q(M)$ stable, $x$ has to be fixed.

B.3. PROPOSITION. *Let $M$ be a bounded set of $\mathscr{B}$. The stabilizer of $M$ in* Isom $\mathscr{B}$ *has a fixed point $x$, belonging to the closure of the convex hull of $M$.*

*Proof.* Take $E = \mathscr{B}$, $E' =$ convex hull of $M$ and apply the lemma.

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